ON AN INTEGRO-DIFFERENTIAL EQUATION WITH A SYMMETRIC KERNEL

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Abstract. The present paper deals with the solvability of the Cauchy problem of some classes of integro-differential equations.

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Problem of a cylindrical vibration of elastic plate can be reduced to the following integro-differential equation (see, e.g.., [1])

$$\varphi(x,t) + \int\limits_0^l K(x,\xi)\varphi_{,tt}(\xi,t)d\xi = f(x,t), \quad x \in [0,l], \quad t > 0, \tag{1}$$

where $\varphi(\cdot,t) \in C([0,l])$, $\varphi(x,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0)$, $\varphi(x,t) \in C(0 \leq c \leq l,\, t \geq 0)$; $K(x,\xi) \in C([0,l] \times [0,l])$ is defined in the explicit form, and $f(x,t) \in C(0 \leq c \leq l,\, t \geq 0)$ is a given function.

We solve (1) under the following initial conditions

$$\varphi(x,0) = \varphi_1(x), \quad \varphi_{,t}(x,0) = \varphi_2(x), \quad x \in [0,l], \tag{2}$$

where $\varphi_i(x) \in C([0,l])$ $(i = 1,2)$ are given functions. By $\varphi_{,t}(x,t)$ we denote $\varphi_{,t}(x,t) := \frac{\partial\varphi(x,t)}{\partial t}$.

Let us consider the case when $K(x,\xi)$ isn't a degenerate kernel (case of the degenerate kernel is investigated in [2]) and $K(x,\xi)$ is symmetric with respect to x and ξ, i.e., $K(x,\xi) = K(\xi,x)$.

Let firstly $f(x,t) \equiv 0$ and let us consider the following integral equation

$$X(x) = \lambda \int\limits_0^l K(x,\xi)X(\xi)d\xi. \tag{3}$$

We denote by λ_n and X_n corresponding eigenvalues and eigenfunctions of (3). It is known that because of $K(x,\xi)$ isn't a degenerate kernel, the number of eigenvalues of (3) isn't finite, all λ_n are real numbers, and system of $X_n(x)$ is ful (see, e.g., [3]). Without loss of generality it can be assumed that $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \dots$ (see [3]).

Furthermore, from (1) we obtain

$$\varphi(x,t) = -\int\limits_0^l K(x,\xi)\varphi_{,tt}(\xi,t)d\xi =: \int\limits_0^l K(x,\xi)u(\xi,t)d\xi, \tag{4}$$

according to the Hilbert-Schmidt theorem $\varphi(x,t)$ can be expressed as an absolutely and uniformly convergent series

$$\varphi(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t), \tag{5}$$

where

$$T_n(t) = \int\limits_0^l \varphi(x,t)X_n(x)dx. \tag{6}$$

By virtue of (6) and initial conditions (2) we obtain

$$T_n(0) = \int\limits_0^l \varphi_1(x)X_n(x)dx, \quad T_n'(0) = \int\limits_0^l \varphi_2(x)X_n(x)dx. \tag{7}$$

Let us consider expression (6), in view of (4) and (3) we get

$$\begin{aligned} T_n(t) &= -\int\limits_0^l X_n(x)\int\limits_0^l K(x,\xi)\varphi_{,tt}(\xi)d\xi dx = -\frac{1}{\lambda_n}\int\limits_0^l \varphi(x,t)X_n(x)dx \\ &= -\frac{1}{\lambda_n}\int\limits_0^l \varphi_{,tt}(x)X_n(x)dx = -\frac{1}{\lambda_n}T_n''(t). \end{aligned}$$

Hence, for $T_n(t)$ we have the following equation

$$T_n''(t) + \lambda_n T_n(t) = 0, \tag{8}$$

under initial conditions (7). Solution of the problem (8)-(7) has the following form

$$T_n(t) = b_1^n \cos(\sqrt{\lambda_n}t) + b_2^n \sin(\sqrt{\lambda_n}t), \quad b_i^n = \text{const}, \quad \text{i} = 1,2,$$

where

$$b_1^n = \int\limits_0^l X_n(x)\varphi_1(x)dx, \quad b_2^n = \frac{1}{\sqrt{\lambda_n}}\int\limits_0^l X_n(x)\varphi_2(x)dx.$$

So, (5) can be rewritten as follows

$$\varphi(x,t) = \sum_{n=1}^{\infty}\left(b_1^n \cos(\sqrt{\lambda_n}t) + b_2^n \sin(\sqrt{\lambda_n}t)\right)X_n(x). \tag{9}$$

After formal differentiation of the last series with respect to t we have

$$\varphi_{,t}(x,t) = \sum_{n=1}^{\infty} X_n \sqrt{\lambda_n} \left(b_2^n \cos(\sqrt{\lambda_n} t) - b_1^n \sin(\sqrt{\lambda_n} t) \right), \tag{10}$$

$$\varphi_{,tt}(x,t) = -\sum_{n=1}^{\infty} X_n \lambda_n \left(b_1^n \cos(\sqrt{\lambda_n} t) + b_2^n \sin(\sqrt{\lambda_n} t) \right). \tag{11}$$

Theorem 1 *The series (10) and (11) are convergent absolutely and uniformly on $[0,l]$ if $\varphi_i(x)$ $(i = 1,2)$ can be expressed as an absolutely and uniformly convergent series as follows*

$$\varphi_i(x) = \int_0^l K(x,\xi) \int_0^l K(\xi,\eta)\psi_i(\eta)d\eta d\xi =: \int_0^l K(x,\xi)\chi_i(\xi)d\xi, \tag{12}$$

for any $\psi_i(x)$ piece $-$ wise continious function on $[0,l]$, $i = 1,2$.

