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On Deviation Between Kernel-Type Estimators of a Distribution Density in Some Independent Samples

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In the paper, the tests are constructed for the hypotheses that $p \geq 2$ independent samples have the same distribution density (homogeneity hypothesis) or have the same well-defined distribution density (goodness-of-fit test). The limiting power of the constructed tests is found for some local "close" alternatives.

Keywords: Homogeneity hypothesis, Goodness-of-fit test, Power of test, Wiener process, Test consistency, Kernel type estimator of density, Histogram.

AMS Subject Classification: 62G10, 62G20.

1. Introduction

Let $X^{(i)} = (X_1^{(i)}, \dots, X_{n_i}^{(i)})$, $i = 1, \dots, p$, be independent samples of size $n_1, n_2, \dots, n_p$, from $p \geq 2$ general populations with distribution densities $f_1(x), \dots, f_p(x)$. It is based on sample $X^{(i)}$, $i = 1, \dots, p$, checking two hypotheses: the homogeneity hypothesis

$$H_0 : \ f_1(x) = \cdots = f_p(x)$$

and the goodness-of-fit hypothesis

$$H_0' : \ f_1(x) = \cdots = f_p(x) = f_0(x),$$

where $f_0(x)$ is the completely defined density function. In the case of the hypothesis H_0, the general density of the distribution $f_0(x)$ is unknown.

In the present paper, the tests are constructed for the hypotheses H_0 and H_0'

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against the sequence of close alternatives:

$$H_1: \ f_i(x) = f_0(x) + \alpha(n_0)\varphi_i(x),$$
$$\alpha(n_0) \longrightarrow 0, \ \ n_0 = \min(n_1, \dots, n_p) \longrightarrow \infty,$$
$$\int \varphi_i(x)\,dx = 0, \ \ i = 1, \dots, p.$$

We consider the test for the hypotheses H_0 and H_0' based on the statistic

$$T(n_1, n_2, \dots, n_p) = \sum_{i=1}^{p} N_i \int \Big[\widehat{f}_i(x) - \frac{1}{N}\sum_{j=1}^{p} N_j \widehat{f}_j(x)\Big]^2 r(x)\,dx, \tag{1}$$

where $\widehat{f}_i(x)$ is a kernel-type Rosenblatt-Parzen estimator of the density of the function $f_i(x)$:

$$\widehat{f}_i(x) = \frac{a_i}{n_i}\sum_{j=1}^{n_i} K\big(a_i(x - X_j^{(i)})\big), \ \ N_i = \frac{n_i}{a_i}, \ \ N = N_1 + \dots + N_p.$$

The particular case $p = 2$ is considered in [1] and [7]. Then the statistic T takes the explicit form

$$T(n_1, n_2) = \frac{N_1 N_2}{N_1 + N_2} \int \big(\widehat{f}_1(x) - \widehat{f}_2(x)\big)^2 r(x)\,dx.$$

2. Preliminaries

We consider the question concerning the limiting law of the distribution of statistic (1) for the hypothesis H_1 when n_i tends to infinity so that $n_i = nk_i$, where $n \to \infty$, and k_i are constants. Let $a_1 = a_2 = \dots = a_p = a_n$, where $a_n \to \infty$ as $n \to \infty$.

To obtain the limiting law of distribution of the functional $T_n = T(n_1, \dots, n_p)$, we make assumptions as to the functions $K(x)$, $f_0(x)$, $\varphi_i(x)$, $i = 1, \dots, p$, and $r(x)$:

(i) $K(x) \geq 0$, vanishes outside the finite interval $(-A, A)$ and, together with its derivatives, is continuous on this interval or absolutely continuous on $(-\infty, \infty)$, $x^2 K(x)$ is integrable and $K^{(1)}(x) \in L_1(-\infty, \infty)$. In both cases $\int K(x)\,dx = 1$.

(ii) The density function $f_0(x)$ is bounded and positive on $(-\infty, \infty)$ or it is bounded and positive in some finite interval $[c, d]$. Besides, in the domain of positivity it has a bounded derivative.

(iii) Functions $\varphi_j(x)$, $j = 1, \dots, p$, are bounded and have bounded derivatives of first order; also $\varphi_i^{(1)}(x) \in L_1(-\infty, \infty)$.

(iv) The weight function $r(x)$ is piecewise-continuous, bounded and $r(x) \in L_1(-\infty, \infty)$.

