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Application of Deep Neural Network for Numerical Approximation for Averaged Nonlinear Integro-Differential Equation

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Numerical approximation of the nonlinear averaged integro-differential model associated with the penetration of a magnetic field into a substance is studied. The statements for large-time behavior of solutions and uniqueness are given too. A novel strategy involving the application of machine learning techniques is proposed for the numerical estimation. The numerical solutions are compared to the exact ones.

Keywords: Nonlinear integro-differential equations, large time behavior, numerical solution, machine learning algorithms, deep neural networks.

AMS Subject Classification: 45K05, 35K55, 68T05, 68T07.

1. Introduction

Simulation of the wide range of applied problems results to nonlinear integro-differential equations or systems of such equations. These equations, involving partial derivatives of the unknown function, often include integrals of the function and its derivatives. Many scientific works focus on investigating and approximate solution of such models (see, for example, [1] - [7] and references therein). These equations find applications in various fields, including the mathematical modeling of electromagnetic field penetration into substances. A variable magnetic field propagated into a medium induces a corresponding electric field, which generates currents that heat the material, subsequently affecting its resistance. It is crucial to account for this resistance when dealing with significant temperature oscillations. In the quasi-stationary scenario, the corresponding system of Maxwell's equations takes the form (see, for example, [8]):

$$\frac{\partial H}{\partial t} = -rot(\nu_m rotH), \tag{1}$$

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$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m \left(rotH\right)^2. \qquad (2)$$

Here, $H = (H_1, H_2, H_3)$ is the magnetic field vector, θ represents temperature, and c_ν and ν_m are parameters denoting the thermal heat capacity and electro-conductivity of the material, respectively. System (1) characterizes the diffusion of the magnetic field, while equation (2) describes temperature changes due to Joule heating, excluding thermal conductivity considerations.

When c_ν and ν_m are functions of the temperature θ, meaning $c_\nu = c_\nu(\theta)$ and $\nu_m = \nu_m(\theta)$, above-mentioned equations can be reformulated as in [9]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2 \, d\tau\right) rotH\right], \qquad (3)$$

where the function $a = a(S)$ is defined for $S \in [0, \infty)$.

Studies of models like (3) began in [9], where the existence of a generalized solution to the first boundary value problem for the one-dimensional space case was demonstrated for $a(S) = 1 + S$, along with uniqueness proofs for more general cases.

In [10], a generalization of the system similar to (3) was proposed, where the temperature within the body was considered constant and dependent only on time, resulting in an, so called, averaged integro-differential system to model the magnetic field's penetration. This model is expressed as [10]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t \int_0^1 |rotH|^2 \, dx d\tau\right) rotH\right]. \qquad (4)$$

Assuming a magnetic field H of the form $H = (0, 0, U)$, where $U = U(x,t)$ is a scalar function dependent on time and a single spatial variable, the curl of H becomes $rotH = \left(0, -\frac{\partial U}{\partial x}, 0\right)$ and thus, the one-dimensional analog of the system (4) simplifies to:

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right) \frac{\partial^2 U}{\partial x^2}. \qquad (5)$$

It's important to note that the study of integro-differential equations like (3) and (4) is quite complex, and only special cases have been investigated so far. Studies on the existence and uniqueness of solutions for the initial-boundary value problems related to these equations can be found in [9] - [14], where theorems of existence, based on a priori estimates and using methods such as Galerkin's method and compactness arguments, are discussed. Note that the existence and uniqueness of the solutions of the initial-boundary value problems for the equations of type (5) were first studied in [14]. The existence theorems demonstrated in these studies rely on methods involving a priori estimates, utilizing techniques similar to those employed for nonlinear parabolic equations as discussed in [15] and [16].

The semi-discrete and finite difference schemes as well as Galerkin finite element approximation for numerical resolution of the initial-boundary value problems to the (5) type models and models like them have been explored in [17] - [22]. It is also worth mentioning that based on researches given in [23] and [24], in [25] and [26] the machine learning approaches such as Gaussian Processes and Neural Network applications for linear partial differential equations (PDEs) were studied correspondingly. Additionally, the discussions on the questions of existence, uniqueness, regularity as well as an asymptotic behavior of the solutions of the initial-boundary value problems to the models similar to equation (5) can be found, for example, in [22], [27] - [34] and references therein.

