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MAIN ARTICLES

On the Rubio de Francia's Theorem in Variable Lebesgue Spaces

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To the memory of academician Levan Zhizhiashvili

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In this paper we study some generalization of Rubio de Francia's theorem in variable exponent Lebesgue spaces.

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1. Main result

The Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ are of interest for their applications to modeling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth condition (see [4], [3]).

Given a measurable function $p : \mathbb{R}^n \longrightarrow [1, \infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Let $B(x, r)$ denote the open ball in $\mathbb{R}^n$ of radius r and center x. By $|B(x, r)|$ we denote $n-$dimensional Lebesgue measure of $B(x, r)$. The Hardy-Littlewood

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maximal operator M is defined on the locally integrable function f on $\mathbb{R}^n$ by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Define the spherical maximal operator $\mathcal{M}$, by

$$\mathcal{M}f(x) := \sup_{t>0} |\mu_t * f(x)| = \sup_{t>0} \left| \int_{\{y\in\mathbb{R}^n:|y|=1\}} f(x-ty)d\mu_1(y) \right|$$

where μ_t denotes the normalized surface measure on the sphere of center 0 and radius t in $\mathbb{R}^n$. The Hardy-Littlewood maximal operator M, which involves averaging over balls, is clearly related to the spherical maximal operator, which averages over spheres. Indeed, by using polar coordinates, one easily verifies the pointwise inequality $Mf(x) \leq \mathcal{M}f(x)$ for any (continuous) function.

Given a multiplier $m \in L^\infty(\mathbb{R}^n)$, we define the operators $\mathcal{M}_t$, $t > 0$ by $(\mathcal{M}_t f)^\wedge(\xi) = \widehat{f}(\xi)m(t\xi)$ and the maximal multiplier operator $\mathcal{M}_m f(x) = \sup_{t>0} |(\mathcal{M}_t f)(x)|$ (which is well defined a priori for the Schwartz function).

For $\alpha > 0$, let $m_\alpha(x) = (1-|x|^2)^{\alpha-1}/\Gamma(\alpha)$, where $|x| < 1$, and $m_\alpha(x) = 0$ if $|x| \geq 1$. With $m_{\alpha,t}(x) = m_\alpha(x/t)t^{-n}$, $t > 0$, we define spherical means of (complex) order $Re\alpha > 0$, by

$$\mathcal{M}_t^\alpha f(x) = (m_{\alpha,t} * f)(x).$$

Note that the Fourier transform of m_α is given by

$$\widehat{m}_\alpha(\xi) = \pi^{-\alpha+1}|\xi|^{-n/2-\alpha+1}J_{n/2+\alpha}(2\pi|\xi|).$$

The definition of $\mathcal{M}_t^\alpha$ can be extended to the region $Re\alpha \leq 0$ by the analytic continuation. Indeed for complex α in general we can define the operator $\mathcal{M}_t^\alpha$ by

$$(\mathcal{M}_t^\alpha f)^\wedge(\xi) = \widehat{m}_\alpha(t\xi)\widehat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^n).$$

Define the spherical maximal operator of order α by

$$\mathcal{M}^\alpha f(x) = \sup_{t>0} |M_t^\alpha f(x)|.$$

We observe that for $\alpha = 0$ we have $\mathcal{M}^\alpha f(x) = c\mathcal{M}f(x)$ for appropriate constant c.

Theorem 1.1 (Rubio de Francia)**:** *If $m(\xi)$ is the Fourier transform of a compactly supported Borel measure and satisfies $|m(\xi)| \leq (1+|\xi|)^{-a}$ for some $a > 1/2$ and all $\xi \in \mathbb{R}^n$, then the maximal operator $\mathcal{M}_m$ maps $L^p(\mathbb{R}^n)$ to itself when $p > \frac{2a+1}{2a}$.*