3. Statements of the main results

Theorem 3.1: *Let the conditions* (i)–(iv) *be fulfilled. If* $\alpha_n = n^{-1/2} a_n^{1/4}$ $(\alpha_n = \alpha(n_0))$, $n^{-1} a_n^{9/2} \to 0$ *as* $n \to \infty$, *then for the hypothesis* H_1 *the random variable* $a_n^{1/2}(T_n - \mu)$ *has the normal limiting distribution* $(A(\varphi), \sigma^2)$, *where*

$$A(\varphi) = \sum_{i=1}^{p} k_i \int \Big[\varphi_i(x) - \frac{1}{\overline{k}} \sum_{j=1}^{p} k_j \varphi_j(x)\Big]^2 r(x)\, dx,$$

$$\sigma^2 = 2(p-1) \int f_0^2(x) r^2(x)\, dx \cdot R(K_0), \quad K_0 = K * K,$$

$$\mu = (p-1) \int f_0(x) r(x)\, dx \cdot R(K), \quad R(g) = \int g^2(x)\, dx,$$

$$\overline{k} = k_1 + \cdots + k_p, \quad p \geq 2.$$

The conditions of Theorem 3.1 as regards a_n and α_n are fulfilled, for instance, if it is assumed that $a_n = n^{\delta}$, $\alpha_n = n^{-1/2+\delta/4}$ for $0 < \delta < \frac{2}{9}$.

Corollary 3.2: *Let the conditions* (i), (ii) *and* (iv) *be fulfilled for* $K(x)$, $f_0(x)$ *and* $r(x)$, *respectively. If* $n^{-1} a_n^2 \to 0$, *then the random variable* $a_n^{1/2}(T_n - \mu)$ *for the hypothesis* H_0' *has a normal limiting distribution* $(0, \sigma^2)$.

Using this corollary we may construct the test for the hypothesis H_0'; the critical domain for testing this hypothesis is defined by the inequality

$$T_n \geq d_n(\alpha), \tag{2}$$

where

$$d_n(\alpha) = \mu + a_n^{-1/2} \sigma \lambda_\alpha,$$

λ_α is the quantile of the level $1-\alpha$, $0 < \alpha < 1$, of the standard normal distribution $\Phi(x)$.

Remark 1: The particular case $p = 2$ of criteria (2) is considered in [1, p. 43].

Corollary 3.3: *In the conditions of Theorem* 3.1 *the local behavior of the power* $P_{H_1}(T_n \geq d_n(\alpha))$ *is as follows: for* $n \to \infty$

$$P_{H_1}\big(T_n \geq d_n(\alpha)\big) \longrightarrow 1 - \Phi\Big(\lambda_\alpha - \frac{A(\varphi)}{\sigma}\Big).$$

We introduce the notation

$$f_n^*(x) = \frac{1}{\overline{k}} \sum_{j=1}^{p} k_j \widehat{f}_j(x),$$

$$\overline{\mu}_n = \int f_n^*(x) r(x)\, dx,$$

$$\Delta_n^2 = \frac{1}{\overline{k}} \sum_{i=1}^{p} k_i \Delta_{in}^2, \quad \Delta_{in}^2 = \int \widehat{f}_i^{\,2}(x) r^2(x)\, dx.$$

Theorem 3.4: *Let all the conditions of Theorem* 3.1 *be fulfilled. Then* $a_n^{1/2}(T_n - \mu_n)\sigma_n^{-1}$ *for the hypothesis* H_1 *has a normal limiting distribution* $(A(\varphi)\sigma^{-1}, 1)$*, where*

$$\mu_n = (p-1)R(K)\overline{\mu}_n, \quad \sigma_n^2 = 2(p-1)R(K_0)\Delta_n^2.$$

Corollary 3.5: *Let the conditions* (i), (ii), (iv) *and* $n^{-1}a_n^2 \to 0$ *be fulfilled. Then the random variable*

$$a_n^{1/2}(T_n - \mu_n)\sigma_n^{-1}$$

for the hypothesis H_0 *has a normal limiting distribution* $(0,1)$.

This result allows us to construct the asymptotic test of the hypothesis testing for $H_0: \ f_1(x) = \cdots = f_p(x)$ (hypothesis of homogeneity); the critical domain is established by the inequality

$$T_n \geq \widetilde{d}_n(\alpha) = \mu_n + a_n^{-1/2}\sigma_n\lambda_\alpha, \tag{3}$$

where λ_α is a quantile of the level $1-\alpha$ of the standard normal distribution $\Phi(x)$.

