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Computer Modelling of a Probabilistic Solution for the Dirichlet Generalized Harmonic Problem in Some Finite Axisymmetric Bodies with a Cylindrical Hole

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In this paper, we investigate the application of the probabilistic method (PM) for the numerical solution of the Dirichlet generalized harmonic problem in axisymmetric finite homogeneous and isotropic bodies with a right circular cylindrical hole. The term "generalized" indicates that a boundary function has a finite number of first kind discontinuity curves. The suggested algorithm for the numerical solution of boundary problems consists of the following main stages: a) application of the PM, which in turn is based on the computer modeling of the Wiener process; b) finding the intersection point of the trajectory of the simulated Wiener process and the surface of the problem domain; c) development of a code for the numerical implementation and checking the reliability of obtained results; d) finding the probabilistic solution of generalized problems at any fixed points in the considered domains. The algorithm does not require the approximation of a boundary function. The PM is tested on an explicit analytical solution from the literature. To illustrate the effectiveness and simplicity of the proposed method several examples are considered. Numerical results are presented and discussed.

Keywords: Dirichlet generalized harmonic problem, Probabilistic method, Wiener process, Axisymmetric body, Computer modeling.

AMS Subject Classification: 35J05, 35J25, 65C10, 65N75

1. Introduction

The method of probabilistic solution (MPS), or in short, the probabilistic method (PM) is a relatively new technique for the numerical solution of Dirichlet ordinary and generalized harmonic problems. The investigation and application of the PM has been carried out for almost twenty years at the Muskhelishvili Institute of Computational Mathematics. The reason for this mentioned analysis is one theorem of Duenkin and Yushkevich (see Section 3).

In the present paper, the PM for the numerical solution of the Dirichlet harmonic problem with singularities in the boundary data is considered, and some axisymmetric finite domains with circular cylindrical holes are considered.

It is known (see e.g., [1–5]) that in practical stationary problems (for example, for determination of the temperature of the thermal field or the potential of the

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electric field, and so on) there are cases when it is necessary to consider the Dirichlet generalized harmonic problem.

In general, it is known (see e.g., [1, 6, 7]) that the methods used to obtain an approximate solution to ordinary boundary value problems are less suitable (or not suitable at all) for solving generalized boundary value problems. In particular, the convergence of the approximate process is very slow in the neighborhood of boundary singularities and, consequently, the accuracy of the approximate solution of the generalized problem is very low. For example, a similar phenomenon takes place when solving the Dirichlet generalized harmonic problem by the method of fundamental solutions. Therefore researchers have tried to conduct preliminary improvements of the boundary value problem in question.

For plane Dirichlet generalized harmonic problems the following approaches may be used: I) A method of reduction of the Dirichlet generalized problem to an ordinary problem (see e.g., [8–11]); II) A method of conformal mapping (see e.g., [12]); III) The probabilistic method (see e.g., [13, 14]).

In the case of 3D Dirichlet generalized harmonic problems, the difficulties become more significant. In particular, from the above approaches we can apply only III). The choice and construction of computational schemes (algorithms) mainly depend on the problem class, its dimension, geometry and location of singularities on the boundary. In particular, there does not exist a standard scheme which can be applied to a wide class of domains. In the literature (see e.g., [1–5]), simplified, or so called "solvable" generalized problems are considered and some methods, namely, separation of variables, particular solutions and heuristic methods are mainly applied for their solution and therefore the accuracy of the solution is rather low. Since heuristic methods do not guarantee finding the best solution(moreover, in some cases they may give an incorrect solution), it is necessary to check these solutions in order to establish how well they satisfy all conditions of the problem (see e.g., [1]).

Therefore the construction of high accuracy and effectively realizable computational schemes for the approximate solution of 3D Dirichlet generalized harmonic problems (whose application is possible for a wide class of domains) have both theoretical and practical importance.

It should be noted that in the literature (see e.g., [4], pp. 346-348), while solving 3D Dirichlet generalized harmonic problems, the existence of discontinuity curves is often ignored. This fact and the application of classical methods to solving Dirichlet generalized harmonic problems is the reason of this low accuracy. Therefore for the numerical solution of 3D Dirichlet generalized harmonic problems we should apply methods which do not require the approximation of a boundary function and in which the existence of discontinuity curves is not ignored. The probabilistic method (PM) is one such method.

A brief outline of this paper is as follows. The mathematical formulation of the 3D Dirichlet generalized harmonic problem is given in Section 2. In Section 3, the PM and the simulation of the Wiener process are briefly described. In Section 4, one analytical generalized solution for an axisymmetric circular cylindrical ring (from the literature) is considered and its properties are given. In Section 5, several examples are considered and the results of numerical experiments are presented. vestigations are provided. Finally, in Section 6, some conclusions and ideas for future investigations are provided.

2. Mathematical formulation of the problem

Let D be an axisymmetric finite domain with a right circular cylindrical hole. Without loss of generality we assume that the axis of D is the axis Ox_3 of the Cartesian coordinate right-handed system $Ox_1x_2x_3$ and the lower base of D lies in the plane Ox_1x_2. We consider the following Dirichlet generalized harmonic problem.

Problem A. The function $g(y)$ is given on the boundary S of the domain D and is continuous everywhere, except a finite number of curves $l_1, l_2, \dots, l_n$, which represent discontinuity curves of the first kind for the function $g(y)$. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup_{k=1}^{n} l_k)$ satisfying the conditions:

$$\Delta u(x) = 0, \quad x \in D, \tag{2.1}$$

$$u(y) = g(y), \qquad y \in S, \quad y \,\overline{\in}\, l_k \subset S \;\; (k = \overline{1, n}), \tag{2.2}$$

$$|u(x)| < c, \quad x \in \overline{D}, \tag{2.3}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, c is a real constant and S is a closed piecewise smooth surface.

