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A Boundary-Contact Problem for Two Rectangularly Linked Elastic Bars

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Static and dynamical boundary-contact problems for two rectangularly (with respect to their longitudinal axes) linked elastic bars with variable rectangular cross-sections are considered within the framework of the $(0, 0)$ approximation of hierarchical models. They may have a contact interface either really (in this case the bars may have different elastic constants) or mentally (in the case when two bars represent an entire (undivided) body).

Key words: Elastic bars, Cusped bars, Boundary-contact problems.

AMS Subject Classification: 74K10.

1. Introduction

In [1] (see also [2]) I. Vekua constructed hierarchical models for elastic prismatic shells based on the Fourier-Legendre expansions (series) with respect to the thickness variable of the stress tensor X_{ij}, strain tensor e_{ij}, and displacement vector components u_i within the framework of the linear theory of elasticity. Generalizing this idea and using the double Fourier-Legendre expansions (series) of the above mentioned physical and geometrical quantities, in [3] G. Jaiani constructed hierarchical models for elastic prismatic bars with variable rectangular cross-sections. One can find the survey of further developments of these topics in [4].

In the present paper static and dynamical boundary-contact problems for two rectangularly (with respect to their longitudinal axes) linked elastic bars with variable rectangular cross-sections within the framework of the $(0, 0)$ approximation of hierarchical models (see [3]). They can be linked either really (in this case the bars may have different elastic constants) or mentally (in the case when two bars represent an entire (undivided) body). The paper is organized as follows. In the introduction the elastic structure under consideration is described. Section 2 is devoted to construction of governing equations. In Section 3 a boundary-contact problem is formulated and solved in the explicit form. In Section 4 rectangularly linked prismatic bars loaded by self-weight are treated.

Let $Ox_1x_2x_3$ be the anticlockwise Cartesian rectangular frame. Let B_1 and B_3

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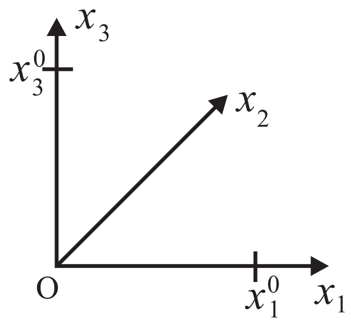

Figure 1.Cartesian rectangular anticlockwise frame; longitudinal axes and lengths of perpendicularly linked bars

be conditionally horizontal and verticall elastic bars with the "longitudinal axes" [1)] lying on the coordinate axes x_1 and x_3, and lengths x_1^0 and x_3^0, respectively (see Fig. 1):

$$B_l := \Big\{(x_1, x_2, x_3) \in R^3 : \ 0 \le x_l \le x_l^0, \ \overset{(-)}{h^l_m}(x_l) \le x_m \le \overset{(+)}{h^l_m}(x_l)\Big\},$$
$$l = 1, 3; \ m \ne l, \ m = 1, 2, 3.$$

Let face surfaces of the bars B_l, $l = 1, 3$, be

$$\begin{aligned} &\pm x_m = \pm \overset{(\pm)}{h^l_m}(x_l) \ge 0, \ 0 \le x_l \le x_l^0, \text{ with} \\ &\Big[\overset{(-)}{h^l_m}(x_l)\Big]^2 + \Big[\overset{(+)}{h^l_m}(x_l)\Big]^2 \ne 0, \ 0 < x_l < x_l^0, \\ &l = 1, 3; \ \ m \ne l, \ \ m = 1, 2, 3, \end{aligned} \tag{1}$$

which are piece-wise smooth.

By

$$2h^l_m(x_l) := \overset{(+)}{h^l_m}(x_l) - \overset{(-)}{h^l_m}(x_l), \ \ l = 1, 3, \ \ m \ne l, \ \ m = 1, 2, 3, \ \ x_l \in [0, x_l^0],$$

we denote the length of the side (which is parallel to the coordinate axis x_m) of the rectangular cross-section of the bar B_l, $l = 1, 3$. According to (1) the lengths of the above mentioned sides and, hence the areas of cross-sections of the bars B_l, $l = 1, 3$, may vanish only at the ends of the bars. In the last case bars are called cusped bars (for cusped bars see [4] and the references given there).

Let the maximal abscissa and maximal x_3-coordinate (see Figures 2 and 3) of

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[1)] which really coincide with the longitudinal axes of symmetry of the bars if

$$\overset{(-)}{h^l_m}(x_l) = -\overset{(+)}{h^l_m}(x_l), \ \ l = 1, 3, \ \ m \ne l, \ \ m = 1, 2, 3.$$

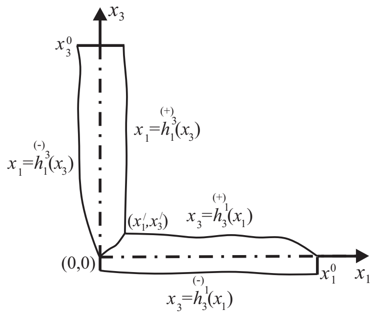

Figure 2. A cross-section $x_2 = c$ with face curves, axes of bars, a mental (or real) interface curve

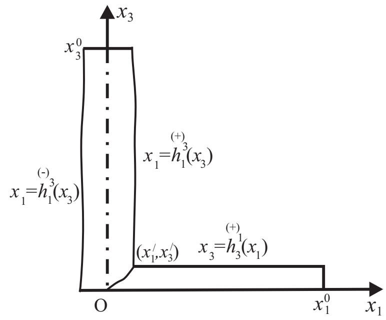

Figure 3. A cross-section $x_2 = c$, when $\overset{(-)}{h}{}_3^1 (x_1) \equiv 0$, $0 < x_1 < x_1^0$

common points of the face surfaces

$$x_3 = \overset{(+)}{h}{}_3^1 (x_1), \quad x_2 \in \Big] \overset{(-)}{h}{}_2^1 (x_1), \ \overset{(+)}{h}{}_2^1 (x_1) \Big[, \quad x_1 \in [0, x_1^0],$$

and

$$x_1 = \overset{(+)}{h}{}_1^3 (x_3), \quad x_2 \in \Big] \overset{(-)}{h}{}_2^3 (x_3), \ \overset{(+)}{h}{}_2^3 (x_3) \Big[, \quad x_3 \in [0, x_3^0],$$

be $0 \leq x_1' << x_1^0$ and $0 \leq x_3' << x_3^0$, respectively[1)], let the linked ends of longitudinal axes of the bars be the origin of the coordinate system, and let the equation of a section of the interface (real or mental) between the bars by the plane

$$x_2 = c = const,$$

be $x_3 = f(x_1)$, $f'(x_1) > 0$[2)] (see, e.g., Figures 2 and 3);

$$h := \max\Big\{ \overset{(-)}{h}{}_2^3 (x_3'), \ \overset{(-)}{h}{}_2^1 (x_1') \Big\} \leq c \leq H := \min\Big\{ \overset{(+)}{h}{}_2^3 (x_3'), \ \overset{(+)}{h}{}_2^1 (x_1') \Big\}$$

If $\overset{(\pm)}{h}{}_2^3 (x_3') = \overset{(\pm)}{h}{}_2^1 (x_1')$, the face surfaces $\overset{(\pm)}{h}{}_2^3$ and $\overset{(\pm)}{h}{}_2^1$ will be continuously connected. In Figures 4 and 5 cross-sections $x_2 = c$ of bars linked along segments of the axis x_2 are shown.

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1) The set of common points of the above face surfaces we will consider as the interface between bars B_1 and B_3.