This paper aims to propose a novel strategy involving the application of machine learning techniques for the numerical approximation of the initial-boundary value problem with homogenous Dirichlet boundary conditions for equation (5) in the case where the function $a(S)$ is defined as $1+S$. In particular, the goal is to use a Deep Neural Network (DNN) to approximate the solution of the initial-boundary value problem of the nonlinear integro-differential equation with a specific diffusion coefficient.

The rest of the article is organized as follows: In the next section, the problem statement is done and theorems of large time behavior of solution and uniqueness are stated. Section three discusses numerical approximation techniques by applying a DNN to solve a nonlinear integro-differential equation (5).

2. Problem statement. Some asymptotic estimations and uniqueness of solution

In the infinite cylinder $(0,1)\times(0,\infty)$ let us consider the following initial boundary value problem for the nonlinear integro-differential equation (5):

$$\begin{aligned} &U(0,t)=U(1,t)=0, \quad t\geq 0, \\ &U(x,0)=U_0(x), \quad x\in[0,1], \end{aligned} \tag{6}$$

where $U_0(x)$ is the given initial condition.

We assume that $U=U(x,t)$ is a solution to the problem (5), (6) that meets certain continuity and integrability prerequisites, which are detailed further in terms of Sobolev spaces. In particular, we presume that $U=U(x,t)$ is a solution of the problem (5), (6) on $[0,1]\times[0,\infty)$ such that $U(\cdot,t)$, $\frac{\partial U(\cdot,t)}{\partial x}$, $\frac{\partial U(\cdot,t)}{\partial t}$, $\frac{\partial^2 U(\cdot,t)}{\partial x^2}$, $\frac{\partial^2 U(\cdot,t)}{\partial t\,\partial x}$ are all in $C^0([0,\infty);L_2(0,1))$, while $\frac{\partial^2 U(\cdot,t)}{\partial t^2}$ is in $L_2((0,\infty);L_2(0,1))$.

The following statements addresses the asymptotic behavior and uniqueness of solution [14], [17], [18], [28] - [32].

Theorem 2.1 : *If $U_0\in H_0^1(0,1)$, then the solution of the problem (5), (6) satisfies the following estimate*

$$\|U\|_{L_2(0,1)}+\left\|\frac{\partial U}{\partial x}\right\|_{L_2(0,1)}\leq C\exp\left(-\frac{t}{2}\right).$$

Here and below in this section C denote positive constants independent from t

and usual $L_2(0,1)$ and Sobolev spaces $H^k(0,1)$, $H_0^k(0,1)$ are used.

Note that Theorem 2.1 gives exponential stabilization of the solution of the problem (5), (6) in the norm of the space $H^1(0,1)$. The following theorem shows the stabilization of solution in the norm of the space $C^1(0,1)$.

Theorem 2.2: *If $U_0 \in H^4(0,1) \cap H_0^1(0,1)$, then the solution of the problem (5), (6) satisfies the following estimates:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C \exp\left(-\frac{t}{2}\right), \qquad \left|\frac{\partial U(x,t)}{\partial t}\right| \leq C \exp\left(-\frac{t}{2}\right).$$

Remark: Globally defined solutions for the problem (5), (6) can be acquired through a standard process. Initially, local solutions are established over a maximal time span. Subsequently, derived a-priori estimates are employed to demonstrate that the solutions remain bounded within a finite time frame, as discussed in [15] and [16].

Theorem 2.3: *If problem (5), (6) has a solution then it is unique.*

As usual, to prove the uniqueness of solution it is assumed that there exist two different U_1 and U_2 solutions of problem (5), (6) and $\|U_1 - U_2\|_{L_2(0,1)} \equiv 0$ equivalency is proven. The following identity is mainly used to prove the uniqueness theorem [22].

$$\left\{\left(1+\int_0^t\int_0^1\left(\frac{\partial U_2}{\partial x}\right)^2 dx d\tau\right)\frac{\partial U_2}{\partial x}\right.$$

$$\left.-\left(1+\int_0^t\int_0^1\left(\frac{\partial U_1}{\partial x}\right)^2 dx d\tau\right)\frac{\partial U_1}{\partial x}\right\}\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)$$

$$=\int_0^1\frac{d}{d\mu}\left\{1+\int_0^t\int_0^1\left[\frac{\partial U_1}{\partial x}+\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right]^2 dx d\tau\right\}$$

$$\times\left[\frac{\partial U_1}{\partial x}+\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right)\right] d\mu\left(\frac{\partial U_2}{\partial x}-\frac{\partial U_1}{\partial x}\right).$$

