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CONVERGENCE OF NEW MODIFIED TRIGONOMETRIC SUMS IN THE METRIC SPACE L

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ABSTRACT. We introduce here new modified cosine and sine sums as

$$\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle(a_j\cos jx)$$

and

$$\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle(a_j\sin jx)$$

and study their integrability and L^1-convergence. The L^1-convergence of cosine and sine series have been obtained as corollary. In this paper, we have been able to remove the necessary and sufficient condition $a_k\log k=o(1)\ \ as\ \ k\to\infty$ for the L^1-convergence of cosine and sine series.

1. INTRODUCTION

Consider cosine and sine series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx \tag{1.1}$$

and

$$\sum_{k=1}^{\infty}a_k\sin kx \tag{1.2}$$

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or together

$$\sum_{k=1}^{\infty} a_k \phi_k(x) \tag{1.3}$$

Where $\phi_k(x)$ is $\cos kx$ or $\sin kx$ respectively. Let the partial sum of (1.3) be denoted by $S_n(x)$ and $t(x) = \lim_{n\to\infty} S_n(x)$. Further, let $t^r(x) = \lim_{n\to\infty} S_n^r(x)$ where $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Definition 1.1. A sequence $\{a_k\}$is said to convex if $\triangle^2 a_k \geq 0$, where $\triangle^2 a_k = \triangle(\triangle a_k)$ and $\triangle a_k = a_k - a_{k+1}$, and quasi-convex sequence if $\sum (k+1)\triangle^2 a_k < \infty$.

The concept of quasi-convex was generalized by Sidon [4] in the following manner:

Definition 1.2. [4] A null sequence $\{a_k\}$ is said to belong to class S if there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, \;\; k \to \infty, \tag{1.4}$$

$$\sum_{k=0}^{\infty} A_k < \infty, \tag{1.5}$$

and

$$|\Delta a_k| \leq A_k, \; \forall \; k. \tag{1.6}$$

A quasi-convex null sequence satisfies conditions of the class S because we can choose

$$A_n = \sum_{m=n}^{\infty} |\triangle^2 a_m|.$$

Concerning L^1-convergence of (1.1) and (1.2), the following theorems are known:

Theorem 1.3. *([1], p. 204) If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.1) in the metric space L^1, it is necessary and sufficient that $a_k \log k = o(1), \quad k \to \infty$.*

This theorem is due to Kolmogorov [2]. Teljakovskii [5] generalized Theorem 1.3 for the cosine series (1.1) with coefficients $\{a_k\}$ satisfying the conditions of the class S in the following form:

Theorem 1.4. *If the coefficient sequence $\{a_k\}$ of the cosine series (1.1) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (1.1) is $a_k \log k = o(1), \; k \to \infty$.*

Theorem 1.5. *([1], p. 201) If $a_k \downarrow 0$ and $\sum_{k=1}^{\infty} \left(\frac{a_k}{k}\right) < \infty$, then (1.3) is a Fourier series.*

In the present paper, we introduce new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle(a_j \cos jx)$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \sin jx)$$

and study their integrability and L^1-convergence under a new class SJ of coefficient sequences defined as follows:

Definition 1.6. A null sequence $\{a_k\}$ of positive numbers belongs to class SJ if there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, \ \ as\ k \to \infty, \tag{1.7}$$

$$\sum_{k=1}^{\infty} A_k < \infty, \tag{1.8}$$

$$\left|\triangle\left(\frac{a_k}{k}\right)\right| \leq \frac{A_k}{k}\ \forall\ k. \tag{1.9}$$

Clearly class SJ $\subset$ class S, Since

$$\left|\triangle\left(\frac{a_k}{k}\right)\right| \leq \frac{A_k}{k} \Rightarrow |\Delta a_k| \leq A_k,\ \forall\ k.$$

Following example shows that the class SJ is proper subclass of class S.

Example 1.7. For $k = I - \{0, 1, 2\}$, where I is set of integers, define $\{a_k\} = \frac{1}{k^3}$, then there exists $\{A_k\} = \frac{1}{k^2}$ such that $\{a_k\}$ satisfies all the conditions of class S but not class SJ. However, for $k = 1, 2, 3...$ the sequence $\{b_k\} = \frac{1}{k^3}$ satisfies conditions of class SJ as well as conditions of class S. Clearly, class SJ is proper subclass of class S.

Now, we define a new class SJ_r of coefficient sequences which is an extension of class SJ.