It is shown (see [7, 15]) that Problem (2.1), (2.2), (2.3) has a unique solution depending continuously on the data, and for a generalized solution $u(x)$ the generalized extremum principle is valid:

$$\min_{x \in S} u(x) < \underset{x \in D}{u(x)} < \max_{x \in S} u(x), \tag{2.4}$$

where it is assumed that $x \overline{\in} l_k$ $(k = \overline{1, n})$ for $x \in S$.

Note (see [7]) that the additional requirement (2.3) of boundedness actually concerns only the neighborhoods of singularities of the function $g(y)$ and it plays an important role in the extremum principle (2.4).

On the basis of (2.3), in general, the values of $u(y)$ are not uniquely defined on the curves l_k. For example, if Problem A concerns the determination of the thermal (or the electric) field, then we must take $u(y) = 0$ when $y \in l_k$, respectively. In this case, in the physical sense the curves l_k are non-conductors (or dielectrics). Otherwise, l_k will not be a discontinuity curve.

It is evident that, the surface S is divided into the parts $S_i (i = \overline{1, m})$ by curves $l_k (k = \overline{1, n})$, for the concrete case, where one of the following conditions holds: $n = m, n < m, n > m$. Thus, the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} g_1(y), & y \in S_1, \\ g_2(y), & y \in S_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ g_m(y), & y \in S_m, \\ 0, & y \in l_k \ (k = \overline{1, n}), \end{cases} \tag{2.5}$$

where: S_i $(i = \overline{1, m})$ are the parts of S without discontinuity curves, respectively; the functions $g_i(y)$, $y \in S_i$ $(i = \overline{1, m})$ are continuous on the parts S_i. It is evident that $S = (\bigcup_{j=1}^{m} S_j) \bigcup (\bigcup_{k=1}^{n} l_k)$.

Remark 1 : If the interior S is empty then we have the generalized problem with respect to the closed shell.

3. The Probabilistic method and simulation of the Wiener process

In this section the essence of the suggested algorithm for the numerical solution of the problem of type A is given, and its detailed description is in [16]. The main theorem in the realization of the PM is the following one(see e.g., [15])

Theorem 3.1 : *If a finite domain $D \subset R^3$ is bounded by a piecewise smooth surface S and $g(y)$ is a continuous (or discontinuous) bounded function on S, then the solution of the Dirichlet ordinary (or generalized) boundary value problem for the Laplace equation at the fixed point $x \in D$ has the form*

$$u(x) = E_x g(x(\tau)). \tag{3.1}$$

In (3.1): $E_x g(x(\tau))$ is the mathematical expectation of the values of the boundary function $g(y)$ at the random intersection points of the trajectory of the Wiener process and the boundary S; τ is the random moment of the first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random intersection points $y^j = (y_1^j, y_2^j, y_3^j) \in S$ $(j = \overline{1, N})$ is sufficiently large, then according to the law of large numbers, from (3.1) we have

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{j=1}^{N} g(y^j) \tag{3.2}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in probability. Thus, if we have the Wiener process, the approximate value of the probabilistic solution to Problem A at a point $x \in D$ is calculated by formula (3.2).

In order to simulate the Wiener process we use the following recursion relations(see e.g., [7, 16]):

$$\begin{aligned} x_1(t_k) &= x_1(t_{k-1}) + \gamma_1(t_k)/nq, \\ x_2(t_k) &= x_2(t_{k-1}) + \gamma_2(t_k)/nq, \\ x_3(t_k) &= x_3(t_{k-1}) + \gamma_3(t_k)/nq, \\ (k = 1, 2, \dots), &\quad x(t_0) = x, \end{aligned} \tag{3.3}$$

according to which the coordinates of the point $x(t_k) = (x_1(t_k), x_2(t_k), x_3(t_k))$ are being determined. In (3.3): $\gamma_1(t_k)$, $\gamma_2(t_k)$, $\gamma_3(t_k)$ are three normally distributed independent random numbers for the k-th step, with zero means and variances equal to one(the generation of the above numbers takes place apart); nq is a quantification number (nq) such that $1/nq = \sqrt{t_k - t_{k-1}}$ and when $nq \to \infty$, then the discrete process approaches the continuous Wiener process. In the implementation, the random process is simulated at each step of the walk and continues until it crosses the boundary.

In the considered case computations and the generation of random numbers are done in MATLAB.

Remark 2 : It is evident that for the PM it does not matter whether the boundary function (2.5) is axisymmetric or not.

4. An analytical solution of a certain Dirichlet generalized harmonic problem

This section is devoted to an explicit analytic solution of the Dirchlet generalized harmonic problem for a right circular axisymmetric cylindrical ring. We intend to use it for testing.

Let the domain D be a right circular cylindrical ring $D(a < r < b, o < x_3 < h)$, where h is its height, $r = \sqrt{{x_1}^2 + {x_2}^2}$, and a, b are the internal and external radii of ring, respectively.

In ([5], p.82) for the ring D a simplified case of Problem A is considered, in particular, when the boundary function $g(y) = g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 0, & y \in S_1 = \{y \in S | a \leq r \leq b, \;\; y_3 = 0\}, \\ 0, & y \in S_2 = \{y \in S | a \leq r \leq b, \;\; y_3 = h\}, \\ 0, & y \in S_3 = \{y \in S | r = a, \;\; 0 < y_3 < h\}, \\ v, & y \in S_4 = \{y \in S | r = b, \;\; 0 < y_3 < h\}, \\ 0, & y \in l_k, \;\; (k = 1, 2). \end{cases} \tag{4.1}$$

In (4.1): v is a real constant; l_1, l_2 are the external circles of the bases S_1 and S_2; l_1, l_2, S_1, S_2, S_3 are non conductors (or dielectrics); S is the full surface of D $(S = (\bigcup\limits_{j=1}^{4} S_j) \bigcup (\bigcup\limits_{k=1}^{2} l_k))$.

