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EXISTENCE AND UNIQUENESS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this article, the recently developed monotonous iterative method is used to investigate fractional differential equations involving Riemann-Liouville differential operators with integral boundary conditions. The existence and uniqueness of solutions are obtained.

1. Introduction

We consider the following fractional differential problem with integral boundary condition

$$\begin{aligned} &D^q x(t) = f(t, x), \quad t \in J = [0, T], \quad T \geq 0, \\ &x(0) = \lambda \int_0^T x(s)ds + d, \quad d \in \mathbb{R}. \end{aligned} \tag{1.1}$$

where $0 < q < 1$, λ is 1 or -1 and $f \in C[J \times \mathbb{R}, \mathbb{R}]$. D^q denotes the fractional derivative of order q in the sense of Riemann-Liouville. Problem (1.1) with $q = 1$ was investigated by Jankowski [5]. Very recently, the basic theory of problem (1.1) with $\lambda = 0$ has been obtained by Lakshmikantham and Vatsala in a series of work [6, 7, 8]. The monotonous iterative method for fractional differential equations and the theory of fractional differential equalities have also been developed.

The significance of fractional differential equations has been displayed in the research of applied mathematics these years, especially in the study on disordered semiconductors and viscoelastic materials; see [1, 2, 13, 14], for instance. Because most of nonlinear fractional differential equations do not exact analytic solutions, various analytic approximation and numerical methods have been proposed and

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developed recently. For example, the He's homotopy perturbation method[9, 10, 12, 15] and variational iteration method[3, 4, 11] have been successfully applied to solve a variety of nonlinear fractional differential equations.

In the present article, we shall discuss the existence and uniqueness of problem (1.1) by employing the monotonous iterative method recently developed by Lakshmikantham and Vatsala. Note that problem (1.1) is equivalent to the following integral equation:

$$x(t) = \lambda \int_0^T x(s)ds + d + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad 0 \le t \le T,$$

where $\Gamma(q)$ is the Gamma Function of q.

The layout of this paper is as follows. In Section 2, we employ the monotonous iterative method to testify the existence and uniqueness of solution to problem (1.1) in the case of $\lambda = 1$. The similar way to establish the corresponding theory of solution to problem (1.1) when $\lambda = -1$ is given in Section 3.

2. Case λ=1

Before the detailed establishment, let us introduce some definitions and assumptions.

Definition 2.1. Assume that there exist $v_0(t)$ and $w_0(t)$ which are locally Hölder continuous and satisfy:

$$D^q v_0(t) \le f(t, v_0(t)), \quad v_0(0) \le \int_0^T v_0(s)ds,$$

$$D^q w_0(t) \ge f(t, w_0(t)), \quad w_0(0) \ge \int_0^T w_0(s)ds.$$

Then we call $v_0(t)$ and $w_0(t)$ lower and upper solutions of problem (1.1), respectively.

We make the following assumptions:

- (H_1) $f \in C(J \times \mathbb{R}, \mathbb{R})$;
- (H_2) $v_0(t), w_0(t) \in C^1(J, \mathbb{R})$ are lower and upper solutions of problem (1.1);
- (H_3) There exists $0 \le M \le \frac{1}{T^q \Gamma(1-q)}$ such that $f(t,x) - f(t,y) \ge -M(x-y)$ for $x \ge y$.

Lemma 2.2. *If H_1 and H_2 hold, and there exists $L \in (0, \frac{1}{T^q\Gamma(1-q)})$ such that $f(t,x) - f(t,y) \le L(x-y)$ for $y \le x$. Then, $v(0) \le w(0)$ implies $v(t) \le w(t)$.*

Actually, the proof in the case of integral boundary condition is the same as that in the initial value condition because the integral condition has the same definition like the initial one used in the proofs. And the proof of the latter can be found in [8], so we omit it here.