Note that for normalized surface measure of the sphere we have $|\widehat{d\mu_1}(\xi)| \leq C(1+|\xi|)^{-(n-1)/2}$ and from Theorem Rubio de Francia follows Stein's theorem on boundedness of the spherical maximal operator in $L^p(\mathbb{R}^n)$ (see [7]). According to Stein's theorem for the corresponding maximal operator (spherical maximal operator)

$$\|\mathcal{M}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

holds if $p > n/(n-1)$, $n \geq 3$, where f is initially taken to be in the class of rapidly decreasing functions. The two-dimensional version of this result was proved by Burgain [1]. The key feature of the spherical maximal operator is the non-vanishing Gaussian curvature of the sphere. Indeed, one obtains the same L^p bounds if the sphere is replaced by a piece of any hypersurface in $\mathbb{R}^n$ with everywhere non-vanishing Gaussian curvature (see [2]). More generally, if σ is smooth compactly supported measure in a hypersurface on $\mathbb{R}^n$ with k non vanishing principal curvatures ($k > 1$), then $|\widehat{\sigma}(\xi)| \leq C(1+|\xi|)^{-k/2}$ and from Theorem of Rubio de Francia follows Greenleaf's theorem (see [2], [8]).

Our aim of the paper is to study boundedness properties of the Rubio de Francia's maximal multiplier operator $\mathcal{M}_m$ in variable Lebesgue spaces. Note that the boundedness of the spherical maximal operator in variable Lebesgue spaces was investigated in [5] and [6].

In many applications a crucial step has been to show that the Hardy-Littlewood maximal operator is bounded on a variable L^p space. Note that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(\cdot}(\mathbb{R}^n)$ whenever the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot}(\mathbb{R}^n)$ (see [3], [4]).

Assume that $p_- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x)$. Let $\mathcal{B}(\mathbb{R}^n)$ be the class of all functions $p(\cdot)$ $(1 < p_- \leq p_+ < \infty)$ for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We say that a function $p : \mathbb{R}^n \to (0,\infty)$ is locally log-Hölder continuous on $\mathbb{R}^n$ if there exists $c_1 > 0$ such that

$$|p(x) - p(y)| \leq c_1 \frac{1}{\log(e + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. We say that $p(\cdot)$ satisfies the log-Hölder decay condition if there exist $p_\infty \in (0,\infty)$ and a constant $c_2 > 0$ such that

$$|p(x) - p_\infty| \leq c_2 \frac{1}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that $p(\cdot)$ is globally log-Hölder continuous in $\mathbb{R}^n$ ($p(\cdot) \in \mathcal{P}_{\log}$) if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

If $p : \mathbb{R}^n \to (1,\infty)$ is globally log-Hölder continuous function in $\mathbb{R}^n$ and $p^- > 1$, then the classical boundedness theorem for the Hardy-Littlewood maximal operator can be extended to $L^{p(\cdot)}$ (see(see [3], [4]).

By $\mathcal{B}_\theta(\mathbb{R}^n)$ $(0 < \theta < 1)$ we denote the class of exponents $p(\cdot)$ such that the following complex interpolation expansion $L^{p(\cdot)}(\mathbb{R}^n) = [L^2(\mathbb{R}^n), L^{\widetilde{p}(\cdot)}(\mathbb{R}^n)]_\theta$ is valid, where $\widetilde{p}(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ (obviously we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$). Note that $p(\cdot) \in \mathcal{B}_\theta(\mathbb{R}^n)$ if and only if $\frac{2\theta p(\cdot)}{2-(1-\theta)p(\cdot)} \in \mathcal{B}(\mathbb{R}^n)$.

Our main results are the following

Theorem 1.2 : *Let $m(\xi)$ be the Fourier transform of a compactly supported Borel measure σ and $|m(\xi)| \leq C(1+|\xi|)^{-\alpha}$, where $\alpha > 1/2$. If $p(\cdot) \in \mathcal{B}_\theta(\mathbb{R}^n)$ for some $0 < \theta < \frac{2\alpha-1}{2\alpha-1+2n}$, then the maximal operator $\mathcal{M}_m$ maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.*

Theorem 1.3 : *If $m(\xi)$ is the Fourier transform of a compactly supported Borel measure and satisfies $|m(\xi)| \leq (1+|\xi|)^{-a}$ for some $a > 1/2$ and all $\xi \in \mathbb{R}^n$. If $p(\cdot) \in \mathcal{P}_{\log}$ and*

$$\frac{2n+2\alpha-1}{n+2\alpha-1} < p_- \leq p_+ < \frac{n+2\alpha-1}{n} p_-.$$

then the maximal operator $\mathcal{M}_m$ maps $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.