In ([5], p.415), it is given that in conditions (4.1) the exact analytical solution to Problem A has the following form (in cylindrical coordinates)

$$\begin{aligned} u(r, x_3) = \frac{4v}{\pi} \sum_{m=0}^{\infty} & \frac{I_0(c_m r) K_0(c_m a) - I_0(c_m a) K_0(c_m r)}{I_0(c_m b) K_0(c_m a) - I_0(c_m a) K_0(c_m b)} \\ & \times \frac{\sin(c_m x_3)}{2m+1} \equiv \frac{4v}{\pi} \sum_{m=0}^{\infty} u_m(r, x_3), \end{aligned} \tag{4.2}$$

where $a < r < b$, $0 < x_3 < h$, $c_m = (2m+1)\pi/h$, I_0 and K_0 are the first and second kind Bessel functions of order zero with an imaginary argument, respectively.

It is known (see, e.g., [17])that

$$I_0(t) \equiv J_0(it) = \sum_{k=0}^{\infty} \frac{(\frac{t}{2})^{2k}}{(k!)^2}, \quad t \in R,$$

$$I_0(0) = 1, \quad I_0(t) \to \frac{e^t}{\sqrt{2\pi t}} \; for \; t \to \infty.$$

$$K_0(t) \equiv K_0(it) = -(ln\frac{t}{2} + C)I_0(t) + \sum_{k=0}^{\infty} \Phi(k) \frac{(\frac{t}{2})^{2k}}{(k!)^2}, \quad t > 0,$$

where

$$\Phi(k) = \sum_{j=1}^{k} \frac{1}{j}, \quad k \geq 1, \quad \Phi(0) = 0,$$

$$K_0(t) \to \sqrt{\frac{\pi}{2t}} e^{-t} \text{ for } t \to \infty$$

and $C = 0.577215664901532$ is the Euler-Mascheroni constant(see e.g., [9], p.592).

Moreover, it is known (see, e.g.,[17]) that $I_0(t)$ and $K_0(t)$ are linearly independent solutions of the following ordinary differential equation

$$y'' + \frac{1}{t} y\prime - y = 0, \quad \text{where} \quad y = y(t), \; t \in (0, \infty). \tag{4.3}$$

Since (4.2) is constructed by methods presented in Section 1, its investigation is necessary. For the solution (4.2), the validity of the following properties are shown in [18]: 1) the general term $u_m(r, x_3)$ of series (4.2) is harmonic; 2) the series (4.2) is uniformly convergent in D; 3) the asymptotical behaviour of the general term of (4.2) is

$$u_m(r, x_3) \to \frac{1}{2m+1} exp(c_m(r - b)) \quad \text{for} \quad m \to \infty \quad \text{and} \quad (r, x_3) \in D;$$

4) It is easy to see that for the solution $u(r, x_3)$ conditions (4.1) are satisfied on S_1, S_2, S_3, l_1, l_2. If $(r, y_3) \in S_4$ or $(r = b, 0 < y_3 < b)$ then from (4.2) we have

$$u(b, y_3) = \frac{4v}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(c_m y_3). \tag{4.4}$$

In [18] it is shown that series (4.4) is uniformly convergent and equals to v when $0 < y_3 < h$.

Besides, since $\sin(c_m(h/2 + t)) = \sin(c_m(h/2 - t))$ for $0 \leq t \leq h/2$, therefore, $u(r, h/2 + t) = u(r, h/2 - t)$ for $a \leq r \leq b$, and this fact is in exact agreement with the real physical picture(or in the considered case the physical field is symmetric with respect to the plane $x_3 = h/2$ and is not dependent on the angle of rotation).

Table 4.1C. Results for partial sum of (4.4)

i	$(b, y_3$	S_{100}	S_{500}	S_{1000}
1	$(2, 1)$	1.00315	1.00064	1.00032
2	$(2, 1.5)$	0.99842	0.99968	0.99984
3	$(2, 1.8)$	0.99166	0.99833	0.99917
4	$(2, 1, 9)$	0.98068	0.99613	0.99807
5	$(2, 1.95)$	0.90301	0.90798	0.98989
6	$(2, 1.995)$	0.87899	0.89175	0.97978
7	$(2, 1.999)$	0.20087	0.87393	0.92922
8	$(2, 1.9999)$	0.01986	0.019910	0.02002

For illustration we calculated the partial sum $S_p(r, x_3)$ of series (4.2) for $m = \overline{0, p}$ at several interesting points. In the numerical experiments we took: $a = 1$, $b = 2$, $h = 2$, $v = 1$. Because of the convergence rate of (4.2) when $(r, x_3) \in D$, the calculations have shown that for $p = 20, 50$ practically: $S_p(1.2, 1) = 0.221517$; $S_p(1.5, 1) = 0.519826$; $S_p(1.8, 1) = 0.846086$; $S_p(1.5, 1.5) = 0.424747$; $S_p(1.5, 0.5) = 0.425002$; $S_p(1.5, 1.8) = 0.217324$; $S_p(1.5, 0.2) = 0.217581$; $S_p(1.8, 1.5) = 0.750919$; $S_p(1.8, 1, 8) = 0.516492$.

Since boundary function (4.1) is symmetric with respect to the plane $x_3 = 1$, the partial sum $S_p(r, x_3)$ is calculated also at the points which are symmetric with respect to the plane $x_3 = 1$, the results are in good agreement with the real physical picture.

It is clear that if a point $y(y_1, y_2, y_3) \equiv (b, y_3) \in S_4$ and tends to the discontinuity curve l_k $(k = 1, 2)$, then all terms of series (4.4) tend to zero. Consequently, series (4.4) converges very slowly, therefore, the accuracy of the satisfaction of the boundary condition is very low.

In Table 4.1C the values of the partial sum $S_p(b, x_3)$ of (4.4) at several points of S_4 for $p = 100, 500, 1000$ and the same parameters- a, b, h, v are given.

