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Application of General Integral of Quasi-Linear Equation to Solving of Non-Linear Cauchy Problem

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In this work we describe the process of construction of solution to Initial Value Problem for some quasi-linear non-strictly hyperbolic equation. Using a General integral, the non-linear version of Cauchy problem is investigated. We also depict the structure of domain where the solution is defined.

Keywords: Quasi-linear hyperbolic equation, parabolic degeneracy, characteristic, general integral, Cauchy problem.

AMS Subject Classification: 35L15, 35L80.

1. Introduction

We consider the new class of second order quasi-linear equations. The second order derivatives of these equations have coefficients, which contain squares of the first derivatives of the unknown solution; both families of characteristics depend on the unknown function. This class of equations is noticeable also for the fact that it admits parabolic degeneration, which also depends on the unknown solution [1]. Generally, some of the equations of this class admit order degeneration either. We were able to construct general integrals in the form of the sum of two arbitrary functions for some equations from this class [2]. Since no general theory of nonlinear equations is available, studying some particular classes and particular equations is valuable, as this widens the class of quasi-linear equations for which the general integrals are constructed and the various problems are posed correctly. Here we present a systematic description of application of the integrals to solving the nonlinear Cauchy problem and to study the geometry of areas of definition of the global solution in specific cases.

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2. Initial value problem

1. In this work a non-linear variant of Cauchy problem is considered for one class of differential equations:

$$L(u) \equiv (u_y^2 - u_y)u_{xx} - (2u_x u_y + u_y - u_x - 1)u_{xy} + (u_x^2 + u_x)u_{yy} = \Phi(u_x, u_y, y) \quad (1)$$

The main part of (1) is a non-strictly hyperbolic second-order operator L. For an equation from this class, in article [2] we construct a general solution and study the characteristic problem.

Characteristic roots defined by operator L

$$\lambda_1 = -\frac{p+1}{q}, \quad \lambda_2 = -\frac{p}{q-1}, \tag{2}$$

where $p = u_x, q = u_y$ are Monge designations, behave differently with different functions $u \in C^2(R^2)$: with some $u(x, y)$ they may coincide at all points. Then along such functions operator (1) ceases to be hyperbolic and parabolically degenerates. This class of the functions is defined by means of the condition

$$p - q + 1 = 0. \tag{3}$$

If the solution of the given equation belongs to this class it will be a parabolic solution. It follows from the structure of the roots (2) that when having parabolic solutions their values not only coincide but they both equal to $\lambda_1 = \lambda_2 = -1$. Accordingly, in such case characteristic directions coincide with the direction of the family of lines $x + y = c$. If the condition (3) is not fulfilled at all points but only at the determined number of points, then the solution is related to the parabolically degenerated hyperbolic class ([4-6]).

In the particular case, when

$$L(u) = -\frac{1}{y}p(p+1)(p-q+1) \tag{4}$$

the equation can be fully integrated and its general integral has the following form ([3])

$$f(u+x) + g(u-y) = y^2, \tag{5}$$

with the arbitrary functions $f, g \in C^2(R^1)$

We will concentrate on the case when $L(u) = 0$, so the equation has the form

$$(u_y^2 - u_y)u_{xx} - (2u_x u_y + u_y - u_x - 1)u_{xy} + (u_x^2 + u_x)u_{yy} = 0. \tag{6}$$

And the general integral is

$$f(u+x) + g(u-y) = x, \tag{7}$$

However, we can easily prove that the equivalent expression of the general integral is

$$f(u+x)+g(u-y)=y, \tag{8}$$

It should be noted that the general integral (8) of the equation (6) as well as the integral (5) of the concrete equation (4) are not connected to any kind of correct problems. Equivalence of general integrals and corresponding to them equations is proved in [1].

In this work the Cauchy problem is considered when the initial data support is entirely the segment of the straight-line $y=0$. The statement of the problem is the following:

Cauchy problem. Find a regular solution of equation (6) together with its domain of definition, satisfying the conditions

$$u|_{y=0}=\varphi(x), u_y|_{y=0}=\psi(x), x\in[a,b], \tag{9}$$

when $\varphi\in C^2[a,b]$, $\psi\in C^1[a,b]$ are the given functions.