Lemma 2.3. *If* (H_1), (H_2) *and* (H_3) *hold, and the mapping* $A: Av_0 = v_1, Aw_0 = w_1$ *is defined as:*

$$D^q v_1 = f(t, v_0) - M(v_1(t) - v_0(t)), \quad v_1(0) = \int_0^T v_0(s)ds + d,$$

$$D^q w_1 = f(t, w_0) - M(w_1(t) - w_0(t)), \quad w_1(0) = \int_0^T w_0(s)ds + d. \tag{2.1}$$

Then the mapping A *has the properties:*

(a) $Av_0 \geq v_1$, $Aw_0 \leq w_1$;
(b) *on* $[v_0, w_0] = [x \in C(J, \mathbb{R}) : v_0 \leq w_0]$, A *is a monotonous operator.*

Proof. (a). Note that the right side of equations in (2.1) meets Lipshitz condition, which warrants the uniqueness of the solution v_1, w_1.

Setting $p = v_1 - v_0$, we have

$$\begin{aligned} D^q p(t) &= f(t, v_0(t)) - M[v_1(t) - v_0(t)] - f(t, v_0(t)) \\ &\geq -M[v_1(t) - v_0(t)] \\ &= -Mp(t), \\ p(0) &= v_1(0) - v_0(0) \geq 0. \end{aligned}$$

Note that the initial value problem $D^q x(t) = 0$, $x(0) = x_0$ has a unique solution $x(t) \equiv 0$ on J. It follows from Lemma 2.2 that $p(t) \geq 0$, which implies $v_1(t) \geq v_0(t)$. Similarly, we can get $w_1(t) \leq w_0(t)$.

(b). For σ_1, $\sigma_2 \in [v_0, w_0]$, $\sigma_2 \geq \sigma_1$, setting $p = A\sigma_2 - A\sigma_1 = x_2 - x_1$, we have

$$\begin{aligned} D^q p &= f(t, \sigma_2(t)) - M(x_2 - \sigma_2) - f(t, \sigma_1(t)) + M(x_1 - \sigma_1) \\ &\geq -M(\sigma_2 - \sigma_1) + M(\sigma_2 - \sigma_1) - M(x_2 - x_1) \\ &= -M(x_2 - x_1) \\ &= -Mp(t), \\ p(0) &= x_2(0) - s_1(0) \\ &= \int_0^T [\sigma_2(s) - \sigma_1(s)]ds \\ &\geq 0. \end{aligned}$$

It follows from Lemma 2.2 that $A\sigma_2 \geq A\sigma_1$, which implies that A is a monotonous operator. The proof of Lemma 2.3 is completed. □

Theorem 2.4. *Assume that* H_1, H_2 *and* H_3 *hold, and the mapping* $A: Av_n = v_{n+1}$, $Aw_n = w_{n+1}$ *are defined as:*

$$D^q v_{n+1} = f(t, v_n) - M(v_{n+1}(t) - v_n(t)), \quad v_{n+1}(0) = \int_0^T v_n(s)ds + d,$$

$$D^q w_{n+1} = f(t, w_n) - M(w_{n+1}(t) - w_n(t)), \quad w_{n+1}(0) = \int_0^T w_n(s)ds + d.$$

Then there exist monotone sequences $\{v_n\}$, $\{w_n\}$ *such that* $v_n \to v(t)$, $w_n \to w(t)$ *as* $n \to \infty$, *where* (v, w) *are the extremal solutions of (1.1) on* $0 \leq t \leq T$.

Proof. The definition of the mapping A gives $Av_{n-1} = v_n, Aw_{n-1} = w_n$. From (a) and (b) in Lemma 2.3, it is not difficult to get:

$$v_0 \leq\ v_1 \leq\ \ldots \leq\ v_{n-1} \leq\ v_n \leq\ w_n \leq\ w_{n-1} \leq \ldots \leq\ w_2 \leq\ w_1 \leq w_0,\ t \in J.$$

Apparently, there exist v, w such that $\lim_{n\to\infty} v_n(t) = v(t), \lim_{n\to\infty} w_n(t) = w(t)$. And v, w are solutions of problem (1.1). Next we will prove that v, w are the minimal and maximal solutions of problem (1.1), respectively.