From Table 4.1C it is clear that the accuracy of the satisfaction of the boundary condition is very low in the neighborhood of the discontinuity curves, as expected (see Section 1).

Our calculations showed that the analytic solution (4.2) is sufficiently accurate for a wide group of practical problems. In addition, the results of calculations for inner control points are in good accordance with the real physical picture of the field. Finally, we note that the considered problem can be used as a test with the help of the above-mentioned analytic solution.

In Section 5, for comparison, the PM is tested on the solution (4.2) (see Example 2).

Remark 3 : If we consider the simple case, when $g(y) = v$ in (4.1) on the lower base of the cylindrical ring, and $g(y) = 0$ on the remaining surface, then the analytic form of the exact solution of Problem A is so difficult in the sense of numerical implementation, that it has only theoretical significance (see [5], p.82, pp.416-417).

5. Numerical examples

It should be noted that in the 3D case, in general (except for a special case), there are no test solutions for generalized problems of type A, therefore, for the verification of the scheme needed for the numerical solution of Problem A, the reliability of obtained results can be demonstrated in the following way.

If we take $g_i(y) = 1/|y - x^0|$ in boundary conditions (2.5), where $y \in S_i$ $(i = \overline{1,m})$, $x^0 = (x_1^0, x_2^0, x_3^0) \,\overline{\in}\, \overline{D}$, and $|y - x^0|$ denotes the distance between the points y and x^0, then it is evident that the curves l_k $(k = \overline{1,n})$ represent removable discontinuity curves for the boundary function $g(y)$. Actually, in the mentioned case instead of generalized problem A we obtain the following Dirichlet ordinary harmonic problem.

Problem B. Find a Function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D})$ satisfying the conditions:

$$\Delta u(x) = 0, \;\; x \in D,$$
$$u(y) = 1/|y - x^0|, \;\; y \in S, \; x^0 \,\overline{\in}\, \overline{D}.$$

We solve this problem (by the PM) with the use of the program used for Problem A. It is well-known that Problem B is well posed, i.e., its solution exists, is unique and depends on data continuously. Evidently, an exact solution of Problem B is

$$u(x^0, x) = \frac{1}{|x - x^0|}, \quad x \in \overline{D}, \quad x^0 \,\overline{\in}\, \overline{D}. \tag{5.1}$$

It should be noted that the numerical solution of the Dirichlet ordinary harmonic problems by the PM is interesting and important (see e.g., [19, 20]). In this paper, Problem B has an auxiliary role. In particular, for Problem B, the verification of the scheme needed for the numerical solution of Problem A and the corresponding program (comparison of the obtained results with exact solution) are carried out first of all, and then Problem A is solved under boundary conditions (2.5).

In the present paper the PM is applied to four examples. In the tables, N is the number of implementation of the Wiener process for the given points $x^i = (x_1^i, x_2^i, x_3^i) \in D$, and nq is the number of quantification. For simplicity, in the considered examples the values nq and N are the same. In the tables for problems of type B we present the absolute errors Δ^i at the points $x^i \in D$ of $u_N(x)$, in the PM approximation, for $nq = 200$ and various values of N, and under the notation $(E \pm k)$ for $10^{\pm k}$. In particular, $\Delta^i = |u_N(x^i) - u(x^0, x^i)|$, where $u_N(x^i)$ is the approximate solution of Problem B at the point x^i, which is defined by formula (3.2), and the exact solution $u(x^0, x^i)$ of the test problem is given by (5.1). In the tables, for problems of type A, the probabilistic solution $u_N(x)$ is presented at the points x^i, defined by (3.2).

Remark 4: The Problems A and B for ellipsoidal, spherical, cylindrical, conic, prismatic and pyramidal domains are considered in [7, 16, 21, 22].

Example 1: In the first example in the role of the axisymmetric domain D is we take the right circular cylindrical ring $D(a < r < b,\; 0 < x_3 < h)$, where h is a height of the ring, $r = \sqrt{x_1^2 + x_2^2}$, and a, b are the internal and external radii of the ring, respectively.

We consider Problem A for D, when the boundary function $g(y) \equiv g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 2, & y \in S_1 = \{y \in S|\ a < r < b,\ \ y_3 = 0\}, \\ 0.5, & y \in S_2 = \{y \in S|\ a < r < b,\ \ y_3 = h\}, \\ 1.5, & y \in S_3 = \{y \in S|\ r = b,\ y_1 > 0,\ y_2 > 0,\ 0 < y_3 < h\}, \\ 1, & y \in S_4 = \{y \in S|\ r = b,\ y_1 < 0,\ y_2 > 0,\ 0 < y_3 < h\}, \\ 1.5, & y \in S_5 = \{y \in S|\ r = b,\ y_1 < 0,\ y_2 < 0,\ 0 < y_3 < h\}, \\ 1, & y \in S_6 = \{y \in S|\ r = b,\ y_1 > 0,\ y_2 < 0,\ 0 < y_3 < h\}, \\ 1, & y \in S_7 = \{y \in S|\ r = a,\ 0 < y_3 < h/2\}, \\ 0.8, & y \in S_8 = \{y \in S|\ r = a,\ h/2 < y_3 < h\}, \\ 0, & y \in l_k\ (k = \overline{1,9}). \end{cases} \tag{5.2}$$

In (5.2): l_1, l_2 are the external circles of the bases S_1 and S_2; l_3, l_4 are the internal circles of the bases S_1 and S_2; l_5 is an intersection circle of surface hole with the plane $x_3 = h/2$; l_i $(i = \overline{6,9})$ are the generatrices of the ring, passing through the points $(b, 0)$, $(0, b)$, $(-b, 0)$, $(0, -b)$, respectively; It is evident that in the physical sense in the considered case l_k $(k = \overline{1,9})$ are non-conductors(or dielectrics). S is the full surface of D $(S = (\bigcup\limits_{j=1}^{8} S_j)\bigcup(\bigcup\limits_{k=1}^{9} l_k)$.