2) We also consider the cases when adjoint parts of a face surface of one bar and a base of another one serve as an an interface between the bars (see, e.g., Fig 6).

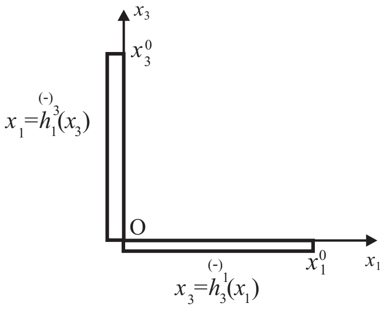

Figure 4. A cross-section $x_2 = c$, when $\overset{(+)}{h}{}_3^1\,(x_1) \equiv 0$, $\overset{(+)}{h}{}_1^3\,(x_3) \equiv 0$

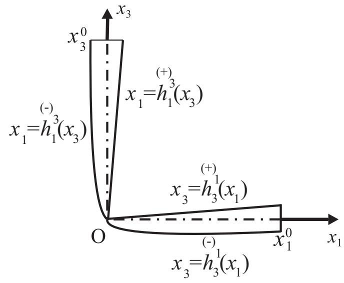

Figure 5. A cross-section $x_2 = c$, when the bars are linked with blunt cusped ends

2. Governing equations of the $(0, 0)$ approximation

In the (0,0) approximation of hierarchical models constructed on the basis of the three-dimensional (3D) linear theory of elasticity in the class of twice continuously differentiable functions the governing equations of the horizontal bar B_1 with the longitudinal axis lying on the coordinate axis x_1 have the following form (see [3])

$$\Big(h_2^1(x_1)h_3^1(x_1)v_{j,1}^1(x_1,t)\Big)_{,1} + \overset{0,0}{Y}{}_j^1(x_1,t) = \Big(\overset{1}{\Lambda}_j\Big)^{-1} \rho h_2^1(x_1)h_3^1(x_1)\frac{\partial^2 v_j^1(x_1,t)}{\partial t^2}, \quad (2)$$

$$x_1 \in]0, x_1^0[, \quad j = 1,2,3,$$

where

$$v_j^1(x_1,t) := \frac{u_{j00}^1(x_1,t)}{h_2^1(x_1)h_3^1(x_1)},$$

$$\overset{0,0}{Y}{}_1^1(x_1,t) := \frac{\overset{0,0}{X}{}_1^1(x_1,t)}{\lambda+2\mu}, \quad \overset{0,0}{Y}{}_j^1(x_1,t) := \frac{\overset{0,0}{X}{}_j^1(x_1,t)}{\mu}, \quad j = 2,3, \quad (3)$$

$$\overset{0,0}{X}{}_j^1(x_1) = \int\limits_{\overset{(-)}{h}{}_2^1}^{\overset{(+)}{h}{}_2^1} \Bigg[\sqrt{1+\Big(\overset{(+)}{h}{}_{3,1}^1\Big)^2}\; X^1_{\overset{(+)}{\nu}_3 j}(x_1,x_2,\overset{(+)}{h}{}_3^1)$$

$$+\sqrt{1+\Big(\overset{(-)}{h}{}_{3,1}^1\Big)^2}\; X^1_{\overset{(-)}{\nu}_3 j}(x_1,x_2,\overset{(-)}{h}{}_3^1)\Bigg]\, dx_2$$

$$+\int\limits_{\overset{(-)}{h}{}_3^1}^{\overset{(+)}{h}{}_3^1} \Bigg[\sqrt{1+\Big(\overset{(+)}{h}{}_{2,1}^1\Big)^2}\; X^1_{\overset{(+)}{\nu}_2 j}(x_1,\overset{(+)}{h}{}_2^1,x_3)$$

$$+\sqrt{1+\left(\overset{(-)}{h}{}^1_{2},_1\right)^2}\, X^1_{\overset{(-)}{\nu}_2 j}(x_1,\overset{(-)}{h}{}^1_2,x_3)\Bigg]\, dx_3+X^1_{j00}(x_1),\quad j=1,2,3,$$

$X^1_{\overset{(+)}{\nu}_l j}$, $X^1_{\overset{(-)}{\nu}_l j}$, $l=2,3$, are the components of continuously differentiable tractions acting on the face surfaces of the bar B_1, X^1_{j00} are the components of the double $(0,0)$ moment of the continuous body force X^1_j per unit volume of B_1,

$$\overset{1}{\Lambda}_j := \begin{cases} \lambda+2\mu, & j=1,\\ \mu, & j=2,3, \end{cases} \tag{4}$$

The double $(0,0)$ moments of displacements u^l_j, stresses X^l_{ij}, and strains e^l_{ij}, $l=1,3$, are defined as follows

$$\left(u^l_{j00},X^l_{ij00},e^l_{ij00},X^l_{j00}\right)(x_l,t):=\int\limits_{\overset{(-)}{h}{}^l_m(x_l)}^{\overset{(+)}{h}{}^l_m(x_l)} dx_m \int\limits_{\overset{(-)}{h}{}^l_n(x_l)}^{\overset{(+)}{h}{}^l_n(x_l)} \left(u^l_j,X^l_{ij},e^l_{ij},X^l_j\right)(x_1,x_2,x_3,t)dx_n,$$
$$l=1,3;\ m\neq l,\ m\neq n,\ n\neq l;\ m,n,i,j=1,2,3.$$

Subscripts preceded by a comma mean differentiation with respect to the corresponding variables.

In the class of twice continuously differentiable functions the governing equations of the vertical bar B_3 with the longitudinal axis lying on the coordinate axis x_3 have the following form

$$\left(h^3_2(x_3)h^3_1(x_3)v^3_{j,3}(x_3,t)\right)_{,3}+\overset{0,0}{Y^3_j}(x_3,t)$$
$$=\left(\overset{3}{\Lambda}_j\right)^{-1}\rho h^3_1(x_3)h^3_2(x_3)\frac{\partial^2 v^3_j(x_3,t)}{\partial t^2},\ x_3\in]0,x^0_3[,\ j=1,2,3, \tag{5}$$

where

$$\overset{3}{\Lambda}_j := \begin{cases} \lambda+2\mu, & j=3,\\ \mu, & j=1,2. \end{cases} \tag{6}$$

$$\overset{0,0}{Y^3_\alpha}(x_3,t):=\frac{\overset{0,0}{X^3_\alpha}(x_3,t)}{\mu},\ \alpha=1,2;\ \overset{0,0}{Y^3_3}(x_1,t):=\frac{\overset{0,0}{X^3_3}(x_3,t)}{\lambda+2\mu}, \tag{7}$$

$$\overset{0,0}{X^3_j}(x_3)=\int\limits_{\overset{(-)}{h}{}^3_2}^{\overset{(+)}{h}{}^3_2}\Bigg[\sqrt{1+\left(\overset{(+)}{h}{}^3_{1},_3\right)^2}\, X^3_{\overset{(+)}{\nu}_1 j}(\overset{(+)}{h}{}^3_1,x_2,x_3)$$

$$+ \sqrt{1+\left(\overset{(-)}{h}{}^3_{1},_3\right)^2}\, X^3_{\overset{(-)}{\nu}_1 j}(\overset{(-)}{h}{}^3_1, x_2, x_3)\Bigg] \, dx_2$$

$$+ \int\limits_{\overset{(-)}{h}_1}^{\overset{(+)}{h}{}^3_1} \Bigg[\sqrt{1+\left(\overset{(+)}{h}{}^3_{2},_3\right)^2}\, X^3_{\overset{(+)}{\nu}_2 j}(x_1, \overset{(+)}{h}{}^3_2, x_3)$$

$$+ \sqrt{1+\left(\overset{(-)}{h}{}^3_{2},_3\right)^2}\, X^3_{\overset{(-)}{\nu}_2 j}(x_1, \overset{(-)}{h}{}^3_2, x_3)\Bigg] \, dx_1 + X^3_{j00},$$

$X^3_{\overset{(+)}{\nu}_l j}$, $X^3_{\overset{(-)}{\nu}_l j}$, $l = 1, 2$, are the components of continuously differentiable tractions acting on to the face surfaces of the bar B_3, X^3_{j00} are the components of the double $(0, 0)$ moment of the continuous body force X^3_j per unit volume of B_3.