Theorem 2.1 : *Implicit solution to the problem* (6), (9) *is written in the following form:*

$$\int_{\nu(u-y)}^{\tau(u+x)} F(t)\,dt=y, \tag{10}$$

where

$$F(t)=\frac{\varphi(t)(\varphi\prime(t)+1)}{\varphi\prime(t)-\psi(t)+1} \tag{11}$$

and $\tau(\xi)=x,\ \nu(\eta)=x$ *are inverse functions of* $\varphi(x)+x$ *and* $\varphi(x)$ *consequently. Here we assume, that functional equations* $\varphi(x)+x=\xi,\ \varphi(x)=\eta$ *are uniquely solvable on the segment* $[a,b]$ *and*

$$\xi\in[\varphi(a),\varphi(b)],\ \eta\in[\varphi(a)+a,\varphi(b)+b].$$

Proof : Let us apply our conditions (9) to the general integral (8). For arbitrary functions f,g we can write an expression:

$$f(\varphi(x)+x)+g((\varphi(x))=0. \tag{12}$$

After that we differentiate the general integral with respect to y and taking into account (9):

$$f\prime(\varphi(x)+x)\psi(x)+g\prime((\varphi(x))(\psi(x)-1)=1, \tag{13}$$

Besides, if we differentiate (12) with respect to x, we have:

$$f\prime(\varphi(x)+x)(\varphi\prime(x)+1)+g\prime((\varphi(x))\varphi\prime(x)=0. \tag{14}$$

We see that conditions (9) together with the general integral (8) yielded equations (13), (14). From these equations we have to determine functions f, g.

Let us consider equations (13) and (14) together as a linear algebraic system with respect to derivatives of functions $f\prime, g\prime$. We assume, that parabolic degeneration is excluded for equation (6), It means that we have condition $p - q + 1 \neq 0$ and the determinant of our algebraic system is not zero. In this case we can determine derivatives $f\prime, g\prime$ of arbitrary functions uniquely:

$$f\prime(\varphi(x) + x) = \frac{\varphi\prime(x)}{\varphi\prime(x) - \psi(x) + 1},$$

$$g\prime((\varphi(x)) = \frac{-(\varphi\prime(x) + 1)}{\varphi\prime(x) - \psi(x) + 1}.$$

Taking into account (11) we have:

$$f\prime(\varphi(x) + x)(\varphi\prime(x) + 1) = F(x), \tag{15}$$

$$g\prime((\varphi(x))\varphi\prime(x) = -F(x), \tag{16}$$

The arbitrary functions f, g are to be determined by integration of (15), (16) in limits (x_0, x), where x_0 is an arbitrary value of x from the segment (a, b) :

$$f(\varphi(x) + x) - f(\varphi(x_0) + x_0) = \int_{x_0}^{x} F(t)\,dt \tag{17}$$

$$g(\varphi(x)) - g(\varphi(x_0)) = -\int_{x_0}^{x} F(t)\,dt \tag{18}$$

As we assumed, the functional equations $\varphi(x) = \eta$ and $\varphi(x) + x = \xi$ are uniquely solvable and their solutions are:

$$x = \tau(\xi), \ x = \nu(\eta).$$

Taking into account all these notations, we have

$$f(\xi) = f(\xi_0) + \int_{x_0}^{\tau(\xi)} F(t)\,dt \tag{19}$$

$$g(\eta) = g(\eta_0) - \int_{x_0}^{\nu(\eta)} F(t)\,dt \tag{20}$$

Inserting into (1) the obtained formulas (18), (19), for the functions f, g, we write the integral of problem (1), (6):

$$\int_{x_0}^{\tau(u+x)} \frac{\varphi\prime(t)(\varphi\prime(t)+1)}{\varphi\prime(t)-\psi(t)+1}\,dt + \int_{x_0}^{\nu(u-y)} \frac{-\varphi\prime(t)(\varphi\prime(t)+1)}{\varphi\prime(t)-\psi(t)+1}\,dt + f(\xi_0) + g(\eta_0) = y$$

Taking into account that everywhere on the segment $[a, b] : f(\xi_0) + g(\eta_0) = 0$, whatever is the value of x on the segment $[a, b]$ including $x = x_0$, after some simple manipulations we can achieve the final form of the integral:

$$\int_{\nu(u-y)}^{\tau(u+x)} \frac{\varphi\prime(t)(\varphi\prime(t)+1)}{\varphi\prime(t)-\psi(t)+1}\,dt = y \tag{21}$$

It means, we proved the integral of our problem does not depend on an arbitrary constant. So, solution to the problem (6), (9) exists and it is expressed implicitly by (21). □

Remark 1 : The integral (21) can be written in an equivalent form:

$$\int_{\nu(u-y)}^{\tau(u+x)} -\frac{\varphi\prime(\varphi\prime+1)}{\varphi\prime-\psi+1}\,dt + \tau(u+x) = x \tag{22}$$

It is easy to show, because if we take an equivalent form (7) of the general integral and apply the same the process of constructing the integral for the problem, we obtain exactly (22). However, it is also easy to prove that from (21) we can obtain (22) directly.

