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On the Heun Equation Induced from Schwarz-Cristoffel Mapping

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We demonstrate that the Schwarz-Cristoffel mapping, which maps the given quadrilateral to the upper half-plane, is a solution to the general Heun equation. Furthermore, we provide an explicit formula for the accessory parameter of the Heun equation as a function of the singular point.

Keywords: General Heun equation, Hypergeometric function, Schwarz-Cristoffel mapping, quadrilateral

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1. Formulation of the main result

We are examining the simplest Fuchsian second-order equations, with particular attention to the role of the Schwarz-Cristoffel mapping for polygonal quadrilaterals [4]. The final form of the obtained differential equation is a Fuchsian equation with four regular singular points. It is known that the canonical form for such equations is the Heun equation [7].

In recent years, the family of Heun equations (general, confluent, double confluent, biconfluent, triconfluent) has become the subject of intensive research, not only due to its various applications in engineering sciences and mechanics (see [3], [8] and literature cited there), astrophysics (see for example [6],[5]) and mathematical physics [9] but also as an intriguing and significant entity in mathematics itself [7] [2]. The numerical analysis of Heun functions and their integration into computer algebra systems is still far from completion. On the other hand, achieving highly accurate numerical values of hypergeometric functions is feasible through computer algebra systems. Therefore, identifying a class of Heun equations whose solutions are expressed through special functions seems important to us.

The general Heun equation depends on one accessory parameter. In this case, as is known, the problem of monodromy arises depending on the accessory parameter (and therefore the solution), which is known as the isomonodromy problem. For

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example, for the Heun equation, monodromy-preserving deformation leads to the so-called Painleve transcendents [1].

A monodromy-preserving deformation of the Fuchsian system is the problem of determining the coefficient matrix in such a way that the monodromy of the given system remains unchanged when the locations of the singular points are varied. The deformation equation becomes a system of differential equations for the accessory parameters as functions of the positions of the singular points.

In this paper, we focus on a special type of the general Heun equation. In this case, the solutions of the Heun equation are expressed in terms of hypergeometric functions, with these functions arising from Schwarz-Christoffel mapping and possessing a profound geometric character. Namely, let Q be a polygonal quadrilateral with vertices w_1, w_2, w_3, w_4 enumerated in the clockwise order. Suppose the vertex w_4 lies at the origin and one side coincides with a line segment of the positive direction of the x-axis. Let the inner angles of Q at the vertices w_1, w_2, w_3, w_4 be $\pi\tau_1, \pi\tau_2, \pi\tau_3, \pi\tau_4$, respectively.

In this notation the following theorem hold.

Theorem 1.1 : *The solution of the Heun equation*

$$f''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{2}{z-a}\right) f'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} f(z) = 0,$$

are

$$f_j(z) = \frac{(z+1)^{\tau_2+\tau_4-1}}{z+a} w_j(z),$$

where $w_j, j = 1, 2, 3, 4$ are the vertices of the polygonal quadralateral Q and

$$w_1(z) = {}_2F_1\left(\tau_4, 1-\tau_3; \tau_1+\tau_4; -z\right),$$

$$w_2(z) = z^{\tau_3-1} {}_2F_1\left(\tau_2, 1-\tau_3; \tau_1+\tau_2; -\frac{1}{z}\right),$$

$$w_3(z) = z^{\tau_2+\tau_3-1} {}_2F_1\left(\tau_2, 1-\tau_1; \tau_2+\tau_3; -z\right),$$

$$w_4(z) = z^{-\tau_4} {}_2F_1\left(\tau_4, 1-\tau_1; \tau_3+\tau_4; -\frac{1}{z}\right).$$

The coefficients of the equation satisfy the relation $\alpha + \beta = \gamma + \delta + 1$,

$$\alpha = 1 + \tau_1, \quad \beta = 2 - \tau_2, \quad \gamma = \tau_1 + \tau_4, \quad \delta = \tau_1 + \tau_3$$

and the accessory parameter q and singular points a connected by the relation

$$q = (1 + a\tau_1)(\tau_2 - 1) + (\tau_3 - 1). \tag{1}$$

In the next section we give the proof of this Theorem.

2. Proof the theorem

Consider the conformal mapping of the lower half-plane onto a polygonal domain realized by the Schwarz-Cristoffel mapping (SC-mapping). Namely, suppose P be the interior of a polygon Γ with clockwise enumerated vertices $w_1, \ldots, w_n$ in the complex plane and with external angles $\pi\beta_1, \ldots, \pi\beta_n$. Let f be a conformal map of the lower half-plane $\mathbb{H}_-$ onto P with $f(\infty) = w_n$. Then

$$f(x) = A + C \int_{\infty}^{x} \prod_{j=1}^{n-1} \left(1 - \frac{\zeta}{z_j}\right)^{-\beta_j} d\zeta, \tag{2}$$

for some complex constants A and C, where $w_k = f(z_k)$ and the preimages z_j of the vertices w_j, for $j = 1, \ldots, n-1$ satisfy the conditions $1 = z_1 < z_2 < \cdots < z_{n-1} < z_n = \infty$.

