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Global Deformations of Witt and Virasoro Algebra

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In this paper I consider deformations with the base being a general commutative algebra with identity. It turns out that in infinite dimension such global deformations give a much richer picture, depending on the augmentation of the base algebra. It is of course not the case if one considers deformations with complete local algebra base. In infinite dimension, even rigidity is not kept by considering global deformations, as the Witt/Virasoro algebra shows. It is formally rigid, but it has plenty nontrivial nice global deformations, like the Krichever-Novikov type algebras.

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1. Introduction

In the previous talk we gave the definition of a *deformation with base*. Recall it for a genaral commutative algebra base A with identity. Fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$, and set $\operatorname{Ker}\varepsilon = m$.

Definition 1.1: A deformation λ of $\mathcal{L}$ with base (A, m) is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ such that

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism [1, 2].

Let us call a deformation with a general commutative algebra base *global*.

We considered complete local algebra base, which is the simplest base where versal defomation theory can be discussed. The reason we considered local algebras, because it has a unique maximal ideal, so only one augmentation.

In a general commutative algebra with identity, there are more than one augmentations, and each one defines a deformation set-up. This means that in this larger category of commutative algebras, versal deformation depends on choosing an augmentation.

We also saw that nontrivial deformations of a Lie algebra $\mathcal{L}$ are in one to one correspondence with elements of the cohomology space $H^2(\mathcal{L}, \mathcal{L})$. We call a Lie algebra *rigid*, if it has no nontrivial deformations.

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A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the *base* of deformation – which is a commutative algebra of functions – with a *smooth manifold*, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations [5].

In finite dimension global deformations coincide with formal deformations, so we can use cohomology theory. We saw that cohomology and versal deformations make it possible to get a geometric description of the moduli space of a certain type of algebraic objects in a given dimension.

In infinite dimension there is no tight relation between global and formal deformations, consequently, a Lie algebra being formally rigid does not mean that it can not have nontrivial deformations.

In this talk I will give an example, where in fact the infinite dimensional Lie algebra is formally rigid, yet is has beautiful nontrivial global deformations.

2. Rigidity

The relationship between trivial cohomology and rigidity has an old history. In this respect, one clear success story is the classification of complex analytic structures on a fixed topological manifold. Also in algebraic geometry one has well-developed results in this direction. One of these results is that the local situation at a point $[C]$ of the moduli space is completely governed by the cohomological properties of the geometric object C.

As typical example recall that for the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g over $\mathbb{C}$ (or equivalently, compact Riemann surfaces of genus g) the tangent space $T_{[C]}\mathcal{M}_g$ can be naturally identified with $H^1(C, T_C)$, where T_C is the sheaf of holomorphic vector fields over C. This extends to higher dimension. In particular, for compact complex manifolds M, the condition $H^1(M, T_M)$ implies that M is rigid (Kodaira). Rigidity means that any differentiable family $\pi : M \to B \subseteq \mathbb{R}$, $0 \in B$ which contains M as the special member $M_0 := \pi^{-1}(0)$ is trivial in a neighbourhood of 0, i.e. for t small enough $M_t := \pi^{-1}(t) \cong M$.

Even more generally, for M a compact complex manifold and $H^1(M, T_M) \neq \{0\}$ there exists a versal family which can be realized locally as a family over a certain subspace of $H^1(M, T_M)$ such that every appearing deformation family is "contained" in this versal family.

These positive results lead to the impression that the vanishing of the relevant cohomology spaces will imply rigidity with respect to deformations also in the case of other structures.

The goal of this lecture is to shed some light on this in the context of deformations of the Witt or Virasoro algebra.