The approximate values of displacements u^l_j, $l = 1, 3$, $j = 1, 2, 3$, in the $(0, 0)$ approximation are defined as

$$u^1_j(x_1, x_2, x_3, t) \cong \frac{u^1_{j00}(x_1, t)}{4h^1_2(x_1)h^1_3(x_1)} =: \frac{1}{4} v^1_j(x_1, t), \quad j = 1, 2, 3. \tag{8}$$

and

$$u^3_j(x_1, x_2, x_3, t) \cong \frac{u^3_{j00}(x_3, t)}{4h^3_1(x_3)h^3_2(x_3)} =: \frac{1}{4} v^3_j(x_3, t), \quad j = 1, 2, 3. \tag{9}$$

For the horizontal bar B_1 and the vertical bar B_3 we have the following formulas for the double moments:

– of the strains

$$\begin{aligned}
&e^1_{1100}(x_1, t) = h^1_2(x_1)h^1_3(x_1)v^1_{1,1}(x_1, t); \quad e^1_{2200}(x_1, t) \equiv 0; \quad e^1_{3300}(x_1, t) \equiv 0;\\
&e^1_{3200}(x_1, t) = e^1_{2300}(x_1, t) \equiv 0;\\
&2e^1_{3100}(x_1, t) = 2e^1_{1300}(x_1, t) = h^1_2(x_1)h^1_3(x_1)v^1_{3,1}(x_1, t);\\
&2e^1_{2100}(x_1, t) = 2e^1_{1200}(x_1, t) = h^1_2(x_1)h^1_3(x_1)v^1_{2,1}(x_1, t),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&e^3_{3300}(x_3, t) = h^3_2(x_3)h^3_1(x_3)v^3_{3,3}(x_3, t); \quad e^3_{2200}(x_3, t) \equiv 0; \quad e^3_{1100}(x_3, t) \equiv 0;\\
&e^3_{1200}(x_3, t) = e^3_{2100}(x_3, t) \equiv 0;\\
&2e^3_{1300}(x_3, t) = 2e^3_{3100}(x_3, t) = h^3_2(x_3)h^3_1(x_3)v^3_{1,3}(x_3, t);\\
&2e^3_{2300}(x_3, t) = 2e^3_{3200}(x_3, t) = h^3_2(x_3)h^3_1(x_3)v^3_{2,3}(x_3, t),
\end{aligned} \tag{11}$$

correspondingly;

– of the stresses

$$\begin{aligned}
&X^1_{1100}(x_1,t) = (\lambda+2\mu)h^1_2(x_1)h^1_3(x_1)v^1_{1,1}(x_1,t);\\
&X^1_{2200}(x_1,t) = \lambda h^1_2(x_1)h^1_3(x_1)v^1_{1,1}(x_1,t);\\
&X^1_{3300}(x_1,t) = \lambda h^1_2(x_1)h^1_3(x_1)v^1_{1,1}(x_1,t);\\
&X^1_{3200}(x_1,t) = X^1_{2300}(x_1,t) = 2\mu e^1_{2300}(x_1,t) \equiv 0;\\
&X^1_{2100}(x_1,t) = X^1_{1200}(x_1,t) = 2\mu e^1_{1200}(x_1,t) = \mu h^1_2(x_1)h^1_3(x_1)v^1_{2,1}(x_1,t);\\
&X^1_{3100}(x_1,t) = X^1_{1300}(x_1,t) = 2\mu e^1_{1300}(x_1,t) = \mu h^1_2(x_1)h^1_3(x_1)v^1_{3,1}(x_1,t),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&X^3_{3300}(x_3,t) = (\lambda+2\mu)h^3_2(x_3)h^3_1(x_3)v^3_{3,3}(x_3,t);\\
&X^3_{2200}(x_3,t) = \lambda h^3_2(x_3)h^3_1(x_3)v^3_{3,3}(x_3,t);\\
&X^3_{1100}(x_3,t) = \lambda h^3_2(x_3)h^3_1(x_3)v^3_{3,3}(x_3,t);\\
&X^3_{1200}(x_3,t) = X^3_{2100}(x_3,t) \equiv 0;\\
&X^3_{2300}(x_3,t) = X^3_{3200}(x_3,t) = \mu h^3_2(x_3)h^3_1(x_3)v^3_{2,3}(x_3,t);\\
&X^3_{3100}(x_3,t) = X^3_{1300}(x_3,t) = \mu h^3_2(x_3)h^3_1(x_3)v^3_{1,3}(x_3,t),
\end{aligned} \tag{13}$$

correspondingly.

The formulas (9), (11), (13), (7), (6), and (5) for the vertical bar B_3 with quantities signed by "3" immediately follow from the formulas with quantities signed by "1" for the horizontal bar B_1, if we introduce a new coordinate system $O\tilde{x}_1\tilde{x}_2\tilde{x}_3$ by taking

$$\tilde{x}_1 := x_3, \quad \tilde{x}_2 := x_1, \quad \tilde{x}_3 := x_2; \tag{14}$$

for the vertical bar B_3, write all the formulas in the $O\tilde{x}_1\tilde{x}_2\tilde{x}_3$ system which will coincide with (8), (10), (12), (3), (4), (2) with "$\sim$" indices and, using (14), returning to the system $Ox_1x_2x_3$.

E.g., by virtue of (12), (8), and similar to (8) approximate formulas for 3D stresses

$$4h^3_2(\tilde{x}_1)h^3_3(\tilde{x}_1)X^3_{\tilde{1}\tilde{1}}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,t) \cong X^3_{\tilde{1}\tilde{1}00}(\tilde{x}_1,t) = (\lambda+2\mu)h^3_2(\tilde{x}_1)h^3_3(\tilde{x}_1)v^3_{\tilde{1},\tilde{1}}(\tilde{x}_1,t),$$

whence,

$$4h^3_1(x_3)h^3_2(x_3)X^3_{33}(x_3,x_1,x_2,t) \cong X^3_{3300}(x_3,t) = (\lambda+2\mu)h^3_1(x_3)h^3_2(x_3)v^3_{3,3}(x_3,t);$$

$$4h^3_2(\tilde{x}_1)h^3_3(\tilde{x}_1)X^3_{\tilde{2}\tilde{2}}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,t) \cong X^3_{\tilde{2}\tilde{2}00}(\tilde{x}_1,t) = \lambda h^3_2(\tilde{x}_1)h^3_3(\tilde{x}_1)v^3_{\tilde{1},\tilde{1}}(\tilde{x}_1,t),$$

whence,

$$4h^3_1(x_3)h^3_2(x_3)X^3_{11}(x_3,x_1,x_2,t) \cong X^3_{1100}(x_3,t) = \lambda h^3_1(x_3)h^3_2(x_3)v^3_{3,3}(x_3,t);$$