Definition 1.8. A null sequence $\{a_k\}$ of positive numbers belongs to class SJ_r if there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, \ \ as\ k \to \infty, \tag{1.10}$$

$$\sum_{k=1}^{\infty} k^r A_k < \infty, \qquad (r = 0, 1, 2, ...) \tag{1.11}$$

$$\left|\triangle\left(\frac{a_k}{k}\right)\right| \leq \frac{A_k}{k}\ \forall\ k. \tag{1.12}$$

clearly, for $r = 0$, $\text{SJ}_r = \text{SJ}$. It is obvious that $\text{SJ}_{r+1} \subset \text{SJ}_r$, but converse is not true.

Example 1.9. For $k = 1, 2, 3...$, define $b_k = \frac{1}{k^{r+2}}$, $\quad r = 0, 1, 2, 3, ...$ Firstly, we shall show that $\{b_k\}$ does not belong to SJ_{r+1}.

Really, $b_n = \dfrac{1}{n^{r+2}} \to 0 \quad as \quad n \to \infty.$

Let there exists $\{A_k\} = \frac{1}{k^{r+2}}, \quad r = 0, 1, 2, 3, ...$ s.t. $\sum_{k=1}^{\infty} k^{r+1} A_k = k^{r+1} \frac{1}{k^{r+2}} = \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, i.e. $\{b_k\}$ does not belong to SJ_{r+1}.

But, $A_k \downarrow 0$, *as* $k \to \infty$, and $\sum_{k=1}^{\infty} k^r A_k = k^r \frac{1}{k^{r+2}} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$,

Also $\left| \triangle \left(\frac{b_k}{k} \right) \right| \leq \frac{A_k}{k}, \quad \forall \ k.$
Therefore, $\{b_k\}$ belongs to SJ_r.

In what follows, $t_n(x)$ will represents $f_n(x)$ or $g_n(x)$.

2. Lemmas

We require the following lemmas in the proof of our result.

Lemma 2.1. [3] *Let $n \geq 1$ and let r be a nonnegative integer, $x \in [\epsilon, \pi]$. Then $|\tilde{D}_n^r(x)| \leq C_\epsilon \frac{n^r}{x}$ where C_ϵ is a positive constant depending on ϵ, $0 < \epsilon < \pi$ and $\tilde{D}_n(x)$ is the conjugate Dirichlet kernel.*

Lemma 2.2. [5] *Let $\{a_k\}$ be a sequence of real numbers such that $|a_k| \leq 1$ for all k. Then there exists a constant $M > 0$ such that for any $n \geq 1$*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k \tilde{D}_k(x) \right| \, dx \leq M(n+1).$$

Moreover by Bernstein's inequality, for $r = 0, 1, 2, 3.....$

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k \tilde{D}_k^r(x) \right| \, dx \leq M(n+1)^{r+1}.$$

Lemma 2.3. [3] $||\tilde{D}_n^r(x)||_{L^1} = O(n^r \log n)$, $r = 0, 1, 2, 3....$, *where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet-Kernel.*

3. Main Results

In this paper we shall prove the following main results:

Theorem 3.1. *Let the coefficients of the series (1.3) belongs to class SJ, then the series (1.3) is a Fourier series.*

Proof. Making Use of Abel's transformation on $\sum_{k=1}^{n} \left(\frac{a_k}{k} \right)$, we get

$$\begin{aligned} \sum_{k=1}^{n} \left(\frac{a_k}{k} \right) &= \sum_{k=1}^{n-1} k \triangle \left(\frac{a_k}{k} \right) - a_n \\ &\leq \sum_{k=1}^{n-1} k \left(\frac{A_k}{k} \right) - a_n \end{aligned}$$

But (1.3) belongs to class SJ, therefore, the series $\sum_{k=1}^{\infty}\left(\frac{a_k}{k}\right)$ converges.
Hence the conclusion of theorem follows from Theorem (1.5. □

Theorem 3.2. *Let the coefficients of the series (1.3) belongs to class SJ, then*

$$\lim_{n\to\infty} t_n(x) = t(x) \text{ exists for } x \in (0,\pi). \tag{3.1}$$

$$t \in L^1(0,\pi) \tag{3.2}$$

$$||t(x) - S_n(x)|| = o(1),\ n \to \infty \tag{3.3}$$

Proof. We will consider only cosine sums as the proof for the sine sums follows the same line.
To prove (3.1), we notice that