In a more compact manner a Lie algebra $\mathcal{L}$, i.e. its bracket $[.,.]$, might be written with an anti-symmetric bilinear form

$$\mu_0 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, \qquad \mu_0(x, y) = [x, y],$$

fulfilling certain additional conditions corresponding to the Jacobi identity. Consider on the same vector space $\mathcal{L}$ is modeled on, a family of Lie structures

$$\mu_t = \mu_0 + t \cdot \phi_1 + t^2 \cdot \phi_2 + \cdots \ , \tag{1}$$

with bilinear maps $\phi_i : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that $\mathcal{L}_t := (\mathcal{L}, \mu_t)$ is a Lie algebra and $\mathcal{L}_0$ is the Lie algebra we started with. The family $\{\mathcal{L}_t\}$ is a *deformation* of $\mathcal{L}_0$.

Up to this point we did not specify the "parameter" t. Indeed, different choices are possible.

(1) The parameter t might be a variable which allows to plug in numbers $\alpha \in \mathbb{C}$. In this case $\mathcal{L}_\alpha$ is a Lie algebra for every α for which the expression (1) is defined. The family can be considered as deformation over the affine line $\mathbb{C}[t]$ or over the convergent power series $\mathbb{C}\{\{t\}\}$. The deformation is called a *geometric* or an *analytic deformation* respectively.

(2) We consider t as a formal variable and we allow infinitely many terms in (1). It might be the case that μ_t does not exist if we plug in for t any other value different from 0. In this way we obtain deformations over the ring of formal power series $\mathbb{C}[[t]]$. The corresponding deformation is a *formal deformation.*

(3) The parameter t is considered as an infinitesimal variable, i.e. we take $t^2 = 0$. We obtain *infinitesimal deformations* defined over the quotient $\mathbb{C}[X]/(X^2) = \mathbb{C}[[X]]/(X^2)$.

A Lie algebra $(\mathcal{L}, \mu_0)$ is called *rigid* if every deformation μ_t of μ_0 is locally equivalent to the trivial family. Intuitively, this says that $\mathcal{L}$ cannot be deformed.

The word "locally" in the definition of rigidity means that we only consider the situation for t "near 0". Of course, this depends on the category we consider. As on the formal and the infinitesimal level there exists only one closed point, i.e. the point 0 itself, every deformation over $\mathbb{C}[[t]]$ or $\mathbb{C}[X]/(X^2)$ is already local. This is different on the geometric and analytic level. Here it means that there exists an open neighbourhood U of 0 such that the family restricted to it is equivalent to the trivial one. In particular, this implies $\mathcal{L}_\alpha \cong \mathcal{L}_0$ for all $\alpha \in U$.

Clearly, a question of fundamental interest is to decide whether a given Lie algebra is rigid. Moreover, the question of rigidity will depend on the category we consider. Depending on the set-up we will have to consider infinitesimal, formal, geometric, and analytic rigidity. If the algebra is not rigid, one would like to know whether there exists a moduli space of (inequivalent) deformations. If so, what is its structure, dimension, etc?

For Lie algebra deformations the relevant cohomology space is $H^2(\mathcal{L}, \mathcal{L})$, the space of Lie algebra two-cohomology classes with values in the adjoint module $\mathcal{L}$.

Recall that these cohomology classes are classes of two-cocycles modulo coboundaries. An antisymmetric bilinear map $\phi : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a Lie algebra *two-cocycle* if $d_2\phi = 0$, or expressed explicitely

$$\phi([x,y],z)+\phi([y,z],x)+\phi([z,x],y)-[x,\phi(y,z)]+[y,\phi(z,x)]-[z,\phi(x,y)] = 0. \quad (2)$$

The map ϕ will be a *coboundary* if there exists a linear map $\psi : \mathcal{L} \to \mathcal{L}$ with

$$\phi(x,y) = (d_1\psi)(x,y) := \psi([x,y]) - [x, \psi(y)] + [y, \psi(x)]. \quad (3)$$

If we write down the Jacobi identity for μ_t given by (1) then it can be immediately verified that the first non-vanishing ϕ_i has to be a two-cocycle in the above sense. Furthermore, if μ_t and μ'_t are equivalent then the corresponding ϕ_i and ϕ'_i are cohomologous, i.e. their difference is a coboundary.