Consider a particular case of the mapping (2). Namely, let Q be a quadrilateral with inner angles $\pi\tau_j$ enumerated in the clockwise order. Suppose

$$z_1 = 1, \quad z_2 = 1 + \theta, \quad z_3 = 1 + \theta + r\theta,$$

where $\theta, r > 0$ is a parametrization of preimages of the vertices of the quadrilateral Q, i.e., the inequality

$$1 = z_1 < z_2 < z_3$$

is satisfied. Then the Schwarz-Christoffel transformation is given by

$$f(x) = A + C \int_{\infty}^{x} (1-\zeta)^{\tau_1 - 1} \left(1 - \frac{\zeta}{1+\theta}\right)^{\tau_2 - 1} \left(1 - \frac{\zeta}{1+\theta+r\theta}\right)^{\tau_3 - 1} d\zeta, \tag{3}$$

$A, C \in \mathbb{C}$.

Let Q be a polygonal quadrilateral with vertices w_1, w_2, w_3, w_4 enumerated in the clockwise order. As in the section 1 suppose the vertex w_4 lies at the origin and one side coincides with a line segment of the positive direction of the x-axis. Let the inner angles of Q at the vertices w_1, w_2, w_3, w_4 are $\pi\tau_1, \pi\tau_2, \pi\tau_3, \pi\tau_4$, respectively. If we take $A = 0$ and $C = 1$ in (3), then the function

$$g(x) = \int_{\infty}^{x} (1-\zeta)^{\tau_1 - 1} \left(1 - \frac{\zeta}{1+\theta}\right)^{\tau_2 - 1} \left(1 - \frac{\zeta}{1+\theta+r\theta}\right)^{\tau_3 - 1} d\zeta, \quad r, \theta > 0.$$

maps the lower half-plane onto quadrilateral Q, described above, and the side lengths of Q are given by the following formulas

$$l_1 = cB(\tau_4, \tau_1)\, {}_2F_1\left(\tau_4, 1-\tau_3; \tau_4+\tau_1; -r\right),$$

$$l_2 = r^{\tau_3 - 1} cB(\tau_1, \tau_2)\, {}_2F_1\left(\tau_2, 1-\tau_3; \tau_1+\tau_2; -\frac{1}{r}\right),$$

$$l_3 = r^{\tau_2+\tau_3-1} cB(\tau_2,\tau_3)\,{}_2F_1\left(\tau_2, 1-\tau_1; \tau_2+\tau_3; -r\right),$$

$$l_4 = r^{-\tau_4} cB(\tau_3,\tau_4)\,{}_2F_1\left(\tau_4, 1-\tau_1; \tau_3+\tau_4; -\frac{1}{r}\right),$$

where

$$c = \theta^{-\tau_4}(1+\theta)^{1-\tau_2}(1+\theta+r\theta)^{1-\tau_3}$$

and the ratio of the lengths of the adjacent sides of the quadrilateral is independent of θ. Here B is the Euler beta-function [4].

If we denote by

$$w(z) = {}_2F_1\left(a, b; c; z\right)$$

the solution of the standard hypergeometric differential equation

$$z(z-1)\frac{d^2w}{dz^2} + ((a+b+1)z - c)\frac{dw}{dz} + abw = 0 \tag{4}$$

then the function

$$w(z) = {}_2F_1\left(\tau_4, 1-\tau_3; \tau_1+\tau_4; -z\right)$$

will be the solution of the following equation

$$z(1+z)\frac{d^2w}{dz^2} + ((\tau_4+1-\tau_3+1)z + \tau_1+\tau_4)\frac{dw}{dz} + \tau_4(1-\tau_3)w = 0,$$

which can be rewritten in the following form

$$\frac{d^2w}{dz^2} + \left(\frac{\tau_4-\tau_3+2}{z+1} + \frac{\tau_1+\tau_4}{z} - \frac{\tau_1+\tau_4}{z+1}\right)\frac{dw}{dz} + \frac{\tau_4(1-\tau_3)}{z(1+z)}w = 0.$$

After some simplification, we obtain

$$\frac{d^2w}{dz^2} + \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1}\right)\frac{dw}{dz} + \frac{\tau_4(1-\tau_3)}{z(1+z)}w = 0, \tag{5}$$

where $\tau_1+\tau_2+\tau_3+\tau_4 = 2$.