Corollary 3.6: *In the conditions of Theorem* 3.4 *the local behavior of the power* $P_{H_1}\left(T_n \geq \widetilde{d}_n(\alpha)\right)$ *is as follows*

$$P_{H_1}(T_n \geq \widetilde{d}_n(\alpha)) \longrightarrow 1 - \Phi\big(\lambda_\alpha - A(\varphi)\sigma^{-1}\big).$$

Suppose $\inf\limits_{0\leq x\leq 1} f_0(x) > 0$ and $r(x) = f_0^{-1}(x)$, $x \in [0,1]$ $(= 0,\ x \notin [0,1])$. In this case for the hypothesis H_1, random variable $a_n^{1/2}(T_n - \mu_0)$ has the normal limiting distribution $(A(\varphi), \sigma_0^2)$, where

$$\mu_0 = (p-1)\int K^2(u)\, du, \quad \sigma_0^2 = 2(p-1)\int K_0^2(u)\, du.$$

Let us introduce

$$\widehat{T}_n = \widehat{T}(n_1, n_2) = \frac{n}{a_n} \sum_{i=1}^{p} k_i \int_0^1 \Big[\widehat{f}_i(x) - \frac{1}{\overline{k}} \sum_{j=1}^{p} k_j \widehat{f}_j(x)\Big]^2 r_n(x)\, dx,$$
$$r_n(x) = [f_n^*(x)]^{-1}.$$

Theorem 3.7: *Let the condition* (i)–(iv) *be fulfilled. If* $\alpha_n = n^{-1/2} a_n^{1/4}$ *and* $n^{-1} a_n^{9/2} \ln n \to 0$ *as* $n \to \infty$*, then random variable* $a_n^{1/2}(\widehat{T}_n - \mu_0)$ *for the hypothesis* H_1 *has the normal limiting distribution* $(A(\varphi), \sigma_0^2)$.

Corollary 3.8: *Let the conditions* (i), (ii) *and* (iv) *be fulfilled. If* $n^{-1} a_n^3 \ln n \to 0$*, then the random variable* $a_n^{1/2}(\widehat{T}_n - \mu_0)$ *for the hypothesis* H_0 *has normal distribution* $(0, \sigma_0^2)$.

This corollary allows us to construct the asymptotic test of the hypothesis testing for $H_0 :\ f_1(x) = \cdots = f_p(x)$; the critical domain is established by the inequality

$$\widehat{T}_n \geq \overset{\approx}{d}_n(\alpha) = \mu_0 + a_n^{-1/2} \lambda_\alpha \sigma_0, \tag{4}$$

where λ_α is a quantile of the level $1-\alpha$ of the standard normal distribution $\Phi(x)$.

Corollary 3.9: *In conditions of Theorem* 3.7 *the local behaviour of the power* $P_{H_1}(\widehat{T}_n \geq \overset{\approx}{d}_n(\alpha))$ *is as follows*

$$P_{H_1}\big(\widehat{T}_n \geq \overset{\approx}{d}_n(\alpha)\big) \longrightarrow 1 - \Phi\Big(\lambda_\alpha - \frac{A(\varphi)}{\sigma_0}\Big). \tag{5}$$

Remark 2:

(a) The test (2) of testing hypothesis H_0' against alternative $H_1 :\ f_1(x) = f_0(x)$, $f_j(x) = f_0(x) + \alpha_n \varphi_j(x)$, $j = 2, \dots, p$, is asymptotically strictly unbased, since $A(\varphi) > 0$ and equals 0 if and only if, when $\varphi_j(x) = 0$, $j = 2, \dots, p$.

(b) The tests (3) and (4) of testing hypothesis H_0 against H_1 is asymptotically strictly unbased, since $A(\varphi) > 0$ and equals 0 if and only if, when $\varphi_i(x) = \varphi_j(x)$, $i \neq j$, $i, j = 1, \dots, p$.

4. Proofs of Theorems 3.1, 3.4, 3.7

Proof of Theorem 3.1: Let us represent T_n as the sum

$$T_n = T_n^{(1)} + A_{1n} + A_{2n},$$

where

$$T_n^{(1)} = \frac{n}{a_n}\sum_{i=1}^{p} k_i \int \Big[\widehat{f}_i(x) - E\widehat{f}_i(x) - \frac{1}{\overline{k}}\sum_{j=1}^{p} k_j\big(\widehat{f}_j(x) - E\widehat{f}_j(x)\big)\Big]^2 r(x)\,dx,$$

$$A_{1n} = 2\,\frac{n}{a_n}\sum_{i=1}^{p} k_i \int \big[\widehat{f}_i(x) - E\widehat{f}_i(x)\big]\Big[E\widehat{f}_i(x) - \frac{1}{\overline{k}}\sum_{j=1}^{p} k_j E\widehat{f}_j(x)\Big] r(x)\,dx,$$

$$A_{2n} = \frac{n}{a_n}\sum_{i=1}^{p} k_i \int \Big[E\widehat{f}_i(x) - \frac{1}{\overline{k}}\sum_{j=1}^{p} k_j E\widehat{f}_j(x)\Big]^2 r(x)\,dx.$$

Here and in what follows $E(\,\cdot\,)$ is a mathematical expectation with respect to the hypothesis H_1.