The following results are well-known:

(1) $H^2(\mathcal{L},\mathcal{L})$ classifies infinitesimal deformations of $\mathcal{L}$ [7].
(2) If $\dim H^2(\mathcal{L},\mathcal{L}) < \infty$ then all formal deformations of $\mathcal{L}$ up to equivalence can be realised in this vector space [4].
(3) If $H^2(\mathcal{L},\mathcal{L}) = 0$ then $\mathcal{L}$ is infinitesimally and formally rigid (this follows directly from ((1)) and ((2))).
(4) If $\dim \mathcal{L} < \infty$ then $H^2(\mathcal{L},\mathcal{L}) = 0$ implies that $\mathcal{L}$ is also rigid in the geometric and analytic sense [7], [9], [10].

3. Witt algebra and Virasoro algebra

Consider the complexification $\mathcal{W}$ of the Lie algebra of polynomial vector fields on the circle:

$$e_k \to e^{ik\varphi}\frac{d}{d\varphi},$$

where φ is the angular parameter. The bracket is

$$[e_n, e_m] = (m-n)e_{n+m}.$$

The Lie algebra $\mathcal{W}$ is called the *Witt algebra* defined by E. Cartan. It is infinite dimensional and graded with $\deg e_n = n$. It is well-known (Gelfand, Fuchs, [6]) that $\mathcal{W}$ has a unique nontrivial one-dimensional central extension. It is generated by $e_n (n \in \mathbb{Z})$ and the central element c, and its bracket operation is

$$[e_n, e_m] = (m-n)e_{n+m} + 1/12(m^3 - m)\delta_{n,-m}c, \quad [e_n, c] = 0.$$

The extended Lie algebra is the *Virasoro algebra*.

The cohomology "responsible" for deformations is $H^2(\mathcal{W},\mathcal{W})$.

Theorem 3.1 : *([2, 3]) The cohomology space $H^2(\mathcal{W},\mathcal{W})$ is trivial.*

Hence, guided by the experience in the theory of deformations of complex manifolds, one might think that $\mathcal{W}$ is rigid in the sense that all families containing $\mathcal{W}$ as a special element will be trivial. But this is not the case as we will show. Certain natural families of Krichever-Novikov type algebras of geometric origin will appear which contain $\mathcal{W}$ as special element. But none of the other elements will be isomorphic to $\mathcal{W}$.

In the past decades, much attention has been paid to infinite dimensional Lie algebras, mainly because of their applications in mathematical physics. There are basically two kinds of infinite dimensional objects which are intensively studied: Lie algebras of geometric origin, like vector fields on a smooth manifold, and the so called Kac-Moody algebras, the theory of which is closely related to the theory of finite dimensional semisimple Lie algebras.

There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. One has to clarify the difference between deformations of *geometric* origin and so called *formal* deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions there exists a versal formal deformation, which induces all other deformations. In infinite dimension there is no unique cohomology theory which would sense global deformations.

In our work with Martin Schlichenmaier we constructed global deformations of the Witt algebra by considering certain families of algebras for the genus one case (i.e. the *elliptic curve case*) and let the elliptic curve degenerate to a singular cubic [5]. The two points, where poles are allowed, are the zero element of the elliptic curve and a 2-torsion point. In this way we obtain families parameterized over the affine line with the peculiar behavior that every family is a global deformation of the Witt algebra, i.e. $\mathcal{W}$ is a special member, whereas all other members are mutually isomorphic but not isomorphic to $\mathcal{W}$. Globally these families are non-trivial, but infinitesimally and formally they are trivial. The construction can be extended to the Virasoro algebra.

The results obtained do not have only relevance to the deformation theory of algebras but also to the theory of two-dimensional conformal fields and their quantization. It is well-known that the Witt algebra, the Virasoro algebra, and their representations are of fundamental importance for the local description of conformal field theory on the Riemann sphere (i.e. for genus zero). Krichever and Novikov proposed in the case of higher genus Riemann surfaces the use of global operator fields which are given with the help of the Lie algebra of vector fields of Krichever-Novikov type, certain related algebras, and their representations.