Let $x(t) \in [v_0, w_0]$ is a solution of (1.1) different from $v(t)$ and $w(t)$, so there exist $k \in N$ such that $v_k(t) \leq x(t) \leq\ w_k(t)$. Setting $p(t) = x - v_{k+1}$, we have

$$\begin{aligned} D^q p(t) &= f(t, x(t)) - f(t, v_k(t)) + M[v_{k+1}(t) - v_k(t)] \\ &\geq -M[x(t) - v_k(t)] + M[v_{k+1}(t) - v_k(t)] \\ &= M[v_{k+1}(t) - x(t)] \\ &= Mp(t), \\ p(0) &= x(0) - v_{k+1}(0) \\ &= \int_0^T [x(s) - v_k(s)]ds \\ &\geq 0. \end{aligned}$$

By Lemma 2.2, we get that $p(t) = x(t) - v_{k+1}(t) \geq 0$, which implies $x(t) \geq v(t)$ by letting $k \to \infty$. For the same sake, we can also get $x(t) \leq w(t)$. Now we can see that v, w are the minimal solution and the maximal solution to (1.1). □

Theorem 2.5. *Assume that H_1, H_2 and H_3 hold, and there exists $L \in (0, \frac{1}{T^q\Gamma(1-q)})$ such that $f(t, x) - f(t, y) \leq L(x - y)$ for $y \leq x$. Suppose further that $\lim_{n\to\infty} \|w_n - v_n\| = 0$ where the norm is defined as $\|f\| = \int_0^T |f(s)|ds$. Then the solution of problem (1.1) is unique.*

Proof. Setting $p = w - v$, we have

$$\begin{aligned} D^q p(t) &= f(t, w(t)) - f(t, v(t)) \\ &\leq M[w(t) - v(t)] \\ &= Mp(t), \\ p(0) &= w(0) - v(0) \\ &= \lim_{n\to\infty} [w_n(0) - v_n(0)] \\ &= \lim_{n\to\infty} \|w_{n-1} - v_{n-1}\| \\ &= 0. \end{aligned}$$

Again with Lemma 2.2, we could get $p(t) \leq 0$. As a result, it follows $v(t) \geq w(t)$. But considering $v(t) \geq w(t)$, we have $v(t) \equiv w(t)$. □

3. Case $\lambda = -1$

Definition 3.1. Assume that there exist $v_0(t)$ and $w_0(t)$ which are locally Hölder continuous and satisfy:

$$D^q v_0(t) \leq f(t, v_0(t)), \quad v_0(0) \leq -\int_0^T w_0(s)ds,$$

$$D^q w_0(t) \geq f(t, w_0(t)), \quad w_0(0) \geq -\int_0^T v_0(s)ds.$$

Then we call $v_0(t)$ and $w_0(t)$ the weakly coupled lower and upper solutions of problem (1.1), respectively.

We make assumptions:

- (H_4) There exist weakly coupled upper and lower solutions w_0 and v_0 of problem (1.1);
- (H_5) There exists $M \in [0, \frac{1}{T^q \Gamma(1-q)}]$ such that $M(x-y) \geq f(t,x) - f(t,y) \geq -M(x-y)$ for $x \geq y$;
- (H_6) The sequences $\{v_n\}$ and $\{w_n\}$ given by the mapping A are weakly coupled, and $\lim\limits_{n\to\infty} \|w_n - v_n\| = 0$ where the norm is defined as $\|f\| = \int_0^T |f(s)|ds$.