2. Proofs

Proof (of Theorem 1.2): We set $m(\xi) = \widehat{d\sigma}(\xi)$. Obviously $m(\xi)$ is a C^∞ function. To study the maximal multiplier operator $\mathcal{M}_m f(x)$ we decompose the multiplier $m(\xi)$ into radial pieces as follows: we fix a radial C^∞ function φ_0 in $\mathbb{R}^n$ such that $\varphi_0(\xi) = 1$ when $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ when $|\xi| \leq 2$. For $j \geq 1$ we let

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{1-j}\xi)$$

and we observe that φ_j is localized near $|\xi| \approx 2^j$. Then we have

$$\sum_{j=0}^{\infty} \varphi_j = 1.$$

Set $m_j = \varphi_j m$ for all $j \geq 0$. Then m_j are C_0^∞ functions that satisfy

$$m = \sum_{j=0}^{\infty} m_j.$$

Also, the following estimate is valid:

$$\mathcal{M}_m f \leq \sum_{j=0}^{\infty} \mathcal{M}_j f$$

where

$$\mathcal{M}_j f(x) = \sup_{t>0} |\mathcal{F}^{-1}\left(\widehat{f}(\xi) m_j(t\xi)\right)(x)|.$$

Note that for any $j \geq 0$ we have (see [8]) the estimate

$$\|\mathcal{M}_j f\|_{L^2} \leq C 2^{(1/2-a)j} \|f\|_{L^2} \tag{2.1}$$

for all $f \in L^2(\mathbb{R}^n)$.

Note also that since $\widetilde{p}(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, we have the estimate

$$\|\mathcal{M}_j f\|_{\widetilde{p}(\cdot)} \leq C2^{j(n)} \|f\|_{\widetilde{p}(\cdot)} \tag{2.2}$$

for any $j \geq 0$. The proof of estimate (2.2) is based on the estimate

$$\mathcal{M}_j f(x) \leq C2^{j(n)} Mf(x), \tag{2.3}$$

where M is a Hardy-Littlewood maximal operator.

To establish (2.3), it suffices to show that for any $M > n$ there is a constant $C_M < \infty$ such that

$$\left|\left(\mathcal{F}^{-1}(\varphi_j) * d\sigma\right)(x)\right| \leq \frac{C2^{j(n)}}{(1+|x|)^M.} \tag{2.4}$$

Using the fact that φ is a Schwartz function, we have for every $N > 0$,

$$\left|\left(\mathcal{F}^{-1}(\varphi_j) * d\sigma\right)(x)\right| \leq C_N 2^{nj} \int_{\mathbb{R}^n} \frac{d\sigma(y)}{(1+2^j|x-y|)^N}. \tag{2.5}$$

Let $N > M$. We split the last integral into the regions

$$S_{-1}(x) = S^{n-1} \cap \{y \in \mathbb{R}^n : 2^j|x-y| \leq 1\}$$

and for $k > 0$,

$$S_k(x) = S^{n-1} \cap \{y \in \mathbb{R}^n : 2^k < 2^j|x-y| \leq 2^{k+1}\}.$$

We obtain the following estimate for the expression $\left|\left(\mathcal{F}^{-1}(\varphi_j) * d\sigma\right)(x)\right|$

$$\begin{aligned} &\sum_{k=-1}^{j} \int_{S_k(x)} \frac{C_N 2^{nj} d\sigma(y)}{(1+2^j|x-y|)^N} + \sum_{k=j+1}^{\infty} \int_{S_k(x)} \frac{C_N 2^{nj} d\sigma(y)}{(1+2^j|x-y|)^N} \\ &\leq C'_N 2^{nj} \sum_{k=-1}^{j} \frac{\sigma(S_k(x))\chi_{B(0,3)}(x)}{2^{kN}} + C_N 2^{nj} \sum_{k=j+1}^{\infty} \frac{\sigma(S_k(x))\chi_{B(0,2^{k+1-j}+1)}(x)}{2^{kN}} \\ &=: I + II. \end{aligned} \tag{2.6}$$