4. Krichever-Novikov type algebras

Algebras of Krichever-Novikov types are generalizations of the Virasoro algebra.

Let M be a compact Riemann surface of genus g, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$. Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$ be numbers. Fix

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_{N-K})$$

disjoint ordered tuples of distinct points ("marked points", "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i,j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets and $A = I \cup O$ as a set.

Denote by $\mathcal{L}$ the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A, equipped with the Lie bracket $[.,.]$ of vector fields. Its local form is

$$[e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] := \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z) \right) \frac{d}{dz} \ .$$

To avoid cumbersome notation we will use the same symbol for the section and its representing function.

For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$ and $O = \{\infty\}$, the introduced vector field algebra is the Witt algebra. We denote for short this situation as the *classical situation*.

For infinite dimensional algebras and modules and their representation theory a graded structure is usually of importance to obtain structure results.

In the classical situation the algebras are obviously graded by taking as degree $\deg l_n := n$ and $\deg x \otimes z^n := n$. For higher genus there is usually no grading. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory of representations (Verma modules, etc.). Let A be an (associative or Lie) algebra admitting a direct decomposition as vector space $A = \bigoplus_{n\in\mathbb{Z}} A_n$. The algebra A is called an *almost-graded* algebra if (1) $\dim A_n < \infty$ and (2) there are constants R and S such that

$$A_n \cdot A_m \quad \subseteq \bigoplus_{h=n+m+R}^{n+m+S} A_h, \qquad \forall n,m \in \mathbb{Z} \ . \tag{4}$$

The elements of A_n are called *homogeneous elements of degree n.*

It is possible to introduce an almost gading by fixing the order of the basis elements at the points in I in a certain manner and in O in a complementary way to make them unique up to scaling.

In the following we will give an explicit description of the basis elements for those genus zero and one situation we need. The Witt algebra is a graded Lie algebra. In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory of representations.

For the 2-point situation for M a higher genus Riemann surface and $I = \{P\}$, $O = \{Q\}$ with $P, Q \in M$, Krichever and Novikov introduced an almost-graded structure of the vector field algebras $\mathcal{L}$ by exhibiting a special basis and defining their elements to be the homogeneous elements.

We consider the *genus one* case, i.e. the case of one-dimensional complex tori or equivalently the elliptic curve case.

Recall that the elliptic curves can be given in the projective plane by

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3, \quad g_2, g_3 \in \mathbb{C},$$

where $\Delta := {g_2}^3 - 27{g_3}^2 \neq 0$. This condition $\Delta \neq 0$ assures that the curve will be nonsingular.

Instead of the above elliptic curve expression we can use the description

$$Y^2Z = 4(X - e_1Z)(X - e_2Z)(X - e_3Z)$$

with

$$e_1 + e_2 + e_3 = 0, \quad \text{and} \quad \Delta = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 \neq 0.$$

We set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0, \quad e_i \neq e_j \text{ for } i \neq j\}.$$

In the product $B \times \mathbb{P}^2$ we consider the family of elliptic curves $\mathcal{E}$ over B defined via the second expression (in product form). The family can be extended to

$$\widehat{B} := \{e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0\}.$$

The fibers above $\widehat{B} \setminus B$ are singular cubic curves. Resolving the one linear relation in $\widehat{B}$ via $e_3 = -(e_1 + e_2)$ we obtain a family over $\mathbb{C}^2$.

Consider the complex lines in $\mathbb{C}^2$

$$\begin{aligned} D_s &:= \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\},\ s \in \mathbb{C}, \\ D_\infty &:= \{(0, e_2) \in \mathbb{C}^2\}. \end{aligned}$$

Set also

$$D_s^* = D_s \setminus \{(0,0)\}$$

for the punctured line. Now

$$B \cong \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}.)$$

We have to introduce the points where poles are allowed. For our purpose it is enough to consider two marked points. We will always put one marking to $\infty = (0:1:0)$ and the other one to the point with the affine coordinate $(e_1, 0)$. These markings define two sections of the family $\mathcal{E}$ over $\widehat{B} \cong \mathbb{C}^2$.