In the numerical experiments for the considered example, we took: 1) $a = 1, b = 2, h = 3$; 2) in test Problem B, the boundary function $h(y) = 1/|y - x^0|$, $y \in S$, $x^0 = (0, 0, -4)$.

In order to determine the intersection points $y^j = (y_1^j, y_2^j, y_3^j$ $(j = \overline{1,N})$ of the trajectory of the Wiener process and the surface S, we operate in the following way. During the implementation of the Wiener process, for each current point $x(t_k)$, defined from (3.3), its location with respect to S is checked, i.e., for the point $x(t_k)$ the value

$$d = \sqrt{x_1^2 + x_2^2}$$

is calculated and the following conditions: 1)$a < d < b$ and $0 < x_3(t_k) < h$; 2)$d = a$ or $d = b$ and $0 < x_3(t_k) < h$; 3) $d < a$ or $d > b$ and $0 < x_3(t_k) < h$; 4)$x_3(t_k) < 0$ or $x_3(t_k) > h$ and $a < d < b$; 5) $x_3(t_k) = 0$ or $x_3(t_k) = h$ and $a < d < b$, are checked.

In the first case $x(t_k) \in D$ and the process continues until it crosses the boundary of D. In the second case $x(t_k) \in S$ and $y^j = x(t_k)$. It is evident that in the third case $x(t_k) \overline{\in} \overline{D}$. In this case, let $x(t_{k-1}) \in D$ for the moment $t = t_{k-1}$ and $x(t_k) \overline{\in} \overline{D}$ for the moment $t = t_k$. For the determination of the point y^j, a parametric equation of the line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is firstly obtained, and it has the following form

$$\begin{cases} x_1 = x_1^{k-1} + (x_1^k - x_1^{k-1})\theta, \\ x_2 = x_2^{k-1} + (x_2^k - x_2^{k-1})\theta, \\ x_3 = x_3^{k-1} + (x_3^k - x_3^{k-1})\theta, \end{cases} \tag{5.3}$$

where (x_1, x_2, x_3) is the current point of L and θ is a parameter $(-\infty < \theta < \infty)$, and $x_i^{k-1} \equiv x_i(t_{k-1})$, $x_i^k \equiv x_i(t_k)$ $(i = 1, 2, 3)$. After this, for definition of the

Table 5.1B. Results for Problem B (in Example 1)

x^i	$(0,1.2,1.5)$	$(0,1.5,1.5)$	$(0,1.8,1.5)$	$(0,1.5,0.2)$	$(0,1.5,2.8)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.13E-3$	$0.37E-3$	$0.20E-3$	$0.20E-3$	$0.12E-3$
$1E+4$	$0.29E-3$	$0.12E-3$	$0.17E-3$	$0.42E-3$	$0.17E-3$
$5E+4$	$0.79E-4$	$0.11E-4$	$0.49E-4$	$0.13E-4$	$0.73E-4$
$1E+5$	$0.77E-4$	$0.12E-4$	$0.33E-4$	$0.44E-4$	$0.67E-4$
$5E+5$	$0.32E-4$	$0.34E-4$	$0.29E-4$	$0.21E-4$	$0.29E-4$
$1E+6$	$0.96E-6$	$0.69E-5$	$0.16E-4$	$0.15E-4$	$0.25E-4$

intersection points x^* and x^{**} of line L with the cylindrical surface $x_1^2+x_2^2=a^2$ (or $x_1^2+x_2^2=b^2$) is solved with respect to θ.

It is easy to see that for the parameter θ we obtain the following equation

$$A\theta^2+2B\theta+C=0, \tag{5.4}$$

whose discriminant $d^*=B^2-AC>0$.

Since the discriminant of (5.4) is positive, the points x^* and x^{**} are defined respectively on the basis of (5.3) for the solutions of (5.4) (θ_1 and θ_2). In the role of the point y^j we choose the one (from x^* and x^{**}) for which $|x(t_k)-x|$ is minimal.

In the case 4) we find the intersection point $y=(y_1,y_2,0)$ (or ($y=(y_1,y_2,h)$) of the plane $x_3=0$ (or $x_3=h$) and the line L passing through the points $x(t_{k-1})$ and $x(t_k)$, if $a^2<y_1^2+y_2^2<b^2$ then $y^j=(y_1,y_2,0)$ (or $y^j=(y_1,y_2,h)$). In the case 5) $y^j=(y_1,y_2,0)$ (or $y^j=(y_1,y_2,h)$).

In all examples, considered by us for the determination of the intersection points $y^i=(y_1^i,y_2^i,y_3^i)$ $(i=\overline{1,N})$ of the trajectory of the Wiener process and the surface S the above scheme is used. As noted above, for verification at first we solve the auxiliary Problem B with the program of Problem A.

In Table 5.1B the absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points $x^i\in D$ $(i=\overline{1,5})$ are presented.

On the basis of the results presented in Table 5.1B, we can conclude that the program for Problem A is correct.

We also conducted a verification experiment. Namely, we calculated the probabilistic solution of Problem B at the point (0,1.2,1.5) for $N=1E+5$, $nq=400$ and we obtained $\Delta^1=0.38E-4$ (see, Table 5.1B). The result is improved, as expected (see Section 3). In general, if more accuracy is needed, then calculations for sufficiently large values of nq and N must be realized. In this case, the numerical implementation on a PC takes more time. We can avoid this difficulty if we apply a parallel calculation method. For this, an appropriate computing technique is needed. Respectively, significantly less time will be needed for the numerical implementation and, besides, the accuracy of the obtained results will be improved.