$$4h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)X^3_{\tilde{3}\tilde{3}}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,t) \cong X^3_{\tilde{3}\tilde{3}00}(\tilde{x}_1,t) = \lambda h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)v^3_{\tilde{1},\tilde{1}}(\tilde{x}_1,t),$$

whence,

$$4h^3_1(x_3)h^3_2(x_3)X^3_{22}(x_3,x_1,x_2,t) \cong X^3_{2200}(x_3,t) = \lambda h^3_1(x_3)h^3_2(x_3)v^3_{3,3}(x_3,t);$$
$$4h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)X^3_{\tilde{2}\tilde{3}}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,t) \cong X^3_{\tilde{2}\tilde{3}00}(\tilde{x}_1,t) \equiv X^3_{\tilde{3}\tilde{2}00}(\tilde{x}_1,t) = 0,$$

whence,

$$4h^3_1(x_3)h^3_2(x_3)X^3_{12}(x_3,x_1,x_2,t) \cong X^3_{1200}(x_3,t) \equiv X^3_{2100}(x_3,t) = 0;$$
$$4h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)X^3_{\tilde{1}\tilde{2}}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,t) \cong X^3_{\tilde{1}\tilde{2}00}(\tilde{x}_1,t) \equiv X^3_{\tilde{2}\tilde{1}00}(\tilde{x}_1,t)$$
$$\mu h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)v^3_{\tilde{2},\tilde{1}}(\tilde{x}_1,t),$$

whence,

$$4h^3_1(x_3)h^3_2(x_3)X^3_{31}(x_3,x_1,x_2,t) \cong X^3_{3100}(x_3,t) \equiv X^3_{1300}(x_3,t)$$
$$= \mu h^3_1(x_3)h^3_2(x_3)v^3_{1,3}(x_3,t);$$
$$4h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)X^3_{\tilde{3}\tilde{1}}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,t) \cong X^3_{\tilde{3}\tilde{1}00}(\tilde{x}_1,t) \equiv X^3_{\tilde{1}\tilde{3}00}(\tilde{x}_1,t)$$
$$= \mu h^3_{\tilde{2}}(\tilde{x}_1)h^3_{\tilde{3}}(\tilde{x}_1)v^3_{\tilde{3},\tilde{1}}(\tilde{x}_1,t),$$

whence,

$$4h^3_1(x_3)h^3_2(x_3)X^3_{23}(x_3,x_1,x_2,t) \cong X^3_{2300}(x_3,t) \equiv X^3_{3200}(x_3,t)$$
$$\mu h^3_1(x_3)h^3_2(x_3)v^3_{2,3}(x_3,t).$$

Thus, from (12) we have obtained (13). Analogously, we get (9), (11), (7), (6), (5) from (8), (10), (3), (4), (2), correspondingly.

Evidently,

$$(u_i, e_{ij}, X_{ij})(x_1,x_2,x_3,t)$$

$$\cong \begin{cases} \dfrac{1}{4h^1_2(x_1)h^1_3(x_1)}(u^1_{i00}, e^1_{ij00}, X^1_{ij00})(x_1,t),\ (x_1,x_2,x_3)\in B_1, \\[2ex] \dfrac{1}{4h^3_1(x_3)h^3_2(x_3)}(u^3_{i00}, e^3_{ij00}, X^3_{ij00})(x_3,t),\ (x_1,x_2,x_3)\in B_3, \end{cases}$$

$$u_j(x_1,x_2,x_3,t) \cong \frac{1}{4}\begin{cases} v^1_j(x_1,t),\ (x_1,x_2,x_3)\in B_1, \\ v^3_j(x_3,t),\ (x_1,x_2,x_3)\in B_3, \end{cases}$$

$$i,j = 1,2,3.$$

On the interface $x_3 = f(x_1)$ in $3D$ setting natural contact conditions would be continuity of the stress and displacement vectors on the interface. Since the above-mentioned quantities in B_1 and B_3 depend only on x_1 and x_3, correspondingly, in the $(0,0)$ approximation conditions on the interface take the following form:

$$X^1_{\nu j00}(x_1,t) = X^3_{\nu j00}(f(x_1),t), \quad v^1_j(x_1,t) = v^3_j(f(x_1),t), \tag{15}$$

$$0 \le x_1 \le x^1_1, \quad j = 1,2,3,$$

where ν is the normal to the interface directed into B_1 for clearness. However, since the lengths of the bars are much more than the dimensions of the interface, we replace the contact conditions (15) by the following conditions

$$X^1_{1j00}(x_1,t)|_{x_1=0} = X^3_{1j00}(x_3,t)|_{x_3=0}, \text{ i.e., } X^1_{1j}(0,t) = X^3_{1j}(0,t), \ j = 1,2,3, \tag{16}$$

and

$$v^1_j(x_1,t)|_{x_1=0} = v^3_j(x_3,t)|_{x_3=0}, \text{ i.e., } v^1_j(0,t) = v^3_j(0,t), \ j = 1,2,3. \tag{17}$$

Here we have taken into account $f(0) = 0$ and replaced the normal at the point $O := (0,0,0)$ by the unit basis vector e_1 corresponding to the axis $0x_1$ (i.e., we assume the plane curve $x_3 = f(x_1)$ orthogonal to the axis $0x_1$ at the origin O).

In order to set an initial boundary-contact problem for the rectangularly joint two bars we have to add to the contact conditions (16), (17) the boundary conditions

$$v^1_j(x^0_1,t) = v^1_j(t), \quad v^3_j(x^0_3,t) = v^3_j(t), \quad t > 0, \ \ j = 1,2,3,$$

and the initial conditions

$$v^1_j(x_1,0) = v^{10}_j(x_1), \quad x_1 \in]0, x^0_1[, \ \ j = 1,2,3,$$
$$v^3_j(x_3,0) = v^{30}_j(x_3), \quad x_3 \in]0, x^0_3[, \ \ j = 1,2,3,$$

with prescribed functions $v^{11}_j(t)$, $v^{31}_j(t)$, $v^{10}_j(x_1)$, $v^{30}_j(x_3)$, $j = 1,2,3$.

From (16), by virtue of (12), (13), we obtain

$$X^1_{1100}(x_1,t)|_{x_1=0} = (\lambda + 2\mu)\left[h^1_2(x_1)h^1_3(x_1)v^1_{1,1}(x_1,t)\right]|_{x_1=0}$$
$$= X^3_{1100}(x_3,t)|_{x_3=0} = \lambda\left[h^3_1(x_3)h^3_2(x_3)v^3_{3,3}(x_3,t)\right]|_{x_3=0};$$
$$X^1_{1200}(x_1,t)|_{x_1=0} = \mu\left[h^1_2(x_1)h^1_3(x_1)v^1_{2,1}(x_1,t)\right]|_{x_1=0} = X^3_{1200}(x_3,t)|_{x_3=0} = 0;$$
$$X^1_{1300}(x_1,t)|_{x_1=0} = \mu\left[h^1_2(x_1)h^1_3(x_1)v^1_{3,1}(x_1,t)\right]|_{x_1=0}$$
$$= X^3_{1300}(x_3,t)|_{x_3=0} = \mu\left[h^3_1(x_3)h^3_2(x_3)v^3_{1,3}(x_3,t)\right]|_{x_3=0}.$$