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx)$$

$$\begin{aligned} t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}[a_k\ \cos kx - a_{k+1}\ \cos(k+1)x + a_{k+1}\ \cos(k+1)x \\ -a_{k+2}\cos(k+1)x + ... + a_n \cos nx - a_{n+1}\cos(n+1)x] \end{aligned}$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \sum_{k=1}^{n} a_{n+1}\cos(n+1)x$$

$$t_n(x) = S_n(x) - na_{n+1}\cos(n+1)x \tag{3.4}$$

Since $A_k \downarrow 0,\ as\ k \to \infty$ and $\sum_{k=1}^{\infty} A_k < \infty$, therefore, by Oliver's theorem we have, $kA_k \to 0,\ as\ k \to \infty$ and so

$$na_n = n^2 \sum_{k=n}^{\infty} \triangle\left(\frac{a_k}{k}\right) \le \sum_{k=n}^{\infty} k^2\left(\frac{A_k}{k}\right) = o(1) \tag{3.5}$$

Also $\cos(n+1)x$ is finite in $(0,\pi)$. Hence

$$\lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) = t(x)$$

Moreover,

$$\begin{aligned} t(x) = \lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty}\left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\right) \\ &= \frac{a_0}{2} + \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \cos kx\right) \end{aligned}$$

Use of Abel's transformation yields

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \cos kx\right) = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \triangle\left(\frac{a_k}{k}\right)\tilde{D}'_k(x) + \frac{a_n}{n}\tilde{D}'_n(x)\right]$$

where $\tilde{D}'_n(x)$ is the derivative of conjugate Dirichlet kernel.

$$= \sum_{k=1}^{\infty} \triangle\left(\frac{a_k}{k}\right) \tilde{D}'_k(x)$$

$$\leq \sum_{k=1}^{\infty} \left(\frac{A_k}{k}\right) \tilde{D}'_k(x)$$

By the given hypothesis and lemma 2.1, the series $\sum_{k=1}^{\infty} \left(\frac{A_k}{k}\right) \tilde{D}'_k(x)$ converges. Therefore, the limit $t(x)$ exists for $x \in (0, \pi)$ and thus (3.1) follows.

For $x \neq 0$, it follows from (3.4) that

$$\begin{aligned} t(x) - t_n(x) &= \sum_{k=n+1}^{\infty} a_k \cos kx + n a_{n+1} \cos(n+1)x \\ &= \lim_{m\to\infty} \left[\sum_{k=n+1}^{m} \left(\frac{a_k}{k}\right) k \cos kx\right] + n a_{n+1} \cos(n+1)x \end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned} &= \sum_{k=n+1}^{\infty} \triangle\left(\frac{a_k}{k}\right) \tilde{D}'_k(x) - \frac{a_{n+1}}{n+1}\tilde{D}'_n(x) + n a_{n+1} \cos(n+1)x \\ &\leq \sum_{k=n+1}^{\infty} \left(\frac{A_k}{k}\right) \frac{\triangle\left(\frac{a_k}{k}\right)}{\left(\frac{A_k}{k}\right)}\tilde{D}'_k(x) + \frac{a_{n+1}}{n+1}\tilde{D}'_n(x) + n a_{n+1} \cos(n+1)x \\ &\leq \sum_{k=n+1}^{\infty} \triangle\left(\frac{A_k}{k}\right) \sum_{j=1}^{k} \frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}'_j(x) - \left(\frac{A_{n+1}}{n+1}\right) \sum_{j=1}^{n} \frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}'_j(x) \\ &\quad + \frac{a_{n+1}}{n+1}\tilde{D}'_n(x) + n a_{n+1} \cos(n+1)x \end{aligned}$$

Thus from lemma 2.2 and 2.3, we obtain

$$\begin{aligned} ||t(x) - t_n(x)|| \leq & \sum_{k=n+1}^{\infty} \triangle\left(\frac{A_k}{k}\right) \int_0^{\pi} \left|\sum_{j=1}^{k} \frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}'_j(x)\right| \, dx \\ & + \left(\frac{A_{n+1}}{n+1}\right) \int_0^{\pi} \left|\sum_{j=1}^{n} \frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}'_j(x)\right| \, dx + \int_0^{\pi} \left|\frac{a_{n+1}}{n+1}\tilde{D}'_n(x)\right| \, dx \\ & + n|a_{n+1}| \int_0^{\pi} |\cos(n+1)x| \, dx \end{aligned}$$

$$= O\left(\sum_{k=n+1}^{\infty} k^2 \triangle\left(\frac{A_k}{k}\right)\right) + O\left(n^2\left(\frac{A_{n+1}}{n+1}\right)\right)$$
$$+O(a_{n+1}\log n) + n|a_{n+1}|\int_0^{\pi} |\cos(n+1)x|\ dx$$

But

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n-1} \frac{k(k+1)}{2} \triangle\left(\frac{A_k}{k}\right) + \frac{n(n+1)}{2}\frac{A_n}{n}$$

since $\{a_k\} \in$SJ, we have

$$k(k+1)\frac{A_k}{k} = (k+1)A_k = o(1)\ \ as\ \ k \to \infty.$$

and therefore the series $\sum_{k=n+1}^{\infty} k^2 \triangle\left(\frac{A_k}{k}\right)$, converges.