In Table 5.1A the values of the approximate solution $u_N(x)$ to Problem A at the same points x^i $(i=\overline{1,5})$ are given. The boundary function (5.2) is symmetric with respect to the axis Ox_3, respectively, in the role of x^4 and x^5, the points which are symmetric with respect to the axis Ox_3 are taken. The results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

As noted above, the program for Problem A (in Example 1) is correct, therefore,

Table 5.1A. Results for Problem A (in Example 1)

x^i	$(0, 1.2, 1.5)$	$(0, 1.5, 1.5)$	$(0, 1.8, 1.5)$	$(1.2, 1.2, 1.5)$	$(-1.2, -1.2, 1.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	0.99190	1.10950	1.19554	1.34906	1.35333
$1E+4$	0.99616	1.10656	1.19309	1.35241	1.35460
$5E+4$	0.99659	1.10669	1.19772	1.35241	1.35187
$1E+5$	0.99563	1.10985	1.19835	1.35337	1.35218
$5E+5$	0.99548	1.10846	1.19791	1.35231	1.35149
$1E+6$	0.99532	1.10861	1.19795	1.35225	1.35119

Table 5.2A. Results for Problem A (in Example 2)

x^i	$(0, 1.2, 1)$	$(0, 1.5, 1)$	$(0, 1.8, 1)$	$(0, 1.5, 1.5)$	$(0, 1.5, 1.8)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	0.22360	0.51878	0.79880	0.41560	0.21260
$1E+4$	0.22960	0.52540	0.81190	0.42570	0.21280
$5E+4$	0.22292	0.51766	0.80844	0.42610	0.22090
$1E+5$	0.22087	0.51925	0.80971	0.42483	0.21900
$5E+5$	0.22389	0.51957	0.80920	0.42520	0.21950
$1E+6$	0.22381	0.51961	0.81025	0.42520	0.21950

in the following example we solve Problem A directly.

Example 2 : Here we consider the test problem which is considered in Section 4. In order to compare the results obtained from the analytical solution (4.2) with results obtained by the PM, we solve Problem A under conditions (4.1) by the PM, for the same cylindrical ring and parameters: $a = 1$, $b = 2$, $h = 2$, $v = 1$ (see Section 4).

Since, Example 2 is a special case of Example 1, and its program and the reliability of the obtained results are checked, we directly solved Problem A for Example 2 by the PM.

In the considered case, for the determination of the intersection points y^J $(j = \overline{1, N})$ of the trajectory of modelling the Wiener process and surface S, the same algorithm, described in Example 1 is applied. For the above mentioned comparison we calculated the values of the approximate solution $u_N(x)$ of Problem A at the same points, in which the partial sums $S_p(r, x_3)$ of the series (4.2) are calculated (see Section 4),and the obtained results are given in Table 5.2A.

From Table 5.2A it is clear that at the control points the results obtained using the PM are in good agreement with the results of the test problem and are reliable, with an accuracy which is sufficient for many practical problems.

Example 3 : Here in the role of axisymmetric domain D we took a regular 4-sided prism $ABCDA_1B_1C_1D_1$ with a right circular cylindrical hole. We assume that an axis of symmetry of D lies on Ox_3 of the Cartesian coordinate right-handed system $Ox_1x_2x_3$ and the base of the prism lies in the plane Ox_1x_2, and its sides are perpendicular to the axes Ox_1 and Ox_2, respectively. It is evident that axes of symmetry of the prism and cylindrical hole are one and the same and lie on Ox_3. Besides: h and $2a$ are a height and a base side of the prism, respectively; r is a radius of a base of the cylindrical hole.

We solved Problems B and A when $h = 3$, $a = 1$, $r = 0.5$, $x^0 = (0.5, 1, -5)$, and the boundary function $g(y) = g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 2, & y \in S_1 = \{y \in S | y_1 \in (-a, a), y_2 \in (-a, a), y_3 = 0, d > r\}, \\ 0.5, & y \in S_2 = \{y \in S | y_1 \in (-a, a), y_2 \in (-a, a), y_3 = h, d > r\}, \\ 1.5, & y \in S_3 = \{y \in S |\ y_1 = a,\ -a < y_2 < a,\ 0 < y_3 < h\}, \\ 3, & y \in S_4 = \{y \in S |\ -a < y_1 < a,\ y_2 = a,\ 0 < y_3 < h\}, \\ 1.5, & y \in S_5 = \{y \in S |\ y_1 = -a,\ -a < y_2 < a,\ 0 < y_3 < h\}, \\ 3, & y \in S_6 = \{y \in S |\ -a < y_1 < a,\ y_2 = -a,\ 0 < y_3 < h\}, \\ 1.5, & y \in S_7 = \{y \in S |\ 0 < y_3 < h/2,\ d = r\}, \\ 1, & y \in S_8 = \{y \in S |\ h/2 < y_3 < h,\ d = r\}, \\ 0, & y \in l_k \ \ (k = \overline{1, 15}), \end{cases} \tag{5.5}$$

In (5.5): $d = \sqrt{y_1^2 + y_2^2}$; l_k $(k = \overline{1, 15})$ are discontinuity curves,in particular l_1, l_2 are bases of the hole, l_3 is an intersection of surface hole with the plane $x_3 = h/2$, and l_k $(k = \overline{4, 15})$ are the edges of the prism; S is the surface of D or $S = (\bigcup\limits_{j=1}^{8} S_j) \bigcup (\bigcup\limits_{k=1}^{15} l_k)$, where $S_j (j = \overline{1, 8})$ are the parts of S without discontinuity curves.

In the considered case, for the determination of the intersection points $y^j (j = \overline{1, N})$ of the trajectory of the Wiener process and the surface S the following approach is used. During the implementation of the Wiener process, for each current point $x(t_k)$, defined by (3.3), its location with respect S is verified, i.e., for the point $x(t_k)$ the following conditions

$$-a < x_1(t_k) < a,\ \ -a < x_2(t_k) < a,\ \ 0 < x_3(t_k) < h,\ \ d > r \tag{5.6}$$

are checked. If the conditions (5.6) are fulfilled then the process (3.3) is continuous. If $x(t_k) \in S$ then $y^j = x(t_k)$. In the case when the trajectory of the Wiener process intersects the cylindrical surface $y_1^2 + y_2^2 = r^2, 0 < y_3 < h$ (or basis S_1 and S_2 of the domain D) then for the determination of the intersection points the same algorithm, described in Example 1 is applied.