If we consider the general case when the normal to the plane curve $x_3 = f(x_1)$ at the point O does not coincide with e_1, then from the interface conditions (15) there follows, for $j = 1$:

$$\begin{aligned} X^1_{\nu 100}(x_1,t)|_{x_1=0} &= \Big[X^1_{i100}(x_1,t)\nu_i(0,0,0)\Big]\Big|_{x_1=0} \\ &= \Big\{h^1_2(x_1)h^1_3(x_1)\Big[(\lambda+2\mu)v^1_{1,1}(x_1,t)\nu_1(0,0,0) \\ &+\mu v^1_{2,1}(x_1,t)\nu_2(0,0,0)+\mu v^1_{3,1}(x_1,t)\nu_3(0,0,0)\Big]\Big\}\Big|_{x_1=0} \\ &= X^3_{\nu 100}(x_3,t)\Big|_{x_3=0} = \Big[X^3_{i100}(x_3,t)\nu_i(0,0,0)\Big]\Big|_{x_3=0} \\ &= \Big\{h^3_1(x_3)h^3_2(x_3)\Big[\lambda v^3_{3,3}(x_3,t)\nu_1(0,0,0)+0\cdot\nu_2(0,0,0) \\ &+\mu v^3_{1,3}(x_3,t)\nu_3(0,0,0)\Big]\Big\}\Big|_{x_3=0}; \end{aligned}$$

in particular, if $\nu \equiv e_1 := (1,0,0)$, as it was already derived:

$$(\lambda+2\mu)\Big[h^1_2(x_1)h^1_3(x_1)v^1_{1,1}(x_1,t)\Big]|_{x_1=0} = \lambda\Big[h^3_1(x_3)h^3_2(x_1)v^3_{3,3}(x_3,t)\Big]\Big|_{x_3=0}\Big]; \quad (18)$$

for $j = 2$:

$$\begin{aligned} X^1_{\nu 200}(x_1,t)|_{x_1=0} &= \Big[X^1_{i200}(x_1,t)\nu_i(0,0,0)\Big]\Big|_{x_1=0} \\ &= \Big\{h^1_2(x_1)h^1_3(x_1)\Big[\mu v^1_{2,1}(x_1,t)\nu_1(0,0,0) \\ &+\lambda v^1_{1,1}(x_1,t)\nu_2(0,0,0)+0\cdot\nu_3(0,0,0)\Big]\Big\}\Big|_{x_1=0} \\ &= X^3_{\nu 200}(x_3,t)|_{x_3=0} = \Big[X^3_{i200}(x_3,t)\nu_i(0,0,0)\Big]\Big|_{x_3=0} \\ &= \Big\{h^3_1(x_3)h^3_2(x_3)\Big[0\cdot\nu_1(0,0,0)+\lambda v^3_{3,3}(x_3,t)\nu_2(0,0,0) \\ &+\mu v^3_{2,3}(x_3,t)\nu_3(0,0,0)\Big]\Big\}\Big|_{x_3=0}, \end{aligned}$$

in particular, if $\nu \equiv e_1$, as it was already derived:

$$\mu\Big[h_2(x_1)h_3(x_1)v_{2,1}(x_1,t)\Big]|_{x_1=0} = 0; \quad (19)$$

for $j = 3$:

$$\begin{aligned} X^1_{\nu 300}(x_1,t)|_{x_1=0} &= \Big[X^1_{i300}(x_1,t)\nu_i(0,0,0)\Big]\Big|_{x_1=0} \\ &= \Big\{h^1_2(x_1)h^1_3(x_1)\Big[\mu v^1_{3,1}(x_1,t)\nu_1(0,0,0) \\ &+0\cdot\nu_2(0,0,0)+\lambda v^1_{1,1}(x_1,t)\nu_3(0,0,0)\Big]\Big\}\Big|_{x_1=0} \end{aligned}$$

$$= X^3_{\nu 300}(x_3,t)|_{x_3=0} = \Big[X^3_{i300}(x_3,t)\nu_i(0,0,0)\Big]\Big|_{x_3=0}$$
$$= \Big\{h^3_1(x_3)h^3_2(x_3)\Big[\mu v^3_{1,3}(x_3,t)\nu_1(0,0,0) + \mu v^3_{2,3}(x_3,t)\nu_2(0,0,0)$$
$$+(\lambda+2\mu)v^3_{3,3}(x_3,t)\nu_3(0,0,0)\Big]\Big\}\Big|_{x_3=0};$$

in particular, if $\nu \equiv e_1$, as it was already derived:

$$\mu\Big[h^1_2(x_1)h^1_3(x_1)v^1_{3,1}(x_1,t)\Big]|_{x_1=0} = \mu\Big[h^3_1(x_3)h^3_2(x_3)v^3_{1,3}(x_3,t)\Big]|_{x_3=0}. \qquad (20)$$

3. Boundary-contact problem

Let us consider the static case, then equations (2) and (5) will get the following forms

$$\Big(h^1_2(x_1)h^1_3(x_1)v^1_{j,1}(x_1)\Big)_{,1} + \overset{0,0}{Y^1_j}(x_1) = 0, \quad x_1 \in]0, x^0_1[, \quad j = 1,2,3, \qquad (21)$$

and

$$\Big(h^3_1(x_3)h^3_2(x_3)v^3_{j,3}(x_3)\Big)_{,3} + \overset{0,0}{Y^3_j}(x_3) = 0, \quad x_3 \in]0, x^0_3[, \quad j = 1,2,3, \qquad (22)$$

respectively. It is easily seen that the general solutions to equations (21), (22) have the forms

$$v^1_j(x_1) = -\int\limits_{x^0_1}^{x_1} \frac{d\tau}{h^1_2(\tau)h^1_3(\tau)} \int\limits_{x^0_1}^{\tau} \overset{0,0}{Y^1_j}(\eta)d\eta + c^1_j \int\limits_{x^0_1}^{x_1} \frac{d\tau}{h^1_2(\tau)h^1_3(\tau)} + c^2_j, \qquad (23)$$
$$x_1 \in]0, x^0_1[, \quad j = 1,2,3,$$

and

$$v^3_j(x_3) = -\int\limits_{x^0_3}^{x_3} \frac{d\tau}{h^3_1(\tau)h^3_2(\tau)} \int\limits_{x^0_3}^{\tau} \overset{0,0}{Y^3_j}(\eta)d\eta + d^1_j \int\limits_{x^0_3}^{x_3} \frac{d\tau}{h^3_1(\tau)h^3_2(\tau)} + d^2_j, \qquad (24)$$
$$x_3 \in]0, x^0_3[, \quad j = 1,2,3,$$

correspondingly.

Let

$$\int\limits_{x^0_1}^{x_1} \frac{d\tau}{h^1_2(\tau)h^1_3(\tau)} \in C([0, x^0_1]), \quad \int\limits_{x^0_3}^{x_3} \frac{d\tau}{h^3_1(\tau)h^3_2(\tau)} \in C([0, x^0_3]),$$

which allows cusped contacts as well (for cusped bodies see [4]) and let

$$h_2^1 h_3^1 v_{j,1}^1(x_1,t) \quad h_1^3 h_2^3 v_{j,3}^3(x_3,t)$$

be continuous from the right hand side at points $x_1 = 0$ and $x_3 = 0$, respectively.

Let us consider the boundary conditions

$$v_j^1(x_1^0) = v_j^1, \;\; v_j^3(x_3^0) = v_j^3, \;\; v_j^1, v_j^3 = const, \;\; j = 1,2,3, \tag{25}$$

along with the contact conditions

$$X_{1j00}^1(0) = X_{1j00}^3(0), \tag{26}$$

$$v_j^1(x_1)|_{x_1=0} = v_j^3(x_3)|_{x_3=0}, \quad j = 1,2,3. \tag{27}$$

It is very easy to solve boundary-contact problem (2), (5), (25)-(27) in the explicit form. We do it for readers convenience.