The integral (21) (as well as (22)), allows us to describe the structure of both characteristic families. For the family of characteristics on which invariant $u + x$ preserves is a constant value, we can write the equation describing the family of curves:

$$\int_{\nu(\varphi(c)+c-x-y)}^{\tau(\varphi(c)+c)} \frac{\varphi\prime(\varphi\prime+1)}{\varphi\prime-\psi+1}\,dt = y. \tag{23}$$

This is equation of the family of characteristics which depends on one parameter c.

For the second family of characteristics we have the following expression:

$$\int_{\nu(\varphi(c))}^{\tau(\varphi(c)+x+y)} \frac{\varphi\prime(\varphi\prime+1)}{\varphi\prime-\psi+1}\,dt = y. \tag{24}$$

This equation also describes the family of characteristic curves with one parameter. This second family corresponds to the invariant $u - y$.

The obtained results allow us to describe the structure of areas of definition of solution in each specific case. For example, let us take the following functions:

$$\varphi = kx + l,\ \psi = p, \tag{25}$$

where k, l and p are constants. In other wards let φ function be linear and let ψ function be constant from (23) we obtain that the characteristic family of curve corresponding to invariant $u + x$ will be

$$\int_{(kc+c-x-y)/k}^{c} \frac{k(k+1)}{k-p+1}\, dt = y.$$

After simple calculations we have the equation of straight line:

$$y = -\frac{k+1}{p}(x-c),$$

where c is a parameter. Similarly, from (24) for the family of characteristics corresponding to invariant $u - y$ we can write

$$\int_{c}^{(kc+x+y)/(k+1)} \frac{k(k+1)}{k-p+1}\, dt = y.$$

Again, after simple calculations we obtain the equation of the straight line:

$$y = -\frac{k(x-c)}{(1-p)}.$$

It means, that in case of initial conditions of form (25), both families of characteristics are the families of straight lines. If we take a more specific example

$$\varphi = x + 1, \ \psi = 1,$$

it is easy to see that the characteristic families are

$$y = -2x + 2c, \ x = c$$

for $u + x$ and $u - y$ invariants consequently. The area of definition of solution in this case will be a parallelogram which consists of intersection points of these two families.

So, after solving the Cauchy problem (6),(9) – in case of the quasi-linear equation, we obtained, that the domain of definition of solution is a parallelogram bounded by straight lines

$$y = -2x + 2a, \ x = a, \ y = -2x + 2b, \ x = b.$$

It appears, that we obtained the similar situation to well-know equation $u_{xx} - u_{xx} = 0$, where the domain of definition of solution is a square, bounded by straight lines from families $x + y = const$ and $x - y = const$ [1].

Situation may appear different when initial conditions are both linear functions. For example, if

$$\varphi = x + 5, \ \psi = x + 2,$$

from (21) we have

$$y = log\frac{4(u-y-5)^2}{(u+x-5)^2}$$

and the regular solution does not exist on the straight line $y=-x$ (Fig.1.) From (23) and (24) we can see that characteristic curves of both families (see Fig. 2) intersect at the point $(-ln4, ln4)$. So, in this case we have a node type singular point.

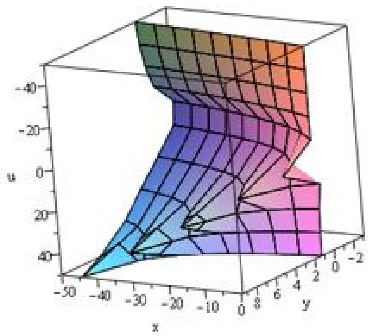

Figure 1. Solution

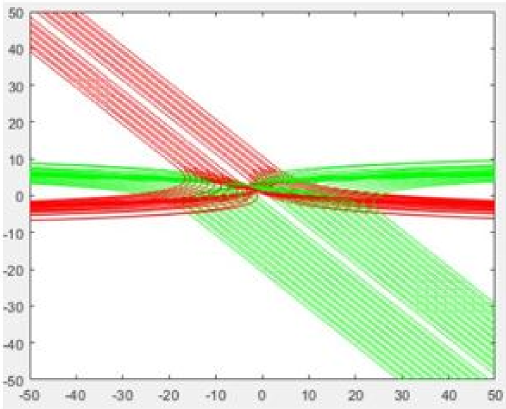

Figure 2. Characteristic curves

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