We interpret the solution

$$w_1(z) = {}_2F_1\left(\tau_4, 1-\tau_3; \tau_1+\tau_4; -z\right)$$

of the equation (5) as the vertice w_1 which we obtain using the Schwarz-Christoffel mapping technique, while the other three vertices also satisfy the equation (5):

$$w_2(z) = z^{\tau_3-1}{}_2F_1\left(\tau_2, 1-\tau_3; \tau_1+\tau_2; -\frac{1}{z}\right),$$

$$w_3(z) = z^{\tau_2+\tau_3-1}{}_2F_1\left(\tau_2, 1-\tau_1; \tau_2+\tau_3; -z\right),$$

$$w_4(z) = z^{-\tau_4}{}_2F_1\left(\tau_4, 1-\tau_1; \tau_3+\tau_4; -\frac{1}{z}\right).$$

Lemma 2.1:
Assume that equation (5) *has a solution* $w(z)$. *Then the function* $f(z) = g(z)w(z)$ *is the solution of the equation*

$$\begin{gathered} f'' + \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1} - \frac{2g'}{g}\right)f' + \\ \left(\frac{2(g')^2 - gg''}{g^2} - \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1}\right)\frac{g'}{g} + \frac{\tau_4(1-\tau_3)}{z(1+z)}\right)f = 0. \end{gathered} \tag{6}$$

Indeed,

$$f' = g'w + gw' = \frac{g'}{g}f + gw' \Rightarrow w' = \frac{f'}{g} - \frac{g'}{g^2}f,$$

$$f'' = g''w + 2g'w' + gw'' = \frac{g''}{g}f + \frac{2g'}{g}f' - \frac{2(g')^2}{g^2}f + gw''$$

$$\Rightarrow w'' = \frac{f''}{g} - \frac{2g'}{g^2}f' + \frac{2(g')^2 - gg''}{g^3}f.$$

Substitute these results into equation (5):

$$\frac{f''}{g} - \frac{2g'}{g^2}f' + \frac{2(g')^2 - gg''}{g^3}f + \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1}\right)\left(\frac{f'}{g} - \frac{g'}{g^2}f\right) + \frac{\tau_4(1-\tau_3)}{z(1+z)}\frac{f}{g} = 0.$$

From this it follows, that $f(z)$ satisfies the equation

$$f'' - \frac{2g'}{g}f' + \frac{2(g')^2 - gg''}{g^2}f + \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1}\right)f' - \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1}\right)\frac{g'}{g}f$$

$$+\frac{\tau_4(1-\tau_3)}{z(1+z)}f = 0.$$

Hence, we obtain equation (6).

Lemma 2.2: *Let $g(z) = (z+1)^m(z+a)^n$. Then the function $f(z)$ from Lemma 2.1 is a solution of the equation*

$$\begin{aligned} f'' &+ \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_1+\tau_3}{z+1} + \frac{2}{z+a}\right) f' \\ &+ \frac{(1+\tau_1)(2-\tau_2)z + ((1+a\tau_1)(1-\tau_2)+(1-\tau_3))}{z(z+1)(z+a)} f = 0. \end{aligned} \tag{7}$$

The proof of Lemma 2.2 follows from the following obvious equalities. Namely,

$$\frac{g'}{g} = \frac{m}{z+1} + \frac{n}{z+a},$$

$$\frac{2(g')^2 - gg''}{g^2} = \frac{m(m+1)}{(z+1)^2} + \frac{n(n+1)}{(z+a)^2} + \frac{2mn}{(z+1)(z+a)}.$$

We need to set n and m non-zero parameters so that the following identity hold

$$\frac{2(g')^2 - gg''}{g^2} - \left(\frac{\tau_1+\tau_4}{z} + \frac{\tau_2+\tau_4}{z+1}\right)\frac{g'}{g} + \frac{\tau_4(1-\tau_3)}{z(1+z)} = \frac{Az+B}{z(z+1)(z+a)},$$

where the left hand side of the last equation is the coefficient of $f(z)$ in the equation (7) and A and B are some constants. After this, we make the choice of the numbers n and m such that they satisfy the identity

$$Az + B = \frac{m((m+1)-(\tau_2+\tau_4))z(z+a)}{(z+1)} + \frac{n(n+1)z(z+1)}{(z+a)}$$

$$+n(2m-(\tau_2+\tau_4))z - n(\tau_1+\tau_4)(z+1) + (\tau_4(1-\tau_3) - m(\tau_1+\tau_4))(z+a).$$

It is possible only when $n(n+1) = 0$ and $m((m+1)-(\tau_2+\tau_4)) = 0$, i.e, $n = -1$ and $m = \tau_2+\tau_4-1$. From this it follows that

$$Az + B = (1+\tau_1)(2-\tau_2)z + ((1+a\tau_1)(1-\tau_2) + (1-\tau_3)).$$

The Lemma 2.2 is proved.

To complete the proof of the Theorem 2.1, it is sufficient to introduce the notation provided in the theorem for the coefficients of the equation (7). In particular, denote by $\alpha = 1+\tau_1$, $\beta = 2-\tau_2$, $\gamma = \tau_1+\tau_4$, $\delta = \tau_1+\tau_3$, then $\alpha+\beta-\gamma-\delta = 1$ and for the accessory parameter and singular point a we obtain the relation (1).

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