With respect to the group structure on the elliptic curve given by ∞ as the neutral element (the first marking) the second marking chooses a two-torsion point. All other choices of two-torsion points will yield isomorphic situations.

We consider the algebras of Krichever-Novikov type corresponding to the elliptic curve and possible poles at $\bar{z} = \bar{0}$ and $\bar{z} = \overline{1/2}$ [1] (respectively in the algebraic-geometric picture, at the points ∞ and $(e_1, 0)$).

First we consider the vector field algebra $\mathcal{L}$. A basis of the vector field algebra is given by

$$V_{2k+1} := (X - e_1)^k Y \frac{d}{dX}, \quad V_{2k} := \frac{1}{2} f(X)(X - e_1)^{k-2} \frac{d}{dX}, \qquad k \in \mathbb{Z}. \tag{5}$$

If we vary the points e_1 and e_2 (and accordingly $e_3 = -(e_1 + e_2)$) we obtain families of curves and associated families of vector field algebras. At least this is the case as long as the curves are non-singular.

The curves are non-singular exactly over the points of B. Over the exceptional D_s at least two of the e_i are the same. For the vector field algebra we obtain

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} & \\ \quad +(e_1 - e_2)(e_1 - e_3)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} & \\ \quad +(m-n-2)(e_1 - e_2)(e_1 - e_3)V_{n+m-4}, & n \text{ odd},\ m \text{ even}. \end{cases}$$

By defining $\deg(V_n) := n$, we obtain an almost-grading.

In fact these relations define Lie algebras for every pair $\mathcal{L}^{(e_1,e_2)}$ the Lie algebra corresponding to (e_1, e_2). Obviously, $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

Proposition 4.1: *([5, Prop. 5.1]) For $(e_1, e_2) \neq (0,0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are not isomorphic to the Witt algebra $\mathcal{W}$, but $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.*

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[1] Here $\bar{z}$ does not denote conjugation, but taking the residue class modulo the lattice.

Consider now the family of algebras obtained by taking as base variety the line D_s (for any s). First consider $s \neq \infty$. We calculate $(e_1-e_2)(e_1-e_3) = e_1^2(1-s)(2+s)$ and can rewrite the brackets for these curves

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} & \\ \qquad +e_1^2(1-s)(2+s)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} & \\ \qquad +(m-n-2)e_1^2(1-s)(2+s)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases}$$

For D_∞ we have $e_3 = -e_2$ and $e_1 = 0$ and obtain

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} - e_2^2 V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} - (m-n-2)e_2^2 V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases}$$

If we take $V_n^* = (\sqrt{e_1})^{-n} V_n$ (for $s \neq \infty$) as generators, we obtain for $e_1 \neq 0$ always the algebra with $e_1 = 1$ in our structure equations. For $s = \infty$ a rescaling with $(\sqrt{e_2})^{-n} V_n$ will do the same (for $e_2 \neq 0$). Hence for fixed s in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0, 0)$.

Theorem 4.2 : *([5]) For every $s \in \mathbb{C} \cup \{\infty\}$ the families of Lie algebras defined via the structure equations for $s \neq \infty$ and the brackets above for $s = \infty$ define global deformations $\mathcal{W}_t^{(s)}$ of the Witt algebra $\mathcal{W}$ over the affine line $\mathbb{C}[t]$.*

Here t corresponds to the parameter e_1 and e_2 respectively.

The Lie algebra above $t = 0$ corresponds always to the Witt algebra, the algebras above $t \neq 0$ belong (if s is fixed) to the same isomorphic type, but are not isomorphic to the Witt algebra.

Hence despite its infinitesimal and formal rigidity, the Witt algebra $\mathcal{W}$ and the Virasoro algebra as well, admits deformations $\mathcal{L}_t$ over the affine line with $\mathcal{L}_0 \cong \mathcal{W}$ which restricted to every (Zariski or analytic) neighbourhood of $t = 0$ are nontrivial.

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