It is not difficult to see that

$$E\widehat{f}_i(x) = a_n \int K(a_n(x-u)) f_0(u)\,du + \alpha_n \varphi_i(x) + \frac{\alpha_n}{a_n}\int tK(t)\int_0^1 \varphi_i^{(1)}\Big(x - \frac{tz}{a_n}\Big)\,dz\,dt.$$

This relation implies

$$A_{2n} = \frac{n\alpha_n^2}{a_n}\,A_n(\varphi) + O\Big(\frac{n\alpha_n^2}{a_n^2}\Big).$$

Hence, since $\frac{n\alpha_n^2}{\sqrt{a_n}} = 1$, we obtain

$$\sqrt{a_n}\,A_{2n} = A(\varphi) + O\Big(\frac{n\alpha_n^2}{a_n^{3/2}}\Big) = A(\varphi) + O\Big(\frac{1}{a_n}\Big), \tag{6}$$

where

$$A(\varphi) = \sum_{i=1}^{p} k_i \int \Big[\varphi_i(x) - \frac{1}{\overline{k}}\sum_{j=1}^{p} k_j \varphi_j(x)\Big]^2 r(x)\,dx.$$

Now let us show that $a_n^{1/2} A_{1n} \longrightarrow 0$ in probability. For this it suffices to show that $a_n^{1/2} E|A_{1n}| \longrightarrow 0$ as $n \to \infty$. We have

$$E|A_{1n}| \le (EA_{1n}^2)^{1/2} =$$

$$= 2\,\frac{n}{a_n}\left\{\sum_{i=1}^{p} k_i^2 E\Big(\int \big(\widehat{f}_i(x) - E\widehat{f}_i(x)\big) A_i(x) r(x)\,dx\Big)^2\right\}^{1/2},$$

where

$$A_i(x) = E\widehat{f}_i(x) - \frac{1}{\overline{k}}\sum_{j=1}^{p} k_j E\widehat{f}_j(x).$$

Further, it is easy to calculate that

$$E\Big[\int \big(\widehat{f}_i(x)-E\widehat{f}_i(x)\big)A_i(x)r(x)\,dx\Big]^2 \le \frac{a_n^2}{k_i n}\,E\Big[\int K\big(a_n(x-X_1^{(i)})\big)A_i(x)r(x)\,dx\Big]^2.$$

Therefore

$$E|A_{1n}| \le c_1\sqrt{n}\,\bigg\{\sum_{i=1}^p k_i\int f_i(u)\,du\,\Big[\int K(a_n(x-u))A_i(x)r(x)\,dx\Big]^2\bigg\}^{1/2}. \quad (7)$$

Since $\sup\limits_x |A_i(x)| \le c_2\alpha_n$ for all $i=1,\dots,p$ and $r(x)$ is bounded, from (7) we obtain

$$a_n^{1/2}E|A_{1n}| \le c_3\,\frac{\sqrt{n}\,\alpha_n}{a_n^{1/2}} = O\Big(\frac{1}{a_n^{1/4}}\Big).$$

Hence

$$a_n^{1/2}A_{1n} = o_p(1). \quad (8)$$

Let us now proceed to the calculation of the limiting distribution of the functional $T_n^{(1)}$:

$$T_n^{(1)} = \frac{n}{a_n}\sum_{i=1}^p k_i\int\Big[\widehat{f}_i(x)-E\widehat{f}_i(x)-\frac{1}{\overline{k}}\sum_{j=1}^p k_j\big(\widehat{f}_j(x)-E\widehat{f}_j(x)\big)\Big]^2 r(x)\,dx, \quad (9)$$

where $\overline{k}=k_1+\dots+k_p$.

After performing some simple transformation in (9), we obtain

$$T_n^{(1)} = \int\bigg[\sum_{i=1}^p\bigg(\sqrt{\frac{n_i}{a_n}}\,\big(\widehat{f}_i(x)-E\widehat{f}_i(x)\big)\bigg)^2-\bigg(\sum_{j=1}^p\alpha_j\sqrt{\frac{n_j}{a_n}}\,\big(\widehat{f}_j(x)-E\widehat{f}_j(x)\big)\bigg)^2\bigg]r(x)\,dx,$$

where $\alpha_i^2=\frac{k_i}{k_1+\dots+k_p}$.

Let

$$\mathbf{Z}(x) = \big(Z_1(x),\dots,Z_p(x)\big)$$

be the vector with the component

$$Z_i(x) = \sqrt{\frac{n_i}{a_n}}\,\big(\widehat{f}_i(x)-E\widehat{f}_i(x)\big),\ \ i=1,\dots,p.$$

Then

$$T_n^{(1)} = \int\Big[|Z(x)|^2-\Big(\sum_{j=1}^p\alpha_j Z_j(x)\Big)^2\Big]r(x)\,dx,$$

where $|a|$ is the length of the vector $a = (a_1, \ldots, a_p)$.

