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On T_2-Negligible S_2-Absolutely Nonmeasurable Sets in the Euclidean Plane

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It is proved that there exists a T_2-negligible set in the plane $\mathbf{R}^2$, which simultaneously is S_2-absolutely nonmeasurable. This result answers one of the questions posed in [5].

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Let $(G, +)$ be a commutative group identified with the group of all its translations and let H be an uncountable subgroup of G. The following classes of subsets of G were studied in connection with some general questions of the theory of nonzero σ-finite H-invariant (H-quasi-invariant) measures on G:

(i) the class of all almost H-invariant subsets of G;

(ii) the class of all H-absolutely negligible sets in G;

(iii) the class of all H-negligible sets in G;

(iv) the class of all H-absolutely nonmeasurable sets in G.

For the precise definitions of these sets and more information about them and their properties, see [4], [5].

Notice that almost H-invariant subsets of G and H-absolutely negligible subsets of G turned out to be useful for constructing proper extensions of nonzero σ-finite H-invariant (H-quasi-invariant) measures on G (see [3], [4], [5], [8], and the references therein).

The following facts should also be mentioned:

(a) there exists a countable family $\{X_i : i \in I\}$ of G-absolutely negligible sets in G such that $G = \cup\{X_i : i \in I\}$;

(b) there exist three G-negligible sets X, Y, Z in G such that $X \cup Y \cup Z = G$;

(c) there exist two G-negligible sets A and B in G such that $A \cup B$ is a G-absolutely nonmeasurable set in G.

The proofs of these facts can be found in [3], [4], [5]. The most interesting cases occur when the role of H is played by one of the following two groups:

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T_G = the group of all translations of G (which may be identified with the original group G);

S_G = the group, generated by all central symmetries of G.

Recall that a central symmetry of G is any transformation s_h of G having the form

$$s_h(g) = 2h - g \qquad (g \in G),$$

where h is a fixed element of G (the center of symmetry). If $h = 0$, then we have the central symmetry s_0 of G whose center coincides with the neutral element 0 in G.

It is clear that the group S_G is generated by s_0 and all translations of G. Besides, the equality

$$s_0 \circ g = (-g) \circ s_0$$

holds true for each element g of G.

Remark 1: Obviously, if for $h \in G \setminus \{0\}$, the relation $2h = 0$ is valid, then $s_h = s_0$. It may happen that in an uncountable commutative group $(G, +)$ the equality $2g = 0$ is fulfilled for all elements g of G. In this case s_0 coincides with the identity transformation of G, so $T_G = S_G$. The standard example of such a $(G, +)$ is Cantor's discontinuum, i.e., the topological group $(\{0, 1\}^\omega, +_2)$, where ω denotes the least infinite cardinal (ordinal) number and $+_2$ denotes the addition operation modulo 2.

We have the following statement (see [3]).

Theorem 1: *For every uncountable commutative group* $(G, +)$ *and for any subset* X *of* G*, these two assertions are equivalent:*

(1) X *is* T_G*-absolutely negligible in* G*;*

(2) X *is* S_G*-absolutely negligible in* G*.*

Proof: In fact, Theorem 1 is proved in [3]. Notice only that the argument presented in [3] is concerned with the case of the additive group $(\mathbf{R}, +)$, where $\mathbf{R}$ denotes the set of all real numbers. However, the same argument works in the general case of all uncountable commutative groups. □

The equivalence of the assertions (1) and (2) of Theorem 1 fails to be true if we replace T_G-absolutely negligible subsets of G and S_G-absolutely negligible subsets of G, respectively, by T_G-negligible subsets of G and S_G-negligible subsets of G.

Consider the special case $(G, +) = (\mathbf{R}^2, +)$, where $\mathbf{R}^2$ is the Euclidean plane. Let T_2 be the group of all translations of $\mathbf{R}^2$ and let S_2 be the group generated by all central symmetries of $\mathbf{R}^2$. The following question was formulated in Chapter 14 of [5]:

Does there exist a T_2-negligible subset of $\mathbf{R}^2$ which, simultaneously, is S_2-absolutely nonmeasurable in $\mathbf{R}^2$?

The main goal of the present paper is to obtain the positive answer to this question not only for $(G, +) = (\mathbf{R}^2, +)$, but also for a more general class of uncountable commutative groups.