Indeed from (25), taking into account (23), (24), we immediately obtain

$$c_j^2 = v_j^1, \quad d_j^2 = v_j^3, \quad j = 1,2,3. \tag{28}$$

From the contact conditions (26) on the interface, i.e., from the conditions (18)-(21), which in the static case are independent of t, according to (23), (24), we have for $j = 1$:

$$\left(-\int_{x_1^0}^{0} \overset{0,0}{Y_1^1}(\eta)d\eta + c_1^1\right)(\lambda + 2\mu) = \left(-\int_{x_3^0}^{0} \overset{0,0}{Y_3^3}(\eta)d\eta + d_3^1\right)\lambda,$$

i.e.,

$$(\lambda + 2\mu)c_1^1 - \lambda d_3^1 = A^1, \tag{29}$$

where

$$A^1 := (\lambda + 2\mu)\int_{x_1^0}^{0} \overset{0,0}{Y_1^1}(\eta)d\eta - \lambda \int_{x_3^0}^{0} \overset{0,0}{Y_3^3}(\eta)d\eta;$$

for $j = 2$:

$$-\int_{x_1^0}^{0} \overset{0,0}{Y_2^1}(\eta)d\eta + c_2^1 = 0,$$

i.e.,

$$c_2^1 = \int_{x_1^0}^{0} \overset{0,0}{Y_2^1}(\eta)d\eta =: E; \tag{30}$$

for $j = 3$:

$$-\int_{x_1^0}^{0} \overset{0,0}{Y_3^1}(\eta)d\eta + c_3^1 = -\int_{x_3^0}^{0} \overset{0,0}{Y_1^3}(\eta)d\eta + d_1^1,$$

i.e.,

$$c_3^1 - d_1^1 = A^2, \tag{31}$$

where

$$A^2 := \int_{x_1^0}^{0} \overset{0,0}{Y_3^1}(\eta)d\eta - \int_{x_3^0}^{0} \overset{0,0}{Y_1^3}(\eta)d\eta.$$

From the contact conditions (27) on the interface, using (23), (24), we have:

$$Cc_j^1 - Dd_j^1 = A_j, \ \ j = 1, 2, 3, \tag{32}$$

where

$$C := \int_{x_1^0}^{0} \frac{d\tau}{h_2^1(\tau)h_3^1(\tau)}, \ \ D := \int_{x_3^0}^{0} \frac{d\tau}{h_1^3(\tau)h_2^3(\tau)},$$

$$A_j := \int_{x_1^0}^{0} \frac{d\tau}{h_2^1(\tau)h_3^1(\tau)} \int_{x_1^0}^{\tau} \overset{0,0}{Y_j^1}(\eta)d\eta - \int_{x_3^0}^{0} \frac{d\tau}{h_1^3(\tau)h_2^3(\tau)} \int_{x_3^0}^{\tau} \overset{0,0}{Y_j^3}(\eta)d\eta + v_j^3 - v_j^1,$$

$$j = 1, 2, 3.$$

As we see from (28), (30) we have found 7 unknown constants c_j^2, d_j^2, c_2^1, $j = 1, 2, 3$, from 12 to be found. Substituting (30) into (32) for $j = 2$ we find the 8th unknown constant

$$d_2^1 = \frac{EC - A_2}{D}.$$

Evidently,

$$D \neq 0.$$

For four remained unknown constants c_1^1, c_3^1, d_1^1, d_3^1 we have the following four equations: (29), (31), and two from (32) for $j = 1, 3$:

$$c_1^1 C - d_1^1 D = A_1, \tag{33}$$

$$c_3^1 C - d_3^1 D = A_3. \tag{34}$$

Calculating c_1^1 and c_3^1 from (29) and (31), we obtain

$$c_1^1 = \frac{A^1}{\lambda + 2\mu} + \frac{\lambda}{\lambda + 2\mu} d_3^1, \tag{35}$$

$$c_3^1 = A^2 + d_1^1, \tag{36}$$

respectively. Substituting (36) and (35) into (34) and (33), respectively, we get the following system

$$d_1^1 C - d_3^1 D = A_3 - CA^2,$$

$$-d_1^1 D + d_3^1 \frac{\lambda C}{\lambda + 2\mu} = A_1 - \frac{A^1 C}{\lambda + 2\mu}.$$

Solving this system with respect to d_1^1 and d_3^1, we get

$$d_1^1 = \Delta^{-1}\Big[(A_3 - CA^2)\frac{\lambda C}{\lambda + 2\mu} + \Big(A_1 - \frac{A^1 C}{\lambda + 2\mu}\Big)D\Big], \tag{37}$$

$$d_3^1 = \Delta^{-1}\Big[C\Big(A_1 - \frac{A^1 C}{\lambda + 2\mu}\Big) + D(A_3 - CA^2)\Big], \tag{38}$$

where

$$\Delta = \frac{\lambda}{\lambda + 2\mu}(C)^2 - (D)^2 = \frac{\sigma}{1 - \sigma}(C)^2 - (D)^2,$$

σ is the Poisson ratio.

If $C = D$, then

$$\Delta = (C)^2 \left(\frac{\lambda}{\lambda + 2\mu} - 1\right) = (C)^2 \frac{-2\mu}{\lambda + 2\mu} \neq 0.$$

In the general case

$$\Delta \neq 0$$

if

$$D \neq \sqrt{\frac{\sigma}{1 - \sigma}} C,$$

i.e., if

$$\int_{x_3^0}^0 \frac{d\tau}{h_1^3(\tau) h_2^3(\tau)} \neq \sqrt{\frac{\sigma}{1 - \sigma}} \int_{x_1^0}^0 \frac{d\tau}{h_2^1(\tau) h_3^1(\tau)}.$$

Substituting (38) and (37) into (35) and (36), respectively, we determine c_1^1 and c_3^1 and the problem under consideration is solved in the explicit form (23), (24).

Using (23), (24), from (12), (13) we easily obtain expressions for the double moments of stresses (in other words, for the integrated on cross-sections stresses):

$$X^1_{1100}(x_1) = (\lambda + 2\mu)\Big[c_1^1 - \int\limits_{x_1^0}^{x_1} \overset{0,0}{Y}{}^1_1(\eta)d\eta\Big],$$

$$X^3_{1100}(x_3) = X^3_{2200}(x_3) = \lambda\Big[d_3^1 - \int\limits_{x_3^0}^{x_3} \overset{0,0}{Y}{}^3_3(\eta)d\eta\Big];$$

$$X^1_{2200}(x_1) = X^1_{3300}(x_1) = \lambda\Big[c_1^1 - \int\limits_{x_1^0}^{x_1} \overset{0,0}{Y}{}^1_1(\eta)d\eta\Big],$$

$$X^3_{3300}(x_3) = (\lambda + 2\mu)\Big[d_3^1 - \int\limits_{x_3^0}^{x_3} \overset{0,0}{Y}{}^3_3(\eta)d\eta\Big];$$

$$X^1_{1200}(x_1) = X^1_{2100}(x_1) = \mu\Big[c_2^1 - \int\limits_{x_1^0}^{x_1} \overset{0,0}{Y}{}^1_2(\eta)d\eta\Big] = \mu\int\limits_{0}^{x_1} \overset{0,0}{Y}{}^1_2(\eta)d\eta,$$

$$X^3_{1200}(x_3) = X^3_{2100}(x_3) = 0;$$

$$X^1_{1300}(x_1) = X^1_{3100}(x_1) = \mu\Big[c_3^1 - \int\limits_{x_1^0}^{x_1} \overset{0,0}{Y}{}^1_3(\eta)d\eta\Big],$$

$$X^3_{1300}(x_3) = X^3_{3100}(x_3) = \mu\Big[d_1^1 - \int\limits_{x_3^0}^{x_3} \overset{0,0}{Y}{}^3_1(\eta)d\eta\Big],$$

$$X^1_{2300}(x_1) = X^1_{3200}(x_1) = 0,$$

$$X^3_{2300}(x_3) = X^3_{3200}(x_3) = \mu\Big[d_2^1 - \int\limits_{x_3^0}^{x_3} \overset{0,0}{Y}{}^3_2(\eta)d\eta\Big].$$