There exists an orthogonal matrix $\mathbf{C} = \|c_{ij}\|$, $i, j = 1, \ldots, p$, depending only on $k_1, k_2, \ldots, k_p$, for which

$$c_{pi} = \alpha_i = \sqrt{\frac{k_i}{k_1 + \cdots + k_p}}\,, \quad i = 1, \ldots, p.$$

Since under orthogonal transformation the vector length does not change, we have

$$\begin{aligned} T_n^{(1)} &= \int \Big[|\mathbf{CZ}|^2 - \Big(\sum_{j=1}^p \alpha_j Z_j(x) \Big)^2 \Big] r(x)\,dx \\ &= \sum_{i=1}^{p-1} \int \Big(\sum_{j=1}^p c_{ij} Z_j(x) \Big)^2 r(x)\,dx. \end{aligned} \tag{10}$$

Let $F_i(x)$ be a distribution function of the random variable $X_1^{(i)}$ and let $\widehat{F}_{n_i}(x)$ be an empirical distribution function of the sample $X^{(i)} = (X_1^{(i)}, \ldots, X_{n_i}^{(i)})$.

Further, by Theorem 3 in [5] we can write that

$$\begin{gathered} \widehat{F}_{n_i}(x) - F_i(x) = n_i^{-1/2} W_i^0(F_i) + \varepsilon_n^{(1)}(x), \\ \sup_{-\infty<x<\infty} |\varepsilon_n^{(1)}(x)| = O\Big(\frac{\ln n}{n}\Big); \end{gathered} \tag{11}$$

$W_i^0(t)$, $i = 1, \ldots, p$, are the independent Brownian bridges depending only on $X^{(i)}$.

Using (11), it is easy to establish ([4], [8]) that

$$\begin{gathered} Z_i(x) = \sqrt{\frac{n_i}{a_n}}\,\big(\widehat{f}_i(x) - E\widehat{f}_i(x)\big) = \xi_i(x) + \varepsilon_n^{(2)}(x), \\ \sup_{-\infty<x<\infty} |\varepsilon_n^{(2)}(x)| = O_p\Big(\frac{\ln n}{\sqrt{n a_n^{-1}}}\Big), \end{gathered} \tag{12}$$

where

$$\xi_i(x) = a_n^{1/2} \int K(a_n(x-u))\,dW_i^0(F_i(u)), \quad i = 1, \ldots, p,$$

then by virtue of (12) we can write

$$\begin{gathered} \sum_{j=1}^p c_{ij} Z_j(x) = \sum_{j=1}^p c_{ij} \xi_j(x) + \varepsilon_n^{(3)}(x), \\ \sup_{-\infty<x<\infty} |\varepsilon_n^{(3)}(x)| = O_p\Big(\frac{\ln n}{\sqrt{n a_n^{-1}}}\Big). \end{gathered} \tag{13}$$

Further, $\xi_j(x)$ can be represented as

$$\xi_j(x) = a_n^{1/2} \int \Big[K(a_n(x-t)) - \int K(a_n(x-u))\, dF_j(u) \Big]\, dW_j(F_j),$$

where $W_j(t)$, $j = 1, \dots, p$ are independent standard Wiener processes on $[0,1]$.

From this representation it follows that

$$\sup_{-\infty<x<\infty} E\, \xi_j^2(x) < \infty, \quad j = 1, 2, \dots, p.$$

From this we have

$$A_n = \sum_{i=1}^{p-1} \int \sum_{j=1}^{p} c_{ij}\, \xi_j(x) r(x)\, dx$$

is uniformly bounded in probability, i.e. $P\{|A_n| \ge M\} \longrightarrow 0$ as $M \to \infty$ uniformly with respect to n.

Therefore

$$A_n \cdot O_p\Big(\frac{a_n \ln n}{\sqrt{n}}\Big) = o_p(1), \tag{14}$$

since, by assumption, $\frac{a_n^{9/2}}{n} \to 0$.

Thus, from representations (10) and (13), and also from relation (14), we find

$$\sqrt{a_n}\, (T_n^{(1)} - T_n^{(2)}) = o_p(1) + O_p\Big(\frac{a_n^{3/2} \ln^2 n}{n}\Big), \tag{15}$$

where

$$T_n^{(2)} = \sum_{i=1}^{p-1} \int \Big(\sum_{j=1}^{p} c_{ij} \xi_j(t) \Big)^2 r(t)\, dt.$$

