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The Range of Critical Numbers for Banach Spaces

George Chelidze *

Faculty of Exact and Natural Sciences, I. Javakhshvili Tbilisi State University, 13 University St., Tbilisi 0186 Georgia

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To consider the intersection of embedded bounded closed sets in infinite-dimensional Banach spaces the numerical parameter was introduced earlier. In the present paper we collect some results that are known around the subject.

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1. Description of the problem

The nonemptiness of the intersection of embedded closed sets in metric spaces is related to some basic concepts of analysis. We mention here the concept of completeness of a space: the metric space X is complete if and only if for any sequence of embedded closed balls $B_n = B(x_n, r_n)$ with $r_n \to 0$ the intersection $\bigcap_{n=1}^{\infty} B_n$ is nonempty. It seems contradictory to our natural intuition that this intersection may be empty if the radii do not tend to zero. A simple example is: $X = \{x_1, x_2, ...\}$, with metric $\rho(x_i, x_j) = (a_i + a_j)(1 - \delta_{ij})$, where $a_i \in R$, $a_i \downarrow a > 0$ and $B_n = B(x_n, a_{n-1} + a_n), n \geq 2$.

But actually this example shows only that the general concept of a metric space is too general to be in full accordance with the intuition accumulated mainly by observations on phenomena for normed spaces. And indeed, if X is a Banach space, this simple problem of embedded closed balls has the natural solution, namely, the intersection is always nonempty regardless the behavior of the radii. However, if we consider a sequence of general embedded bounded closed sets, rather than balls, in a Banach space, then the problem of nonemptiness of the intersection becomes nontrivial and this problem is in fact the subject of the present communication. (Another trivial but still unexpected observation for generals complete metric spaces is that a bigger ball may be a proper part of another smaller ball.) First let us note that this problem is purely infinite-dimensional, because any bounded closed set in a finite-dimensional Banach space is compact. Therefore, the intersection is always nonempty. The converse is also true: If the intersection of all embedded closed sets of a Banach space is nonempty then this space is finite-dimensional. This follows from Riesz Theorem, which asserts that $X$is finite-dimensional if the closed unit

*Email: giorgi.chelidze@tsu.ge

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ball $B = B(0,1)$ of X is a compact set. (Indeed, we should show that for any given sequence $(x_n)_{n\in N}$ of points of B there exists a subsequence of it converging to a point of B. Denote by F_n the closure of the set $\{x_n, x_{n+1}, \cdots\}, n = 1, 2, \cdots$. Clearly, $F_n \subseteq B, n \geq 1$, and for any point y from the intersection $\cap F_n$ there exists a subsequence of $(x_n)_{n\in N}$ that converges to y and compactness of B follows). Note also that if the diameters of the closed sets F_n tend to zero then the intersection is nonempty, and consists of one point only, in any Banach space; this statement is true in any complete metric space as well.

2. Numerical parameter $\varkappa$ and the main theorem

In what follows X will always denote a Banach space. For brevity instead of "a sequence of embedded bounded closed sets $A_1 \supseteq A_2 \supseteq \cdots$ " we will say "an admissible sequence $(A_n)_{n\in N}$ ". To study intersection of admissible sequence the numerical parameter was introduced earlier by N.N.Vakhania and I.N.Kartsivadze [1]. This parameter characterizes in a sense a measure of deviation of A from being ball shaped kind of nonflatness of A. We define first two quantities depending on a point in X

$$r(A,x) = sup\{||x-y|| : y \in A\},$$

$$r^{'}(A,x) = inf\{||x-y|| : y \in X \setminus A\}$$

$$\varkappa(A) = sup\{\frac{r^{'}(A,x)}{r(A,x)} : x \in A\}.$$

It is clear that $0 \leq \varkappa(A) \leq 1$ for any A and it is easy to show that $\varkappa(A) = 1$ if and only if the closure of A is a ball.

In [1] the following Theorem 2.1 and Theorem 2.4. were also proved.

Theorem 2.1: *Let $(A_n)_{n\in N}$ be an admissible sequence in a Banach space. If $\overline{lim}\varkappa(A_n) > \frac{1}{2}$ then the intersection of the sets A_n is nonempty.*

The following statement is an immediate consequence of this theorem.

Corollary 2.2: *The intersection of embedded closed balls in a Banach space is always nonempty.*

The proof of Theorem 2.1 uses the following elementary auxiliary proposition which, unlike the general metric space situation, is in full accordance with the intuition.

Lemma 2.3: *Let $B(x,r)$ and $B(y,R)$ be two balls in a Banach space. If $B(x,r) \subsetneqq B(y,R)$ then $r + ||x-y|| \leq R$.*

Note that a direct simple proof of Corollary to Theorem 2.1 can easily be derived from this lemma.

The next theorem shows that Theorem 2.1 cannot be improved if we want it to be true for all Banach spaces. The number 1/2 turns out to be the best possible in the following natural sense: with the number 1/2 Theorem 2.1 is true but if we write $1/2 - \varepsilon$, $\varepsilon > 0$ then it will not be true for all Banach spaces.