Moreover,

$$\int_0^{\pi} |cos(n+1)x|\ dx = \int_0^{\frac{\pi}{2}} \cos(n+1)x\ dx - \int_{\frac{\pi}{2}}^{\pi} \cos(n+1)x\ dx \leq \frac{2}{n+1}$$

and since a_n's are positive, we have by (3.5) that $a_n \log n \leq n a_n = o(1),\ \ for\ \ n \geq 1$.

Hence, it follows that

$$||t(x) - t_n(x)|| = o(1)\ as\ n \to \infty. \tag{3.6}$$

and since $t_n(x)$ is a polynomial, therefore $t(x) \in L^1$. This proves (3.2).

We now turn to the proof of (3.3), We have

$$\begin{aligned} ||t - S_n|| &= ||t - t_n + t_n - S_n|| \\ &\leq ||t - t_n|| + ||t_n - S_n|| \\ &= ||t - t_n|| + ||n a_{n+1}\cos(n+1)x|| \\ &\leq ||t - t_n|| + n|a_{n+1}|\int_0^{\pi} |\cos(n+1)x|\ dx \end{aligned}$$

Further, $||t(x) - t_n(x)|| = o(1),\ n \to \infty$ (by (3.6)), $\int_0^{\pi} |\cos(n+1)x|\ dx \leq \frac{2}{n+1}$ and $\{a_k\}$ is a null sequence,therefore the conclusion of theorem follows. □

Theorem 3.3. *Let the coefficients of the series (1.3) belongs to class SJ_r, then*

$$\lim_{n\to\infty} t_n^r(x) = t^r(x)\ \textit{exists for}\ x \in (0,\pi). \tag{3.7}$$

$$t^r \in L^1(0,\pi), \qquad (r = 0,1,2,...) \tag{3.8}$$

$$||t^r(x) - S_n^r(x)|| = o(1),\ n \to \infty. \tag{3.9}$$

Proof. We will consider only cosine sums as the proof for the sine sums follows the same line. As in the proof of the Theorem 3.2, we have

$$\begin{aligned} t_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle(a_j \cos jx) \\ &= S_n(x) - n a_{n+1} \cos(n+1)x \end{aligned}$$

we have, then

$$t_n^r(x) = S_n^r(x) - n(n+1)^r a_{n+1} \cos\left((n+1)x + \frac{r\pi}{2}\right)$$

Since $A_k \downarrow 0,\ as\ k \to \infty$ and $\sum_{k=1}^{\infty} k^r A_k < \infty$, therefore, we have, $k^{r+1}A_k \to 0,\ as\ k \to \infty$ and so

$$n^{r+1}a_n = n^{r+2}\sum_{k=n}^{\infty} \triangle\left(\frac{a_k}{k}\right) \leq \sum_{k=n}^{\infty} k^{r+2}\left(\frac{A_k}{k}\right) = o(1) \tag{3.10}$$

Also $\cos\left((n+1)x + \frac{r\pi}{2}\right)$ is finite in $(0, \pi)$. Hence

$$\begin{aligned} t^r(x) &= \lim_{n\to\infty} t_n^r(x) \\ &= \lim_{n\to\infty} S_n^r(x) \\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right) \end{aligned}$$

use of Abel's transformation yields

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right)\right) = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \triangle\left(\frac{a_k}{k}\right)\tilde{D}_k^{r+1}(x) + \frac{a_n}{n}\tilde{D}_n^{r+1}(x)\right],$$

where $\tilde{D}_n^{r+1}(x)$ represents the $(r+1)^{th}$ derivative of conjugate Dirichlet kernel.

$$\begin{aligned} &= \sum_{k=1}^{\infty} \triangle\left(\frac{a_k}{k}\right)\tilde{D}_k^{r+1}(x) + \lim_{n\to\infty}\left[\frac{a_n}{n}\tilde{D}_n^{r+1}(x)\right] \\ &\leq \sum_{k=1}^{\infty}\left(\frac{A_k}{k}\right)\tilde{D}_k^{r+1}(x) + \lim_{n\to\infty}\left[\frac{a_n}{n}\tilde{D}_n^{r+1}(x)\right] \end{aligned}$$