Proof. From (10) we get

$$|\varphi_{,t}(x,t)| \le \sum_{n=1}^{\infty} |\sqrt{\lambda_n} X_n(x) b_1^n| + \sum_{n=1}^{\infty} |\sqrt{\lambda_n} X_n(x) b_2^n|.$$

Let us now consider the first term of the last inequality

$$\sum_{n=1}^{\infty} \left| \sqrt{\lambda_n} X_n(x) b_1^n \right| = \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n} X_n(x) \int_0^l X_n(\xi)\varphi_1(\xi)d\xi \right|$$

$$= \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n} X_n(x) \int_0^l X_n(\xi) \int_0^l K(\xi,\eta)\chi_1(\eta)d\eta d\xi \right|$$

$$= \sum_{n=1}^{\infty} \left| X_n(x)\sqrt{\lambda_n} \int_0^l \chi_1(\eta) \int_0^l K(\xi,\eta)X_n(\xi)d\xi d\eta \right|$$

$$= \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{\lambda_n}} X_n(x) \int_0^l X_n(\eta)\chi_1(\eta)d\eta \right| \le \frac{1}{\sqrt{|\lambda_1|}} \sum_{n=1}^{\infty} \left| X_n(x) \int_0^l X_n(\eta)\chi_1(\eta)d\eta \right|$$

(in view of (12) and Hilbert $-$ Schmidt theorem) $< +\infty$.

Analogously, we can proof the absolutely and uniformly convergence of the second term of (10) and of the series (11).

□

Thus, differentiation of (9) is justified.

Evidently, (9) is the solution of (1), (2) for $f(x,t) \equiv 0$.

Now, let us consider problem (1)-(2) when $f(x,t) \not\equiv 0$, $\varphi_i = 0$, $i = 1,2$, and let $f(x,t)$ be expressed as follows

$$f(x,t) = \int_0^l K(x,\xi) \int_0^l K(\xi,\eta)\psi(\eta)d\eta d\xi =: \int_0^l K(x,\xi)\chi(\xi,t)d\xi, \tag{13}$$

for any $\psi(x)$ piece $-$ wise continious function on $[0,l]$.

Then $f(x,t)$ can be replaced as a convergent series

$$f(x,t) = \sum_{n=1}^{\infty}(f(x,t), X_n(x))X_n(x),$$

here,

$$f(x,t) = \sum_{n=1}^{\infty} X_n(x)f_n(t), \quad f_n(t) := \int_0^l f(x,t)X_n(x)dx.$$

Further, we look for the solution in the form

$$\varphi(x,t) = \sum_{n=1}^{\infty} \Phi_n(x,t),$$

where $\Phi_n(x,t)$ is a solution of the problem (1)-(2) with $f(x,t)$ replaced by $X_n(x)f_n(t)$. Because of (13), we can rewrite (1) as follows

$$\varphi(x,t) = \int_0^l K(x,\xi)\left[-\varphi,_{tt}(\xi,t) + \chi(\xi,t)\right]d\xi$$

So, $\Phi_n(x,t)$ has the following form

$$\Phi_n(x,t) = X_n(x)T_{1n}(t),$$

where

$$T''_{1n}(t) + \lambda_n T_{1n}(t) = f_n(t),$$

$X(x)$ satisfies (3).

Therefore, $\varphi(x,t)$ can be expressed as an absolutely and uniformly convergent series

$$\varphi(x,t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} X_n(x) \int_o^l \sin(\sqrt{\lambda_n}(t-\tau))f_n(\tau)d\tau.$$

Now, similarly to the Theorem 1, if condition (13) is fulfilled, we have the absolutely and uniformly convergent of the following series

$$\varphi,_t(x,t) = \sum_{n=1}^{\infty} X_n(x) \int_0^t \cos(\sqrt{\lambda_n}(t-\tau))f_n(\tau)d\tau,$$

$$\varphi_{,tt}(x,t) = -\sum_{n=1}^{\infty} \sqrt{\lambda_n} X_n(x) \int_0^t \sin(\sqrt{\lambda_n}(t-\tau)) f_n(\tau) d\tau.$$

Remark 2 *Let $f(x,t)$, $\varphi_i(x) \not\equiv 0$ $i = 1, 2$. If the following condition*

$$f(x,t) + \varphi_1(x) + t\varphi_2(x) = \int_0^l K(x,\xi) \int_0^l K(\xi,\eta)\psi(\eta) d\eta d\xi,$$
$$\text{for any } \psi(x) \text{ piece} - \text{wise continious function on } [0,l].$$

is satisfyed then solution of (1)-(2) can be expressed as follows

$$\varphi(x,t) = \sum_{n=1}^{\infty} \Psi_n(x,t),$$

where

$$\Psi_n(x,t) = X_n(x) T_{2n}(t),$$
$$T''_{2n}(t) + \lambda_n T_{2n}(t) = f_n(t),$$
$$f_n(t) := \int_0^l (f(x,t) + \varphi_1(x) + t\varphi_2(x)) X_n(x) dx.$$

Theorem 3 *The solution of the problem (1)-(2) is unique.*

Proof. Let the difference of two admissible solutions of the problem under consideration be $\varphi(x,t)$.

For this function we get the following equation

$$\varphi(x,t) + \int_0^l K(x,\xi) \varphi_{,tt}(\xi,t) d\xi = 0, \tag{14}$$

under the following initial conditions

$$\varphi(x,0) = 0, \quad \varphi_{,t}(x,0) = 0. \tag{15}$$

For $\varphi(x,t)$ we have (5), where $T_n(t)$ is given by the equation (8), and it satisfies homogeneous initial conditions (15), i.e., $T_n(t) \equiv 0$.

Hence, problem (14), (15) has only trivial solution. □

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