Theorem 2.4: *For any number $\varepsilon > 0$ there exist a Banach space X and an admissible sequence $(A_n)_{n\in N}$ such that $\overline{lim}\varkappa(A_n) > \frac{1}{2} - \varepsilon$ but the intersection $\bigcap_{n=1}^{\infty} A_n$ is empty.*

Proof: Let us take c_0(the sequence of all real numbers converging to zero) as X and let $\varepsilon > 0$ be given. Denote $A_n = \{x \in c_0 : 2\varepsilon \leq (-1)^k x_k \leq 1$ for $1 \leq k \leq n$ and $-1 \leq x_k \leq 1$ for $k > n$. It is easy to see that everything can be checked elementarily. □

Note that the space c_0 in which the example was just given to prove Theorem 2.4 is non reflexive. We also know the example in a reflexive space (l_p with an appropriately chosen p depending on ε) but with non-convex sets [2]. Now let us just remark that the example with convex sets cannot exist in a reflexive space.

Proposition 2.5: *Any sequence of embedded bounded closed and convex sets in a reflexive space has nonempty intersection.*

Proof: It is enough to combine the following statements. Any sequence of embedded compact sets in any topology has nonempty intersection. In a dual Banach space any bounded set which is closed in the weak * topology is compact in this topology (the Alaoglu theorem). Weak* topology in a reflexive space is the same as the weak topology. In any Banach space convex sets are closed in the weak topology provided that they are closed in the norm topology. Also let us mention, that the property which was just proved, characterizes reflexive Banach spaces. Indeed, if the intersection of any sequence of embedded bounded closed convex sets is non-empty then the closed unit ball $B(0, 1)$ of X is compact in the weak topology (Schmulyan's theorem) and, moreover, if this is the case then is reflexive (Eberlein's theorem). □

3. Critical numbers (critical values of the parameter $\varkappa$) for individual Banach spaces

As it was indicated in the preamble to Theorem 2.4, the number 1/2 is a kind of critical number for a subclass of Banach spaces consisting of reflexive spaces, as the remark following Theorem 2.4 tells.

In this section we define in a similar manner the critical numbers for individual Banach spaces, and give related results.

Definition 3.1: For Xgiven, say that $\alpha \in R^+$ is the critical value of X, and write $\alpha = cv(X)$ for short, if:

a) any nested sequence of closed, bounded sets $(A_n)_{n\in N}$ such that $\overline{lim}\varkappa(A_n) > \alpha$ has nonempty intersection and

b) for any $\varepsilon > 0$ there exists a nested sequence $(A_n)_{n\in N}$ such that $\overline{lim}\varkappa(A_n) > \alpha - \varepsilon$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

We know critical values of some spaces [3]:

Theorem 3.2: *The critical number for the space l_p for $1 \leq p < \infty$ is expressed* *by the formula*

$$cv(l_p) = \frac{1}{2^{\frac{1}{p}} + 1}.$$

This theorem shows that the set of all critical numbers for Banach spaces contains the interval $[1/3, 1/2)$.

As a consequence of Theorem 2.1 and Theorem 2.4 we have that $cv(c_0) = 1/2$. Since by Theorem 2.1 the number 1/2 is a sufficient number for any Banach space, no number greater than 1/2 can be the critical number for a Banach space (condition(2) of the definition cannot be satisfied).

It is clear that the critical number for any finite-dimensional space is zero. It is easy to see that in any infinite-dimensional X there exists an admissible sequence with the empty intersection. Indeed, let $(x_n)_{n\in N}$ be a bounded sequence in X with $||x_i - x_j|| \geq 2r$ $(i, j = 1, 2, \cdots, n)$ for some positive number r which exists in any infinite-dimensional space X. Denote

$$A_n = \bigcup_{k=n}^{\infty} B(x_k, r - \varepsilon), \varepsilon > 0, n = 1, 2, \cdots.$$

It is easy to check that $(A_n)_{n\in N}$ is an admissible sequence with the empty intersection and with a positive value of $\overline{lim}\varkappa(A_n)$. Only a slightly more involved consideration shows that for any $\varepsilon > 0$ in any infinite-dimensional Banach space we can similarly construct an admissible sequence $(A_n)_{n\in N}$ such that $\overline{lim}\varkappa(A_n) > \frac{1}{3} - \varepsilon$ and the intersection $\bigcap_{n=1}^{\infty} A_n$ is empty [3]. Therefore, no number less than $\frac{1}{3}$ can be the critical number for an infinite-dimensional Banach space (condition (1) of the definition cannot be satisfied.)

Summing up the statement of Theorem 3.2 and the subsequent remarks we get the following assertion.

Theorem 3.3: *The set of critical numbers for all infinite-dimensional Banach spaces is the closed interval [1/3,1/2].*

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