Using the fact that for $y \in S_k(x)$ we have $|x| \leq 2^{k+1-j} + 1$, we obtain the following estimate

$$I \leq C'_N 2^{nj} \sum_{k=-1}^{j} \frac{C2^{(k+1-j)}\chi_{B(0,3)}(x)}{2^{kN}} \leq C_N 2^{(n)j} \chi_{B(0,3)}(x). \tag{2.7}$$

On the other hand

$$II \le C'_N 2^{nj} \sum_{k=j+1}^{\infty} C 2^{-kN} \chi_{B(0,2^{k+1-j}+1)}(x) \tag{2.8}$$
$$\le C'_N \sum_{k=j+1}^{\infty} 2^{nj} 2^{-kN} \frac{(1+2^{k-j+2})^M}{(1+|x|)^M}$$
$$\le C'_M \sum_{k=j+1}^{\infty} \frac{2^{(k-j)(M-N)}}{2^{k(N+1-n)}}$$
$$\le \frac{C''_M 2^j}{(1+|x|)^M},$$

where we used that $N > M > n$. From (2.5)-(2.8) we obtain (2.4) and consequently (2.3).

From (2.1)-(2.2) we obtain

$$\|\mathcal{M}_j\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \le C \|\mathcal{M}_j\|^{1-\theta}_{L^2 \to L^2} \|\mathcal{M}_j\|_{L^{\widetilde{p}(\cdot)} \to L^{\widetilde{p}(\cdot)}} \preceq 2^{(1/2-\alpha)(1-\theta)j} 2^{j(n)\theta}. \tag{2.9}$$

Using the last estimate we obtain if $0 < \theta < \frac{2\alpha-1}{2\alpha-1+2n}$, then

$$\|\mathcal{M}_m\|_{p(\cdot)} \preceq \sum_{j=0}^{\infty} 2^{(1/2-a)(1-\theta)j} 2^{j(n)\theta} \|f\|_{p(\cdot)} \preceq \|f\|_{p(\cdot)}.$$

□

To prove Theorem 1.3 we need the following lemma.

Lemma 2.1: *Suppose $\alpha > 1/2$ and for exponent $p : \mathbb{R}^n \to (1, +\infty)$ we have*

$$\frac{2n+2\alpha-1}{n+2\alpha-1} < p_- \le p_+ < \frac{2n+2\alpha-1}{n}.$$

Then there exists exponent $\widetilde{p} : \mathbb{R}^n \to (1, +\infty)$ such that $1 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$ and $\frac{1}{p(x)} = \frac{1-\theta}{2} + \frac{\theta}{\widetilde{p}(x)}$; $x \in \mathbb{R}^n$ for some θ with property $0 < \theta < \frac{2\alpha-1}{2n+2\alpha-1}$.

Proof: Note that

$$1 < \frac{2n+2\alpha-1}{n+2\alpha-1} < 2 < \frac{2n+2\alpha-1}{n}.$$

We have

$$\frac{n}{2n+2\alpha-1} < \inf_{x\in\mathbb{R}^n} \frac{1}{p(x)} \le \sup_{x\in\mathbb{R}^n} \frac{1}{p(x)} < \frac{n+2\alpha-1}{2n+2\alpha-1}.$$

Let $\frac{1}{p(x)} = \frac{1}{2} + r(x)$. By the assumption we have

$$\frac{n}{2n+2\alpha-1} - \frac{1}{2} < \inf_{x\in\mathbb{R}^n} r(x) \le \sup_{x\in\mathbb{R}^n} r(x) < \frac{n+2\alpha-1}{2n+2\alpha-1} - \frac{1}{2}. \tag{2.10}$$