Denote

$$\eta_i(t) = a_n^{1/2} \int K(a_n(t-u))\, dW_i(F_i(u)),$$

$$T_n^{(3)} = \sum_{i=1}^{p-1} \int \Big(\sum_{j=1}^{p} c_{ij} \eta_j(t) \Big)^2 r(t)\, dt,$$

$$\varepsilon_i(t) = a_n^{1/2} W_i(1) \int K(a_n(t-u)) f_i(u)\, du.$$

Then

$$a_n^{1/2}\big(T_n^{(2)} - T_n^{(3)}\big) = o_p(1). \tag{16}$$

Indeed,

$$E\big|T_n^{(2)}-T_n^{(3)}\big|\le 2\sum_{i=1}^{p-1}E\bigg|\int\sum_{j=1}^{p}c_{ij}\eta_j(t)\sum_{r=1}^{p}c_{ir}\varepsilon_r(t)\bigg|r(t)\,dt$$
$$+\sum_{i=1}^{p-1}E\int\Big(\sum_{j=1}^{p}c_{ij}\varepsilon_j(t)\Big)^2 r(t)\,dt=B_n^{(1)}+B_n^{(2)}. \quad (17)$$

It is easy to check that

$$B_n^{(2)}\le c_4a_n^{-1}.$$

Let us now estimate $B_n^{(1)}$. We have

$$B_n^{(1)}\le 2\sum_{i=1}^{p-1}\bigg[\sum_{j,r}^{p}|c_{ij}c_{ir}|E|W_r(1)|\bigg|\int\Big[\int\Psi_r(t)K(a_n(t-u))r(t)\,dt\Big]\,dW_j(F_j)\bigg|\bigg]$$
$$\le 2\sum_{i=1}^{p-1}\sum_{j=1}^{p}\sum_{r=1}^{p}|c_{ij}c_{ir}|\bigg\{\int\Big(\int\Psi_r(t)K(a_n(t-u))r(t)\,dt\Big)^2\,dF_j(u)\bigg\}^{1/2}\le c_5a_n^{-1},$$

where

$$\Psi_r(t)=\int K(z)f_r(t-za_n^{-1})\,dz.$$

So, substituting the estimators of the expressions $B_n^{(1)}$ and $B_n^{(2)}$ into (17), we obtain statement (16).

Denote

$$\eta_i^0(t)=a_n^{1/2}\int K(a_n(t-x))\,dW_i(F_0),$$

where $F_0(x)$ is a distribution function with density $f_0(x)$. Since $F_i(x)=F_0(x)+\alpha_nU_i(x)$, $U_i^{(1)}(x)=\varphi_i(x)$ and, by assumption, $\varphi_i(x)$ is bounded and $K^{(1)}(x)\in L_1(-\infty,\infty)$, we have

$$\begin{gathered}E\big(\eta_j(t)-\eta_j^0(t)\big)^2=O(a_n\alpha_n),\\ E(\eta_j^0(t))^2=O(1),\end{gathered} \quad (18)$$

uniformly with respect to $t\in(-\infty,\infty)$ and j, $j=1,\dots,p$. Indeed, we have

$$E\big(\eta_j(t)-\eta_j^0(t)\big)^2$$
$$=a_nE\bigg(\int\Big(W_j(F_j(a_n(t-x)))-W_j(F_0(a_n(t-x)))\Big)K^{(1)}(x)\,dx\bigg)^2$$

$$\le a_n \int \Big| F_j(a_n(t-x)) - F_0(a_n(t-x)) \Big| \, |K^{(1)}(z)| \, dx \cdot \int |K^{(1)}(x)| \, dx \le c_6 a_n \alpha_n,$$

and also

$$E(\eta_n^0(t))^2 = a_n \int K^2(a_n(t-x)) f_0(x) \, dx \le \max_x f_0(x) \cdot \int K^2(u) \, du.$$

Further, using (18) and the Cauchy–Schwarz inequality we establish that

$$\sqrt{a_n} \, E|T_n^{(3)} - T_n^{(4)}| = O(a_n \sqrt{\alpha_n}\,) + O(a_n^{3/2} \alpha_n), \tag{19}$$

where

$$T_n^{(4)} = \sum_{i=1}^{p-1} \int \Big(\sum_{j=1}^{p} c_{ij} \eta_j^0(t) \Big)^2 r(t) \, dt.$$

Let us now study the limiting distribution of the functional $T_n^{(4)}$.

The processes $\eta_j^0(t)$, $j = 1, \dots, p$, are independent and Gaussian and therefore the new processes $\sum\limits_{j=1}^{p} c_{ij} \eta_j^0(t)$, $i = 1, \dots, p$, are also independent and Gaussian by virtue of the orthogonality of the matrix $\|c_{ij}\|$. Therefore to find the limiting distribution $T_n^{(4)}$ it remains to establish the limiting distribution of the functional

$$U_n^{(i)} = \int \Big(\sum_{j=1}^{p} c_{ij} \eta_j^0(t) \Big)^2 r(t) \, dt$$

for every fixed i, $i = 1, \dots, p-1$.