Remark 1 : When $(x_1')^2 + (x_3')^2 \neq 0$, in the expressions (23), (24) of the solution the integrands, containing (3) and (7) are not known on the entire interval of integration. In order to avoid this unconvenience we need some more restrictions, e.g., the following:

(i) if $x_1' \neq 0$, $x_3' \neq 0$, taking into account smallness of the sides of the cross-sections, and, excluding fluctuations of forces on the interface, we continue continuously and compatible for both the bars the values of the tractions on the face

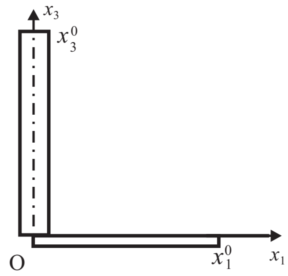

Figure 6. Prismatic bars of constant rectangular cross-sections; $\overset{(+)}{h}{}^1_3(x_1) \equiv 0$, $x_1 \in [0, x_1^0]$, $x_1' \neq 0$, $x_3' = 0$

surfaces

$$x_3 = \overset{(+)}{h}{}^1_3(x_1), \quad \overset{(-)}{h}{}^1_2(x_1) \leq x_2 \leq \overset{(+)}{h}{}^1_2(x_1), \quad x_1 \in [x_1', x_1^0], \tag{39}$$

and

$$x_1 = \overset{(+)}{h}{}^3_1(x_3), \quad \overset{(-)}{h}{}^3_2(x_3) \leq x_2 \leq \overset{(+)}{h}{}^3_2(x_3), \quad x_3 \in [x_3', x_3^0], \tag{40}$$

i.e.,

$$X^1_{\overset{(+)}{\nu_3} j}(x_1, x_2, \overset{(+)}{h}{}^1_3(x_1)), \ j = 1,2,3, \text{ for } \overset{(-)}{h}{}^1_2(x_1) \leq x_2 \leq \overset{(+)}{h}{}^1_2(x_1), \quad x_1 \in [x_1', x_1^0], \tag{41}$$

and

$$X^3_{\overset{(+)}{\nu_1} j}(\overset{(+)}{h}{}^3_1(x_3), x_2, x_3), \ j = 1,2,3, \text{ for } \overset{(-)}{h}{}^3_2(x_3) \leq x_2 \leq \overset{(+)}{h}{}^3_2(x_3), \quad x_3 \in [x_3', x_3^0], \tag{42}$$

respectively, on the surfaces

$$x_3 = f(x_1), \quad \overset{(-)}{h}{}^1_2(x_1) \leq x_2 \leq \overset{(+)}{h}{}^1_2(x_1), \quad x_1 \in [0, x_1'[,$$

and

$$x_1 = f^{-1}(x_3), \quad \overset{(-)}{h}{}^3_2(x_3) \leq x_2 \leq \overset{(+)}{h}{}^3_2(x_3), \quad x_3 \in [0, x_3'[,$$

correspondingly, here f^{-1} is the inverse function of f;

(ii) if $x_1' \neq 0$, $x_3' = 0$ (see Fig. 6), then we continuously continue the traction (41) from the face surface (39) on the plane surface

$$x_3 = 0, \quad \overset{(-)}{h}{}^1_2(x_1) \leq x_2 \leq \overset{(+)}{h}{}^1_2(x_1), \quad x_1 \in [0, x_1'[;$$

(iii) if $x_1' = 0$, $x_3' \neq 0$, then we continuously continue the traction (42) from the face surface (40) on the plane surface

$$x_1 = 0, \quad \overset{(-)}{h}{}_2^3(x_3) \leq x_2 \leq \overset{(+)}{h}{}_2^3(x_3), \quad x_3 \in [0,\, x_3'[.$$

4. Rectangularly linked prismatic bars loaded by self-weight

In this section we consider as an example the vertical prismatic bar, suspended from a fixed support and loaded by self-weight, linked (glued) with the non-fixed end with the horizontal bar with the fixed other end and loaded by self-weight $X_3^3 \equiv X_3^1 = -\gamma = \text{const}$.

Thus, we have the following loadings

$$\begin{aligned} &\overset{0,0}{Y}{}_\alpha^l \equiv 0, \quad l = 1,3, \quad \alpha = 1,2, \\ &\overset{0,0}{Y}{}_3^1(x_1) = -4\gamma h_2^1(x_1) h_3^1(x_1), \quad \overset{0,0}{Y}{}_3^3(x_3) = -4\gamma h_1^3(x_3) h_2^3(x_3), \quad \gamma = const; \end{aligned} \tag{43}$$

boundary conditions

$$v_j^1(x_1^0) = 0, \quad v_j^3(x_3^0) = 0, \quad j = 1,2,3; \tag{44}$$

contact conditions

$$X_{1j}^1(0) = X_{1j}^3(0), \quad v_j^1(0) = v_j^3(0), \quad j = 1,2,3. \tag{45}$$

Taking into account (43)-(45) from (28), (30), and the expressions for A_j, $j = 1,2,3$; d_2^1; A^α, $\alpha = 1,2$; C, D, we get

$$c_j^2 = 0, \quad d_j^2 = 0, \quad j = 1,2,3; \quad c_2^1 = E = 0; \quad A_\alpha = 0, \quad \alpha = 1,2; \quad d_2^1 = 0;$$

$$A_3 = -4\gamma \int_{x_1^0}^{0} \frac{d\tau}{h_2^1(\tau) h_3^1(\tau)} \int_{x_1^0}^{\tau} h_2^1(\eta) h_3^1(\eta) d\eta + 4\gamma \int_{x_3^0}^{0} \frac{d\tau}{h_1^3(\tau) h_2^3(\tau)} \int_{x_3^0}^{\tau} h_1^3(\eta) h_2^3(\eta) d\eta;$$

$$A^1 = 4\lambda\gamma \int_{x_3^0}^{0} h_1^3(\eta) h_2^3(\eta) d\eta; \quad A^2 = -4\gamma \int_{x_1^0}^{0} h_2^1(\eta) h_3^1(\eta) d\eta;$$

$$C = \int_{x_1^0}^{0} \frac{d\tau}{h_2^1(\tau) h_3^1(\tau)}, \quad D = \int_{x_3^0}^{0} \frac{d\tau}{h_1^3(\tau) h_2^3(\tau)}.$$