Below, the symbol $\triangle$ stands for the operation of the symmetric difference of two sets.

As usual, $\mathbf{Q}$ denotes the set (field) of all rational numbers and $(\mathbf{Z}, +)$ denotes the additive group of all integer numbers.

We need several auxiliary statements.

Lemma 1: *If E is an uncountable vector space over $\mathbf{Q}$, then there exists a subset K of E satisfying these two conditions:*

(1) K is almost translation invariant, i.e., for each $e \in E$, the set $K \triangle (K + e)$ has cardinality strictly less than $\mathrm{card}(E)$*;*

(2) $s_0(K) \cap K = \emptyset$ and $s_0(K) \cup K = E \setminus \{0\}$.

Proof: Let α denote the least ordinal number whose cardinality equals $\mathrm{card}(E)$ and let $\{e_\xi : \xi < \alpha\}$ be a Hamel basis of E. For every nonzero vector $e \in E$, we have a unique representation of e in the form

$$e = q_1 e_{\xi_1} + q_2 e_{\xi_2} + ... + q_m e_{\xi_m},$$

where m is a nonzero natural number, $q_1, q_2, ..., q_m$ are some nonzero rational numbers, and ξ_1, ξ_2, . . . ξ_m are some ordinal numbers such that

$$\xi_1 < \xi_2 < ... < \xi_m < \alpha.$$

Denote $q(e) = q_m$, and put $K = \{e \in E : q(e) > 0\}$.

A straightforward verification shows that K satisfies both conditions (1) and (2) of Lemma 1, which ends the proof. □

Remark 2: Sets analogous to the set K of Lemma 1 were considered by several authors (cf., for instance, [1], [7]). Note that, in the case $E = \mathbf{R}^n$, where $n \geq 1$, a Hamel basis of $\mathbf{R}^n$ can be chosen to be a Bernstein subset of $\mathbf{R}^n$. So both sets K and $-K$ also turn out to be Bernstein subsets of $\mathbf{R}^n$.

Lemma 2: *If E is an uncountable vector space over $\mathbf{Q}$, then there exists a set Z in G such that:*

(1) $\cup\{e_i + Z : i \in I\} = E$ for some countable family $\{e_i : i \in I\}$ of vectors from E;

(2) there exists an uncountable family $\{f_j : j \in J\}$ of vectors from E such that, for any two distinct indices $j \in J$ and $k \in J$, in E there are uncountably many pairwise disjoint translates of the set $(f_j + Z) \cap (f_k + Z)$.

The proof of Lemma 2 is given in [5]. It follows from (1) and (2) that the above-mentioned set Z is T_E-absolutely nonmeasurable in E.

Lemma 3: *Let E be an uncountable vector space over $\mathbf{Q}$ whose cardinality is not cofinal with ω, let K be as in Lemma 1, and let Z be as in Lemma 2. Consider the set*

$$X = K \cap Z.$$

Then the following two assertions are valid:

(1) $\mathrm{card}(K \triangle (\cup\{e_i + X : i \in I\})) < \mathrm{card}(E)$ *for some countable family $\{e_i : i \in I\}$ of vectors from E;*

(2) there exists an uncountable family $\{f_j : j \in J\}$ of vectors from E such that, for any two distinct indices $j \in J$ and $k \in J$, in E there are uncountably many pairwise disjoint translates of the set $(f_j + X) \cap (f_k + X)$.

Proof: Taking into account the almost T_E-invariance of K, assertion (1) of Lemma 3 easily follows from assertion (1) of Lemma 2.

Assertion (2) of Lemma 3 is implied by assertion (2) of Lemma 2, because the set X is contained in the set Z. □

Theorem 2: *The set Z of Lemma 3 is T_E-negligible and, simultaneously, S_E-absolutely nonmeasurable.*

Proof: First of all, observe that the properties of K described in Lemma 1 imply that the T_E-invariant σ-ideal of sets generated by $\{K\}$ is proper, i.e., differs from the family of all subsets of E (for example, the set $-K$ does not belong to this σ-ideal).