By making use of the given hypothesis, lemma 2.1 and (3.10), the series $\sum_{k=1}^{\infty}\left(\frac{A_k}{k}\right)\tilde{D}_k^{r+1}(x)$ converges. Therefore, the limit $t^r(x)$ exists for $x \in (0, \pi)$ and thus (3.7) follows. To prove (3.8), we have

$$t^r(x) - t_n^r(x) = \sum_{k=n+1}^{\infty} k^r a_k \cos\left(kx + \frac{r\pi}{2}\right) + n(n+1)^r a_{n+1} \cos\left((n+1)x + \frac{r\pi}{2}\right)$$

Making use of Abel's transformation, we obtain

$$= \sum_{k=n+1}^{\infty} \triangle\left(\frac{a_k}{k}\right)\tilde{D}_k^{r+1}(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) + n(n+1)^r a_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right)$$

$$\leq \sum_{k=n+1}^{\infty} \left(\frac{A_k}{k}\right)\frac{\triangle\left(\frac{a_k}{k}\right)}{\left(\frac{A_k}{k}\right)}\tilde{D}_k^{r+1}(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) + n(n+1)^r a_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right)$$

$$\leq \sum_{k=n+1}^{\infty} \triangle\left(\frac{A_k}{k}\right)\sum_{j=1}^{k}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x) - \left(\frac{A_{n+1}}{n+1}\right)\sum_{j=1}^{n}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x)$$
$$+\frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x) + na_{n+1}\cos\left((n+1)x + \frac{r\pi}{2}\right)$$

Thus from lemma 2.2 and 2.3, we obtain

$$\begin{aligned}
||t^r(x) - t_n^r(x)|| \;\leq\; & \sum_{k=n+1}^{\infty} \triangle\left(\frac{A_k}{k}\right)\int_0^{\pi}\left|\sum_{j=1}^{k}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x)\right| dx \\
& + \left(\frac{A_{n+1}}{n+1}\right)\int_0^{\pi}\left|\sum_{j=1}^{n}\frac{\triangle\left(\frac{a_j}{j}\right)}{\left(\frac{A_j}{j}\right)}\tilde{D}_j^{r+1}(x)\right| dx + \int_0^{\pi}\left|\frac{a_{n+1}}{n+1}\tilde{D}_n^{r+1}(x)\right| dx \\
& + n(n+1)^r|a_{n+1}|\int_0^{\pi}\left|\cos\left((n+1)x + \frac{r\pi}{2}\right)\right| dx \\
= \; & O\left(\sum_{k=n+1}^{\infty} k^{r+2}\triangle\left(\frac{A_k}{k}\right)\right) + O\left(n^{r+2}\left(\frac{A_{n+1}}{n+1}\right)\right) + O(n^r a_{n+1}\log n) \\
& + n(n+1)^r|a_{n+1}|\int_0^{\pi}\left|\cos\left((n+1)x + \frac{r\pi}{2}\right)\right| dx
\end{aligned}$$

Using the argument as in the proof of theorem 3.2, it is easily shown that the series $\sum_{k=n+1}^{\infty} k^{r+2}\triangle\left(\frac{A_k}{k}\right)$, converges. Moreover,

$$\int_0^{\pi}\left|\cos\left((n+1)x + \frac{r\pi}{2}\right)\right| dx \leq \frac{2}{n+1}$$

and for $n \geq 1$, $n^r a_n \log n \leq n^{r+1}a_n = o(1)$ *by* (3.10). Hence it follows that

$$||t^r(x) - t_n^r(x)|| = o(1) \text{ as } n \to \infty. \tag{3.11}$$

and since $t_n^r(x)$ is a polynomial, therefore $t^r(x) \in L^1$. This proves (3.8).

We now turn to the proof of (3.9). We have

$$\begin{aligned} ||t^r - S_n^r|| &= ||t^r - t_n^r + t_n^r - S_n^r|| \\ &\leq ||t^r - t_n^r|| + ||t_n^r - S_n^r|| \\ &= ||t^r - {t_n}^r|| + ||n(n+1)^r a_{n+1} \cos\left((n+1)x + \frac{r\pi}{2}\right)|| \\ &\leq ||t^r - t_n^r|| + n(n+1)^r |a_{n+1}| \int_0^\pi |\cos\left((n+1)x + \frac{r\pi}{2}\right)| dx \end{aligned}$$

Further, $||t^r(x) - t_n^r(x)|| = o(1)$, $n \to \infty$ (by (3.11)) , $\int_0^\pi |\cos\left((n+1)x + \frac{r\pi}{2}\right)| \, dx \leq \frac{2}{n+1}$ and $\{a_k\}$ is a null sequence, the conclusion of theorem follows. □

Remark 3.4. The case $r = 0$, in Theorem 3.3 yields the Theorem 3.2.

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