3. Numerical approximation

The proposed note seeks to build upon the research initiated in [25] and [26], utilizing a Deep Neural Network to address the numerical solution of the following nonlinear integro-differential initial-boundary value problem:

$$
\begin{gathered}
\frac{\partial U(x,t)}{\partial t} - \left(1 + \int_0^t \int_0^1 \left(\frac{\partial U(x,t)}{\partial x}\right)^2 dx d\tau\right) \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t),\ (x,t) \in Q, \\
U(0,t) = U(1,t) = 0, \quad t \in [0,\ T], \\
U(x,0) = U_0(x), \quad x \in [0,\ 1],
\end{gathered} \tag{7}
$$

in which Q defines the domain $(0,\ 1) \times (0,\ T)$ with T being a positive constant, and f and U_0 represent prescribed functions.

The literature extensively covers both the qualitative and quantitative attributes and computational solutions for the problem (7) and their more complex nonlinear variants (see, for example, [9] - [14], [17], [18], [27], [34] and references therein). One of the approaches for numerical solution of parabolic partial differential equation was suggested in [25] using the Gaussian Process as a numerical approximation method for the Heat Equation. As mentioned earlier, we are interested in exploring the capabilities of Machine Learning techniques, particularly DNNs, as an alternative method for solving PDEs. The goal is to train a DNN to predict the solution at any given point $(x,\ t)$ within the domain Q. A DNN may contain several layers, including the input and output layers, with numerous hidden layers in between (as exemplified in Fig. 1). The depth of the network correlates to the number of these hidden layers.

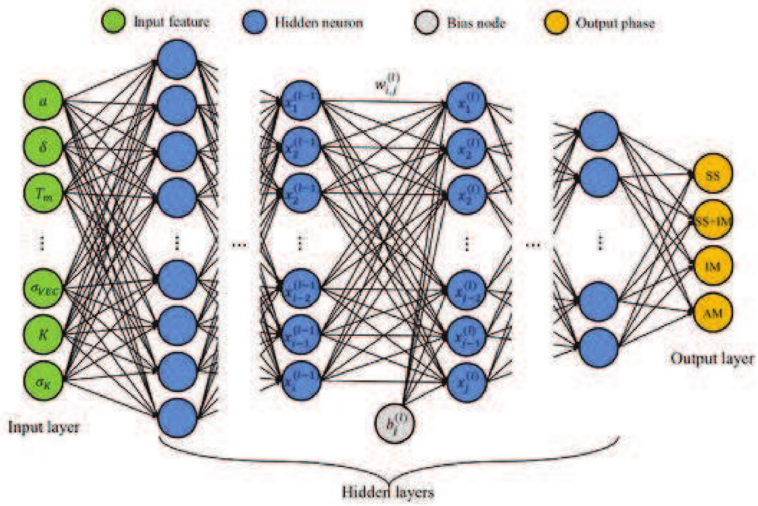

Figure 1. Illustration of Neural Network architecture.

The DNN seeks to formulate an approximation of the problem (7) such that $u(x,\ t,\ \rho) \approx U(x,t)$, where $u(x,t,\rho)$ signifies the output from the DNN, and ρ encompasses all the adjustable parameters of the DNN which are optimized during the learning process. As noted in [26], training a DNN requires a substantial dataset. However, the advantage of employing DNNs in solving PDEs lies in the ability to infuse the training with physical laws, effectively reducing the amount of data required for successful training as discussed in [23], [24].

Adapting the approach from [23], [24], [26], the residual for the nonlinear equation (7) to be evaluated at predefined training points is given as

$$R(x,t,\rho) = \frac{\partial u(x,t,\rho)}{\partial t} - \left(1 + \int_0^t \int_0^1 \left(\frac{\partial u(x,\,t,\,\rho)}{\partial x}\right)^2 dx d\tau\right) \frac{\partial^2 u(x,\,t,\,\rho)}{\partial x^2} - f(x,t). \tag{8}$$

Furthermore, a loss function $\mathcal{F}(x,t,\rho)$ that includes the residual (8) and the initial and boundary conditions can be devised. This function is subject to optimization by a DNN throughout its training phase.