Since $X \subset K$, the same is true for the T_E-invariant σ-ideal of sets generated by $\{X\}$. The latter circumstance implies that there exists a probability T_E-invariant measure ν on E such that $X \in \mathrm{dom}(\nu)$ and $\nu(X) = 0$.

If now μ is an arbitrary σ-finite T_E-quasi-invariant measure on E, then from assertion (2) of Lemma 3 we have

$$X \in \mathrm{dom}(\mu) \Rightarrow \mu(X) = 0.$$

Consequently, X turns out to be a T_E-negligible subset of E.

Let θ be an arbitrary nonzero σ-finite S_E-quasi-invariant measure on E and suppose for a moment that $X \in \mathrm{dom}(\theta)$. Then condition (2) of Lemma 1 and assertion (1) of Lemma 3 imply that

$$E = \cup\{h_i(X) : i \in I\}$$

for a certain countable family $\{h_i : i \in I\}$ of transformations of E, all of which belong to S_E. So, we must have $\theta(X) > 0$. On the other hand, assertion (2) of Lemma 3 implies that $\theta(X)$ must be equal to zero. The obtained contradiction shows that X cannot belong to $\mathrm{dom}(\theta)$ and, consequently, X is S_E-absolutely nonmeasurable in E.

Theorem 2 has thus been proved. □

Remark 3: Let $n \geq 1$ be a natural number. Consider the n-dimensional Euclidean space $\mathbf{R}^n$ as a vector space E over the field $\mathbf{Q}$. Since the cardinality of the continuum is not cofinal with ω, we may apply Theorem 2 to this E. Consequently, there exists a T_n-negligible set in $\mathbf{R}^n$ which simultaneously is S_n-absolutely nonmeasurable (here T_n denotes the group of all translations of $\mathbf{R}^n$ and S_n denotes the group generated by all central symmetries of $\mathbf{R}^n$).

In [5] the following statement was proved:

Let g be a rotation of the plane $\mathbf{R}^2$ (about its origin), which differs from the identity transformation of $\mathbf{R}^2$ and differs from the central symmetry s_0 of $\mathbf{R}^2$. Then there exists a T_2-negligible subset of $\mathbf{R}^2$ which is Γ-absolutely nonmeasurable, where Γ stands for the group generated by $T_2 \cup \{g\}$.

Theorem 2 enables us to strengthen the above statement in the following form.

Theorem 3: *Let g be a rotation of the plane $\mathbf{R}^2$ (about its origin), distinct from the identity transformation of $\mathbf{R}^2$.*

Then there exists a T_2-negligible subset of $\mathbf{R}^2$ which is Γ-absolutely nonmeasurable, where Γ stands for the group generated by $T_2 \cup \{g\}$.

Lemma 4: *Let $(G, +)$ and $(H, +)$ be two commutative groups and let ϕ be a surjective homomorphism from G onto H.*

If Y is an H-negligible subset of H, then $\phi^{-1}(Y)$ is a G-negligible subset of G.

For a proof of Lemma 4, see [4] or [5].

Using Theorem 3 and Lemma 4, it is not difficult to obtain the next statement (cf. Chapter 14 of [5]).

Theorem 4: *Let g be a rotation of the space $\mathbf{R}^3$ (about its origin) distinct from the identity transformation of $\mathbf{R}^3$.*

Then there exists a T_3-negligible subset of $\mathbf{R}^3$ which is Γ-absolutely nonmeasurable, where Γ stands for the group, generated by $T_3 \cup \{g\}$.

Remark 4: The statement analogous to Theorem 2 can be proved for any commutative group $(G, +)$, satisfying the following conditions:

(*) the cardinality of G is not cofinal with ω;

(**) $(G, +)$ is a direct sum of a family of groups, all of which are isomorphic to $(\mathbf{Z}, +)$.

It would be interesting to characterize all those uncountable commutative groups $(G, +)$ in which there exists at least one T_G-negligible S_G-absolutely nonmeasurable set (cf. Remark 1). In this context, it makes sense to recall a profound result of Kulikov (see [2], [6]). According to Kulikov's theorem, any commutative group $(G, +)$ admits a representation in the form

$$G = \cup\{G_m : m < \omega\},$$

where $\{G_m : m < \omega\}$ is an increasing (by inclusion) countable family of subgroups of G, all of which are direct sums of cyclic groups.

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