The covariant function $R_n^{(i)}(t_1, t_2)$ of the Gaussian process $\sum\limits_{j=1}^{p} c_{ij} \eta_j^0(t)$ is equal to

$$R_n^{(i)}(t_1, t_2) = \sum_{j=1}^{p} c_{ij}^2 E \eta_j^0(t_1) \eta_j^0(t_2).$$

However,

$$\begin{aligned} E\eta_j^0(t_1)\eta_j^0(t_2) &= \int K(u) K\big(a_n(t_1 - t_2) + u\big) f_0(t_1 - a_n^{-1} u) \, du \\ &= f_0(t_1) K_0\big(a_n(t_1 - t_2)\big) + O(a_n^{-1}), \end{aligned} \tag{20}$$

where the estimator $O(\,\cdot\,)$ is uniform with respect to t_1, t_2 and $K_0 = K * K$.

From (20) it follows that

$$R_n^{(i)}(t_1, t_2) = f_0(t_1) K_0\big(a_n(t_1 - t_2)\big) + O(a_n^{-1}). \tag{21}$$

A semi-invariant $\chi_n^{(i)}(s)$ of order s of the random variable $U_n^{(i)}$ is defined by the formula [3]

$$\chi_n^{(i)}(s) = (s-1)! \cdot 2^{s-1} \int \cdots \int R_n^{(i)}(x_1, x_2) R_n^{(i)}(x_2, x_3) \cdots R_n^{(i)}(x_s, x_1) \times r(x_1) r(x_2) \cdots r(x_s)\, dx_1\, dx_2 \cdots dx_s. \tag{22}$$

Using (21) and (22) it is not difficult to establish that

$$\begin{aligned} EU_n^{(i)} &= \chi_n^{(i)}(1) = R(K) \int f_0(x) r(x)\, dx + O(a_n^{-1}), \\ \operatorname{Var} U_n^{(i)} &= \chi_n^{(i)}(2) = 2R(K_0) a_n^{-1} \int f_0^2(x) r^2(x)\, dx + o(a_n^{-1}), \end{aligned} \tag{23}$$

and the s-th semi-invariant $\chi_n^{(i)}(s)$ is equal, with an accuracy of terms of higher order smallness, to [3]:

$$(s-1)!\, 2^{s-1} (a_n^{-1})^{s-1} [K * K]^{(s)}(0) \int f_0^s(x) r^s(x)\, dx, \tag{24}$$

where $[K * K]^{(s)}(0)$ means an s-fold convolution of $K_0(x)$ with itself.

From relations (23) and (24) it follows [3] (see also [8]) that

$$a_n^{1/2} \Big(U_n^{(i)} - R(K) \int f_0(x) r(x)\, dx \Big)$$

has a normal limiting distribution with mathematical expectation 0 and dispersion

$$2R(K_0) \int f_0^2(u) r^2(u)\, du, \quad R(g) = \int g^2(x)\, dx,$$

and therefore $\sqrt{a_n}\,(T_n^{(4)} - \mu)$ has a normal limiting distribution $(0, \sigma^2)$.

Finally, taking into account (6), (8), (15), (16), (19) and the representation

$$a_n^{1/2}(T_n - \mu) = a_n^{1/2}(T_n^{(4)} - \mu) + A(\varphi) + O(a_n^{-1/2}) + o_p(1) + O_p\Big(\frac{a_n^{3/2} \ln^2 n}{n}\Big) + O_p(a_n \sqrt{\alpha_n}\,) + O(a_n^{3/2} \alpha_n), \tag{25}$$

we conclude that $a_n^{1/2}(T_n - \mu)$ has a normal limiting distribution $(A(\varphi), \sigma^2)$. □

Proof of Theorem 3.4: It is obvious that

$$a_n^{1/2}(T_n - \mu_n)\sigma_n^{-1} = a_n^{1/2}(T_n - \mu)\sigma^{-1}(\sigma\sigma_n^{-1}) + a_n^{1/2}(\mu - \mu_n)\sigma_n^{-1}.$$

Therefore it suffices to show that

$$a_n^{1/2}\left(\overline{\mu}_n - \int f_0(x) r(x)\, dx\right) = o_p(1) \tag{26}$$

and

$$\Delta_n^2 - \int f_0^2(x) r^2(x)\, dx = o_p(1). \tag{27}$$

But (27) immediately follows from Theorem 2.1 of Bhattacharyya G. K., Roussas G. G. [2] (see also [8], [6]).