Let $x(t) \in D$ for the moment $t = t_{k-1}$ and for the moment $t = t_k$ the trajectory of the Wiener process intersect any lateral face of the prism. In this case, under conditions (5.6) we establish the lateral face where the intersection point y^j is located. After this, for the determination of y^j, a parametric equation of a line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is firstly obtained in the form (5.3). Finally, the intersection point x^* of the line L and that face, which is intersected by the trajectory of the Wiener process is found and respectively, in this case $y^j = x^*$.

In Table 5.3B the absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points $x^i \in D$ $(i = \overline{1, 5})$ are presented.

The values of the approximate solution $u_N(x)$ of Problem A at the points $x^i \in D$ $(i = \overline{1, 5})$ are given in Table 5.3A. Since the boundary function (5.5) is symmetric with respect to the plane Ox_1x_3, therefore, in the role of x^i $(i = 4, 5)$, the points which are symmetric with respect to the plane Ox_1x_3 are taken. The obtained results have sufficient accuracy for many practical problems and are in good

Table 5.3B. Results for Problem B (in Example 3)

x^i	$(0.8, 0.8, 0.2)$	$(0.8, 0.8, 1.5)$	$(0.8, 0.8, 2.8)$	$(0, 0.8, 1.5)$	$(0, -0.8, 1.5)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.48E-4$	$0.51E-4$	$0.44E-5$	$0.27E-3$	$0.42E-4$
$1E+4$	$0.47E-5$	$0.97E-4$	$0.21E-4$	$0.12E-3$	$0.59E-5$
$5E+4$	$0.26E-4$	$0.44E-4$	$0.32E-4$	$0.48E-4$	$0.37E-4$
$1E+5$	$0.49E-4$	$0.35E-4$	$0.19E-4$	$0.74E-4$	$0.34E-4$
$5E+5$	$0.39E-4$	$0.12E-4$	$0.57E-6$	$0.97E-5$	$0.76E-5$
$1E+6$	$0.46E-6$	$0.35E-5$	$0.60E-5$	$0.13E-4$	$0.12E-4$

Table 5.3A. Results for Problem A (in Example 3)

x^i	$(0.8, 0.8, 0.2)$	$(0.8, 0.8, 1.5)$	$(0.8, 0.8, 2.8)$	$(0, 0.8, 1.5)$	$(0, -0.8, 1.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.97120	1.93150	1.49100	2.35180	2.34980
$1E+4$	1.96095	1.92560	1.48725	2.34175	2.33640
$5E+4$	1.98466	1.92592	1.49595	2.34196	2.34166
$1E+5$	1.98034	1.92629	1.49127	2.34004	2.33329
$5E+5$	1.97856	1.93238	1.49168	2.33888	2.33618
$1E+6$	1.98096	1.93188	1.49018	2.33875	2.33818

agreement with the real physical picture(see Table 5.3A).

Example 4 : In this example in the role of the axisymmetric domain D we take a closed truncated right circular cone with a right circular cylindrical hole. We assume that, they have one and the same axis of symmetry and it lies on Ox_3, and the base of D lies in the plane Ox_1x_2. For the considered case, the equations of lateral surfaces of cone- S_c and cylindrical hole - S_h are

$$dc \equiv (x_1)^2 + (x_2)^2 - \Big(\frac{r_1 - r_2}{h}\Big)^2\Big(\frac{r_1 h}{r_1 - r_2} - x_3\Big)^2 = 0, \;\; x \in S_c, \;\; 0 < x_3 < h$$

and

$$dh \equiv (x_1)^2 + (x_2)^2 - (r_3)^2 = 0, \;\; x \in S_h, \;\; 0 < x_3 < h,$$

respectively.

In the above equations h is the height of the truncated cone and hole, r_1 and r_2 are the radii of the lower and upper bases of cone, r_3 is the radius of the base of the hole, and $x = (x_1, x_2, x_3)$ is current point of the noted surfaces, respectively.

It is easy to see that: 1) $dc = 0$, when $x \in S_c$; $dc < 0$, when $x \in D$; $dc > 0$, when $x \overline{\in} \overline{D}$ and $0 < x_3 < h$. 2) $dh = 0$, when $x \in S_h$; $dh < 0$, when $r < r_3$ and $0 < x_3 < h$ $(r = \sqrt{x_1^2 + x_2^2})$; $dh > 0$, when $x \in D$. From 1) and 2) it is evident that if: 3) $dc < 0$ and $dh > 0$, then $x \in D$.

In the numerical experiments for this example, we took: a) $h = 3$, $r_1 = 2$, $r_2 = 1$, $r_3 = 0.5$; b) in the test Problem B the boundary function $h(y) = 1/|y - x^0|$, $y \in S$, $x^0 = (0, 0, -5)$; c) in Problem A the boundary function $g(y) \equiv g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 2, & y \in S_1 = \{y \in S|\ r_3 < r < r_1,\ y_3 = 0\}, \\ 0, & y \in S_2 = \{y \in S|\ r_3 < r < r_2,\ y_3 = h\}, \\ 1.5, & y \in S_3 = \{y \in S_c|\ 0 < y_3 < h/3\}, \\ 1, & y \in S_4 = \{y \in S_c|\ h/3 < y_3 < 2h/3\}, \\ 0.5, & y \in S_5 = \{y \in S_c|\ 2h/3 < y_3 < h\}, \\ 1.5, & y \in S_6 = \{y \in S_h|\ d = r_3,\ 0 < y_3 < h/2\}, \\ 1, & y \in S_7 = \{y \in S_h|\ ,d = r_3,\ h/2 < y_3 < h\}, \\ 0, & y \in l_k (k = \overline{1,7}). \end{cases} \tag{5.7}$$

In (5.7): l_1, l_2 are the external circles of the bases S_1 and S_2; l_3, l_4 are the internal circles of the bases S_1 and S_2; l_5 is the circle, which is obtained by the intersection of the plane $x_3 = h/2$ and the surface S_h; l_6 and l_7 are circles, which are obtained by the intersection of the planes $x_3 = h/3$, $x_3 = 2h/3$ and the surface S_c. It is evident these circles represent discontinuity curves of the first kind for the function $g(y)$. S is the surface of D or $S = (\bigcup_{j=1}^{7} S_j) \bigcup (\bigcup_{k=1}^{7} l_k)$, where S_j $(j = \overline{1,7})$ are the parts of S without discontinuity curves. In the physical sense the above circles are non-conductors(or dielectrics).