It is easy to see that the equation

$$\frac{1}{p(x)} = \frac{1-\theta}{2} + \frac{\theta}{\widetilde{p}(x)}; \tag{2.11}$$

is equivalent to

$$\frac{1}{2} + \frac{r(x)}{\theta} = \frac{1}{\widetilde{p}(x)}. \tag{2.12}$$

Using (2.9) we may take small $\delta > 0$ such that

$$\frac{n}{2n+2\alpha-1} - \frac{1}{2} + \delta < \inf_{x\in\mathbb{R}^n} r(x) \le \sup_{x\in\mathbb{R}^n} r(x) < \frac{n+2\alpha-1}{2n+2\alpha-1} - \frac{1}{2} - \delta.$$

Then for $\theta,\ 0 < \theta < \frac{2\alpha-1}{2\alpha-1}$, where $\theta = \theta < \frac{2\alpha-1}{2\alpha-1} - \theta_0,\ \ \theta_0 > 0$ we have

$$\frac{\frac{n}{2n+2\alpha-1} - \frac{1}{2} + \delta}{\frac{2\alpha-1}{2n+2\alpha+1} - \theta_0} < \inf_{x\in\mathbb{R}^n} \frac{r(x)}{\theta} \le \sup_{x\in\mathbb{R}^n} \frac{r(x)}{\theta} < \frac{\frac{n+2\alpha-1}{2n+2\alpha-1} - \frac{1}{2} - \delta}{\frac{2\alpha-1}{2n+2\alpha+1} - \theta_0}$$

$$-\frac{1}{2}\frac{\frac{2a-1}{2n+2a-1} - 2\delta}{\frac{2a-1}{2n+2a-1} - \theta_0} < \inf_{x\in\mathbb{R}^n} \frac{r(x)}{\theta} \le \sup_{x\in\mathbb{R}^n} \frac{r(x)}{\theta} < \frac{1}{2}\frac{\frac{2a-1}{2n+2a-1} - 2\delta}{\frac{2a-1}{2n+2a-1} - \theta_0}.$$

If we take $\theta_0 < 2\delta$ we obtain

$$-\frac{1}{2} < \inf_{x\in\mathbb{R}^n} \frac{r(x)}{\theta} \le \sup_{x\in\mathbb{R}^n} \frac{r(x)}{\theta} < \frac{1}{2}. \tag{2.13}$$

From (2.11) and (2.12) we get

$$0 < \inf_{x\in\mathbb{R}^n} \frac{1}{\widetilde{p}(x)} \le \sup_{x\in\mathbb{R}^n} \frac{1}{\widetilde{p}(x)} < 1.$$

Consequently we have $1 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$. □

Proof (Proof of Theorem 1.3):
As by the assumption

$$\frac{2n+2\alpha-1}{(n+2\alpha-1)p_-} < \frac{2n+2\alpha-1}{(n)p_+},$$

we can find θ such that

$$\frac{2n+2\alpha-1}{(n+2\alpha-1)p_-} < \theta < \min\left(1, \frac{2n+2\alpha-1}{(n)p_+}\right).$$

It is clear, that

$$\frac{2n+2\alpha-1}{(n+2\alpha-1)} < \theta p_- < \theta p_+ < \frac{2n+2\alpha-1}{(n)}.$$

As we have that if $p(\cdot) \in \mathcal{P}_{\log}$ then $\theta p(\cdot) \in \mathcal{P}_{\log}$ and by Theorem 1.2 we get that the operator $\mathcal{M}_m$ is bounded in $L^{\theta p(\cdot)}(\mathbb{R}^n)$. Using the fact that $[L^\infty(\mathbb{R}^n), L^{p(\cdot)\theta}(\mathbb{R}^n)]_\theta = L^{p(\cdot)}(\mathbb{R}^n)$, $(0 < \theta < 1)$ and the operator $\mathcal{M}_m$ is bounded in $L^\infty(\mathbb{R}^n)$ and $L^{\theta p(\cdot)}(\mathbb{R}^n)$ we obtain that the operator $\mathcal{M}_m$ is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. □

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