In our first experiment for the right-hand side function $f(x,t)$ in problem (7), we have chosen a form that generates a known exact solution, specifically

$$U(x,t) = x(1-x)\exp(-x-t),$$

and the initial condition is similarly derived as

$$U_0(x) = x(1-x)\exp(-x).$$

The neural network's training procedure was executed using the NumPy package, which is tailored for scientific computations, and TensorFlow, a framework dedicated to machine learning applications. These package are wrapped in the Jupyter Notebook script [23].

Figure 2 displays the exact (left) and numerical solution of problem (7), while plots in Figure 3 show error between exact and approximate solutions (left) and asymptotic behavior of solution of problem (7) (right) correspondingly.

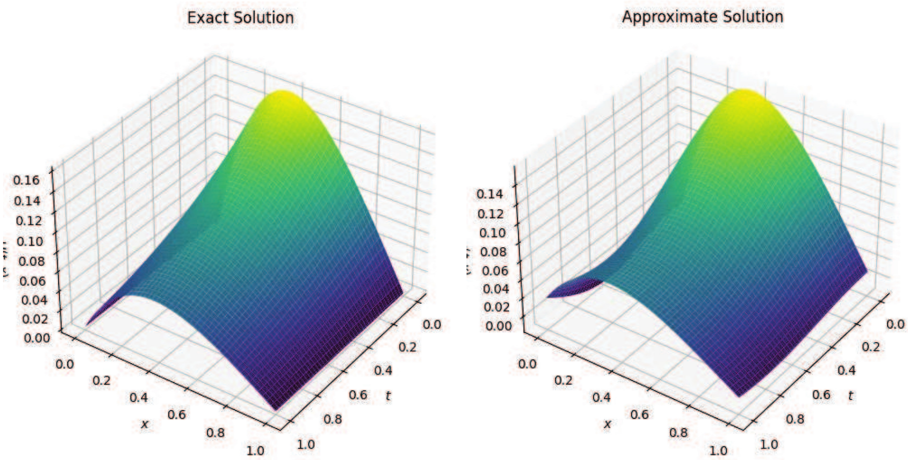

Figure 2. Comparison of exact and approximate solutions.

The picture for asymptotic behavior is drawn from our second experiment once considered right-hand side of the equation in (7) as $f(x,\,y) = 0$, and the initial function is given as follows

$$U(x,\,0) = x(1-x) + x(e^{-x} - e^{-1}cos(10\pi x)).$$

The right plot in Figure 3 shows the vanishing of the solution of problem (7) when $t \to \infty$ as it was expected due to Theorems 2.2 and 2.3.

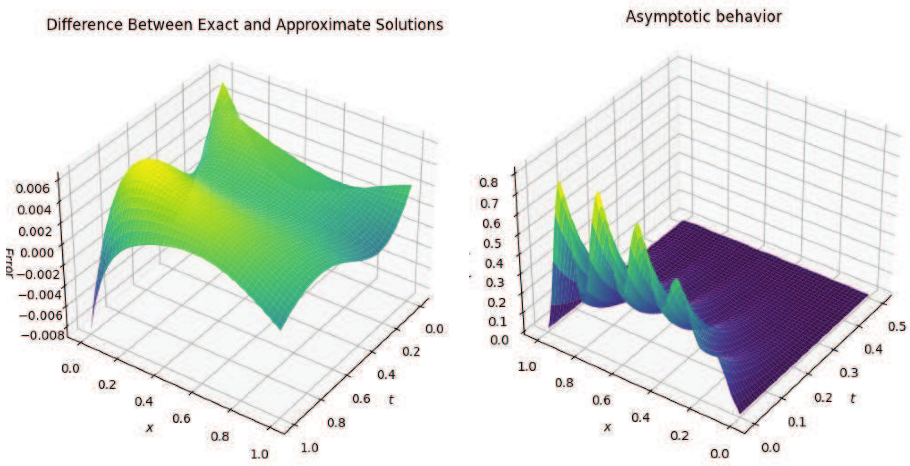

Figure 3. Error surface for exact and approximated (left) and asymptotic behavior of solution (right).

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