Therefore, by virtue of (23), (24), for weighted double moments of the displacements we obtain

$$
\begin{aligned}
&v_1^1(x_1) = c_1^1 \int\limits_{x_1^0}^{x_1} \frac{d\tau}{h_2^1(\tau)h_3^1(\tau)}, \quad v_2^1(x_1) = 0;\\
&v_1^3(x_3) = d_1^1 \int\limits_{x_3^0}^{x_3} \frac{d\tau}{h_1^3(\tau)h_2^3(\tau)}, \quad v_2^3(x_3) = 0;\\
&v_3^1(x_1) = 4\gamma \int\limits_{x_1^0}^{x_1} \frac{d\tau}{h_2^1(\tau)h_3^1(\tau)} \int\limits_{x_1^0}^{\tau} h_2^1(\eta)h_3^1(\eta)d\eta + c_3^1 \int\limits_{x_1^0}^{x_1} \frac{d\tau}{h_2^1(\tau)h_3^1(\tau)};\\
&v_3^3(x_3) = 4\gamma \int\limits_{x_3^0}^{x_3} \frac{d\tau}{h_1^3(\tau)h_2^3(\tau)} \int\limits_{x_3^0}^{\tau} h_1^3(\eta)h_2^3(\eta)d\eta + d_3^1 \int\limits_{x_3^0}^{x_3} \frac{d\tau}{h_1^3(\tau)h_2^3(\tau)},
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
&d_1^1 = \Delta^{-1}\Big[(A_3 - CA^2)\lambda - A^1 D\Big]\frac{C}{\lambda + 2\mu},\\
&d_3^1 = \Delta^{-1}\Big[-\frac{A^1(C)^2}{\lambda + 2\mu} + D(A_3 - CA^2)\Big],\\
&c_1^1 = \Delta^{-1}\Big[(A_3 - CA^2)\lambda - A^1 D\Big]\frac{D}{\lambda + 2\mu},\\
&c_3^1 = A^2 + \Delta^{-1}\Big[(A_3 - CA^2)\lambda - A^1 D\Big]\frac{C}{\lambda + 2\mu}.
\end{aligned}
\tag{47}
$$

By virtue of (12), (13), (46), the double moments of the stresses have the following forms

$$
\begin{aligned}
&X_{1100}^1(x_1) = (\lambda + 2\mu)c_1^1, \quad X_{1100}^3(x_3) = X_{2200}^3(x_3) = \lambda\Big[d_3^1 + 4\gamma \int\limits_{x_3^0}^{x_3} h_1^3(\tau)h_2^3(\tau)d\tau\Big],\\
&X_{2200}^1(x_1) = X_{3300}^1(x_1) = \lambda c_1^1,\\
&X_{3300}^3(x_3) = (\lambda + 2\mu)\Big[d_3^1 + 4\gamma \int\limits_{x_3^0}^{x_3} h_1^3(\tau)h_2^3(\tau)d\tau\Big];\\
&X_{1200}^1(x_1) = X_{2100}^1(x_1) = 0, \quad X_{1200}^3(x_3) = X_{2100}^3(x_3) = 0,\\
&X_{1300}^1(x_1) = X_{3100}^1(x_1) = \mu\Big[c_3^1 + 4\gamma \int\limits_{x_1^0}^{x_1} h_2^1(\tau)h_3^1(\tau)d\tau\Big],
\end{aligned}
$$

$$X^3_{1300}(x_3) = X^3_{3100}(x_3) = \mu d^1_1,$$
$$X^1_{2300}(x_1) = X^1_{3200}(x_1) = 0,$$
$$X^3_{2300}(x_3) = X^3_{3200}(x_3) = 0.$$

Example. Let now (se Fig. 6)

$$2h^3_\alpha(x_3) = 2h^3_\alpha = const, \quad \alpha = 1, 2;$$
$$2h^1_3(x_1) = -\overset{(-)}{h^1_3} = const, \quad 2h^1_2(x_1) = 2h^1_2 = const,$$

then

$$A^1 = -4\lambda\gamma h^3_1 h^3_2 x^0_3, \quad A^2 = 4\gamma h^1_2 h^1_3 x^0_1,$$
$$C = -\frac{x^0_1}{h^1_2 h^1_3}, \quad D = -\frac{x^0_3}{h^3_1 h^3_2}, \quad A_3 = -2\gamma(x^0_1)^2 + 2\gamma(x^0_3)^2 = 2\gamma\Big[(x^0_3)^2 - (x^0_1)^2\Big], \quad (48)$$
$$\Delta = \frac{\sigma}{1-\sigma}\frac{(x^0_1)^2}{(h^1_2 h^1_3)^2} - \frac{(x^0_3)^2}{(h^1_3 h^3_3)^2},$$

and

$$v^1_1(x_1) = \frac{c^1_1}{h^1_2 h^1_3}(x_1 - x^0_1), \quad v^1_2(x_1) = 0, \quad v^1_3(x_1) = 2\gamma(x_1 - x^0_1)^2 + \frac{c^1_3}{h^1_2 h^1_3}(x_1 - x^0_1);$$
$$v^3_1(x_3) = \frac{d^1_1}{h^3_1 h^3_2}(x_3 - x^0_3), \quad v^3_2(x_3) = 0, \quad v^3_3(x_3) = 2\gamma(x_3 - x^0_3)^2 + \frac{d^1_3}{h^3_1 h^3_2}(x_3 - x^0_3);$$
$$X^1_{1100}(x_1) = (\lambda + 2\mu)c^1_1, \quad X^1_{2200}(x_1) = X^1_{3300}(x_1) = \lambda c^1_1,$$
$$X^3_{1100}(x_3) = X^3_{2200}(x_3) = \lambda\Big[d^1_3 + 4\gamma h^3_1 h^3_2(x_3 - x^0_3)\Big],$$

$$X^3_{3300}(x_3) = (\lambda + 2\mu)\Big[d^1_3 + 4\gamma h^3_1 h^3_2(x_3 - x^0_3)\Big],$$
$$X^1_{1200}(x_1) = X^1_{2100}(x_1) = 0,$$
$$X^1_{1300}(x_1) = X^1_{3100}(x_1) = \mu\Big[c^1_3 + 4\gamma h^1_2 h^1_3(x_1 - x^0_1)\Big],$$
$$X^3_{1200}(x_3) = X^3_{2100}(x_3) = 0,$$
$$X^3_{1300}(x_1) = X^3_{3100}(x_3) = \mu d^1_1,$$
$$X^1_{2300}(x_1) = X^1_{3200}(x_1) = 0,$$
$$X^3_{2300}(x_3) = X^3_{3200}(x_3) = 0,$$

with (47) combined with (48).

As we see, the vertical components of the weighted double moments of displacements are quadratic functions, while the horizontal components are linear functions of x_1 for B_1 and of x_3 for B_3; for B_1 the double moments: (i) of the normal stresses are constants, (ii) of the tangent stresses are zero except $X^1_{1300}(x_1) = X^1_{3100}(x_1)$

which are linear functions, while for B_3 the double moments: (i) of the normal stresses are linear functions of x_3, (ii) of the tangent stresses are equal to zero except $X^3_{3200}(x_3) = X^3_{2300}(x_3)$ which are constants. This result was to be expected.

5. Conclusions and Outlook

(1) A model of two rectangularly linked elastic bars within the framework of the (0,0) approximation of hierarchical models of prismatic bars on the basis of the linear theory of elasticity is constructed.
(2) Static and dynamical contact problems are set and the boundary-contact problem, when at the non-contact ends of the bars displacements are prescribed, while on the face surfaces of the bars tractions are known, is solved in the explicit form.
(3) As a concrete example the boundary-contact problem, when the vertical prismatic bar suspended from a fixed support and loaded by self-weight linked (glued) with the non-fixed end with the horizontal bar with the fixed other end and loaded by self-weight, is considered.
(4) Analogously we can construct and investigate similar models within the framework of (1,0) and (1,1) approximations of hierarchical models of prismatic bars on the basis of the linear theory of elasticity and classical and refined models of elastic bars.
(5) In order to investigate dynamical problems along with classical methods an approach developed in [5] can be applied.

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