Let us prove (26). We have

$$\begin{aligned} &a_n^{1/2} E\left| \int f_n^*(x) r(x)\, dx - \int f_0(x) r(x)\, dx\right| \\ &\le a_n^{1/2} E\left|\int \big(f_n^*(x) - Ef_n^*(x)\big) r(x)\, dx\right| + a_n^{1/2} \int \big|Ef_n^*(x) - f_0(x)\big| r(x)\, dx \\ &= A_{1n} + A_{2n}. \end{aligned}$$

It is not difficult to check that

$$A_{2n} \le c_7\big(a_n^{-1/2} + \sqrt{a_n}\, \alpha_n\big).$$

Further, we have

$$\begin{aligned} A_{1n} &\le a_n^{1/2} E^{1/2}\left(\int \big(f_n^*(x) - Ef_n^*(x)\big) r(x)\, dx\right)^2 \\ &\le c_8 a_n^{1/2} \max_{1\le j\le p} \left\{\frac{1}{n}\int f_j(u)\, du \left(\int K(t) r\Big(u - \frac{t}{a_n}\Big)\, dt\right)^2\right\}^{1/2} \le c_9 \Big(\frac{a_n}{n}\Big)^{1/2}. \end{aligned}$$

Therefore

$$A_{1n} + A_{2n} \le c_{10}\left(a_n^{-1/2} + \sqrt{a_n}\, \alpha_n + \Big(\frac{a_n}{n}\Big)^{1/2}\right) \longrightarrow 0. \qquad \square$$

Proof of Theorem 3.7: For proving it is enough to state that $\sqrt{a_n}\,(T_n - \widehat{T}_n) \to 0$ in probability. We have

$$\begin{aligned} &\sqrt{a_n}\, |T_n - \widehat{T}_n| \le L_n^{(1)} \cdot L_n^{(2)}, \\ &L_n^{(1)} = \sqrt{a_n} \sup_{0\le x\le 1} |f_n^*(x) - f_0(x)| \Big(\inf_{0\le x\le 1} \big(f_0(x) f_n^*(x)\big)\Big)^{-1}, \\ &L_n^{(2)} = \frac{n}{a_n} \sum_{i=1}^p k_i \int_0^1 \Big[\widehat{f}_i(x) - \frac{1}{\overline{k}} \sum_{j=1}^p k_j \widehat{f}_j(x)\Big]^2\, dx. \end{aligned}$$

Considering, that $E\widehat{f}_j(x) - f_0(x) = O(a_n^{-1}) + O(\alpha_n)$ uniformly in x, we obtain

$$\sqrt{a_n}\,\sup_x |f_n^*(x) - f_0(x)|$$
$$\leq \sum_{i=1}^{p} k_i \sqrt{a_n}\sup_x |\widehat{f}_i(a) - E\widehat{f}_i(x)| + O(a_n^{-1/2}) + O(\sqrt{a_n}\,\alpha_n).$$

Further, from inequality (2) in [9] (see also [8, p. 43]) it follows that

$$\sqrt{a_n}\sup_x |\widehat{f}_j(x) - E\widehat{f}_j(x)| \leq V_0 a_n^{3/2}\sup_x |\widehat{F}_j(x) - F_j(x)| = O\Big(\frac{a_n^3 \ln n}{n}\Big)^{1/2}$$

with probability 1, where $V_0 = \bigvee_{-\infty}^{\infty}(K)$. From this and by condition $\frac{a_n^{9/2}}{n}\ln n \to 0$ it follows that

$$\sqrt{a_n}\,\sup_x |f_n^*(x) - f_0(x)| \longrightarrow 0 \tag{28}$$

with probability 1.

Further, since

$$\inf_{0\leq x\leq 1} f_0(x) f_n^*(x) \geq \Delta_0 \inf_{0\leq x\leq 1} f_n^*(x), \ \ \Delta_0 = \inf_{0\leq x\leq 1} f_0(x) > 0$$

and

$$\inf_{0\leq x\leq 1} f_n^*(x) \geq \Delta_0 - \sup_{0\leq x\leq 1} |f_n^*(x) - f_0(x)|,$$

then from the last and from (28) it follows that $L_n^{(1)} \to 0$ with probability 1. Further, we have

$$EL_n^{(2)} = \frac{n}{a_n}\sum_{i=1}^{p} k_i \int_0^1 \operatorname{Var}\widehat{f}_i(x)\,dx + \frac{n}{a_n}\sum_{i=1}^{p} k_i \int_0^1 \Big[E\widehat{f}_i(x) - \frac{1}{\bar{k}}\sum_{j=1}^{p} k_j E\widehat{f}_j(x)\Big]^2 dx. \tag{29}$$

It is easity to see that

$$\int_0^1 \operatorname{Var}\widehat{f}_i(x)\,dx = O\Big(\frac{a_n}{n}\Big),$$
$$E\widehat{f}_n(x) = a_n \int K(a_n(x-u)) f_0(u)\,du + O(\alpha_n).$$

Because from (29) we establish that

$$EL_n^{(2)} \leq c_{12} + c_{13}\,\frac{n}{a_n}\,\alpha_n^2 = c_{12} + c_{13} a_n^{-1/2}$$

(by condition $\frac{n\alpha_n^2}{\sqrt{a_n}} = 1$). This means that $L_n^{(2)}$ is bounded by probability. Therefore, $\sqrt{a_n}\,(T_n - \widehat{T}_n) \to 0$ in probability. □

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