In order to determine the intersection points $y^i = (y_1^i, y_2^i, y_3^i)$ $(i = \overline{1,N})$ of the trajectory of the simulated Wiener process and the surface S, we operate in the following way. During the implementation of the simulated Wiener process, for each current point $x(t_k)$, defined from (2.3), its location with respect to S is checked, i. e., for the point $x(t_k)$ the values dc and dh are calculated and as above and the following conditions: 1*) $dc < 0$, $dh > 0$ and $0 < x_3(t_k) < h$; 2*) $dc = 0$ or $dh = 0$ and$0 < x_3(t_k) < h$; 3*) $x_3(t_k) = 0$ or $x_3(t_k) = h$ are checked. In the first case $x(t_k) \in D$ and the process continuous until it crosses the surface S. In the second case $x(t_k) \in S_c$ (or $x(t_k) \in S_h$) and $y^i = x(t_k)$. In the third case: if $r_3 < r < r_2$ then $y^i = (x_1(t_k), x_2(t_k), 0)$; if $r_3 < r < r_1$ then $y^i = (x_1(t_k), x_2(t_k), h)$.

If $x(t_{k-1}) \in D$ for the moment $t = t_{k-1}$ and $x(t_k) \overline{\in} \overline{D}$ for the moment $t = t_k$, then the trajectory of the simulated Wiener process intersects any lateral surface of D (or any base of D). In this case, for the determination of the intersection points the same algorithm, described in Example 1 is applied.

In Table 5.4B the absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points $x^i \in D$ $(i = \overline{1,5})$ are presented.

The values of the approximate solution $u_N(x)$ of problem A at the points $x^i \in D$ $(i = \overline{1,5})$ are given in Table 5.4A. Since the boundary function (5.7) is symmetric with respect to the axis Ox_3, therefore,

In the role of x^k $(k = 4, 5)$, the points which are symmetric with respect to the axis Ox_3 are taken. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

In this work we solved problems of type A when the boundary functions $g_i(y)$ $(i = \overline{1,m})$ are constants. This was motivated by our interest to find out how well the obtained results agreed with the real physical picture. It is evident that solving Problem A under condition (2.5) is as easy as Problem B. In general, we can solve Problem A for all locations of discontinuity curves, which give the possibility

Table 5.4B. Results for Problem B (in Example 4)

x^i	$(0.7, 0, 0.5)$	$(0.7, 0, 1.5)$	$(0.7, 0, 2.5)$	$(1.5, 0, 0.8)$	$(-1.5, 0, 0.8)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.26E-3$	$0.10E-4$	$0.72E-4$	$0.94E-4$	$0.23E-3$
$1E+4$	$0.53E-5$	$0.83E-4$	$0.73E-4$	$0.39E-4$	$0.97E-4$
$5E+4$	$0.32E-4$	$0.79E-4$	$0.22E-4$	$0.16E-4$	$0.41E-4$
$1E+5$	$0.52E-4$	$0.22E-4$	$0.11E-4$	$0.61E-4$	$0.34E-4$
$5E+5$	$0.44E-4$	$0.13E-4$	$0.13E-4$	$0.76E-5$	$0.56E-5$
$1E+6$	$0.13E-4$	$0.19E-4$	$0.44E-5$	$0.27E-4$	$0.49E-4$

Table 5.4A. Results for Problem A (in Example 4)

x^i	$(0.7, 0, 0.5)$	$(0.7, 0, 1.5)$	$(0.7, 0, 2.5)$	$(1.5, 0, 0.8)$	$(-1.5, 0, 0.8)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.58250	1.17140	0.75460	1.42940	1.43050
$1E+4$	1.57505	1.17575	0.76385	1.43260	1.43145
$5E+4$	1.57919	1.17592	0.75844	1.42844	1.43036
$1E+5$	1.57734	1.17522	0.75886	1.42485	1.43017
$5E+5$	1.57751	1.17445	0.75986	1.43074	1.43015
$1E+6$	1.57720	1.17423	0.75966	1.43181	1.43055

to establish the part of surface S where the intersection point is located. The analysis of the results of numerical experiments show that the results obtained by the proposed algorithm are reliable and it is effective for the numerical solution of problems of type A and B. In particular, the algorithm is sufficiently simple for numerical implementation.

6. Concluding Remarks

1. In this work, we have demonstrated that the probabilistic method (PM) is ideally suited for numerically solving of both ordinary and generalized 3D Dirichlet harmonic problems for a wide class of axisymmetric finite domains with cylindrical holes.

2. The PM does not require the approximation of the boundary function, which is one of its important properties.

3. It is easy to program, its computational cost is low, it is characterized by an accuracy which is sufficient for many practical problems.

4. In the future we plan to investigate the following:

* Application of the proposed method to the numerical solution of the Dirichlet generalized harmonic problem for truncated regular n-sided pyramidal domains with cylindrical, conical and prismatic holes.

* Application of the PM for the same type of problem for irregular pyramidal domains.

* Application of the PM for the same type of problem in domains which are bounded by several closed surfaces.

* Application of the PM for the same problem for infinite 3D domains.

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