Lemma 3.2. *Assume that (H_4) and (H_5) hold, and the mapping $A : Av_0 = v_1, Aw_0 = w_1$ is defined as:*

$$D^q v_1 = f(t, v_0) - M(v_1(t) - v_0(t)), \quad v_1(0) = -\int_0^T w_0(s)ds + d,$$
$$D^q w_1 = f(t, w_0) - M(w_1(t) - w_0(t)), \quad w_1(0) = -\int_0^T v_0(s)ds + d. \tag{3.1}$$

Then the conclusions (a) and (b) in Lemma 2.3 hold.

Proof. (a). Note that the right side of equations in (3.1) satisfy Lipshitz condition, which assures the uniqueness of solution to (1.1).

Setting $p = v_1 - v_0$, we have

$$\begin{aligned} D^q p(t) &= f(t, v_0(t)) - M[v_1(t) - v_0(t)] - f(t, v_0(t)) \\ &\geq -M[v_1(t) - v_0(t)] \\ &= -Mp(t), \\ p(0) &= v_1(0) - v_0(0) \\ &= -\int_0^T w_0(s)ds + \int_0^T w_0(s)ds \\ &= 0. \end{aligned}$$

It follows from that $p(t) \geq 0$ which implies $v_1(t) \geq v_0(s)$. For the same sake, $w_1(t) \leq w_0(t)$.

(b). For σ_1, $\sigma_2 \in [v_0, w_0]$, $\sigma_2 \geq \sigma_1$, let $\sigma_1^\star, \sigma_2^\star$ are also coupled solutions, and $\sigma_1^\star \geq \sigma_2^\star$. Setting $p = A\sigma_2 - A\sigma_1 = x_2 - x_1$, we have

$$\begin{aligned}
D^q p &= f(t, \sigma_2(t)) - M(x_2 - \sigma_2) - f(t, \sigma_1(t)) + M(x_1 - \sigma_1) \\
&\geq -M(\sigma_2 - \sigma_1) + M(\sigma_2 - \sigma_1) - M(x_2 - x_1) = -M(x_2 - x_1) \\
&= -Mp(t), \\
p(0) &= x_2(0) - x_1(0) \\
&= \int_0^T [-\sigma_2^\star(s) + \sigma_1^\star(s)] ds \\
&\geq 0.
\end{aligned}$$

It follows from Lemma 2.2 that $x_2(t) \geq x_1(t)$ and A is a monotonous operator. □

Theorem 3.3. *Assume that H_4, H_5 and H_6 hold, and the mapping $A : Av_n = v_{n+1}, Aw_n = w_{n+1}$ are defined as:*

$$D^q v_{n+1} = f(t, v_n) - M(v_{n+1}(t) - v_n(t)), \quad v_{n+1}(0) = -\int_0^T w_n(s) ds + d,$$

$$D^q w_{n+1} = f(t, w_n) - M(w_{n+1}(t) - w_n(t)), \quad w_{n+1}(0) = -\int_0^T v_n(s) ds + d.$$

Then there exist monotone sequences $\{v_n\}$, $\{w_n\}$ which converges to the same function $\phi(t)$ which is the unique solution of problem (1.1) with $\lambda = -1$.

Proof. The definition of A gives $Av_{n-1} = v_n, Aw_{n-1} = w_n$. From (a) and (b) in Lemma 3.2 there is no difficulty to get: for $t \in J, v_0 \leq v_1 \leq \ldots \leq v_{n-1} \leq v_n \leq w_n \leq w_{n-1} \leq \ldots \leq w_2 \leq w_1 \leq w_0$. Apparently, there exist v, w such that $\lim_{n\to\infty} v_n(t) = v(t), \lim_{n\to\infty} w_n(t) = w(t)$. And v, w are solutions to (1.1).

Setting $p = w - v$, we have

$$\begin{aligned}
D^q p &= f(t, w(t)) - f(t, v(t)) \\
&\leq M[w(t) - v(t)] \\
&= p(t), \\
p(0) &= w(0) - v(0) \\
&= \lim_{n\to\infty} [w_n(0) - v_n(0)] \\
&= \lim_{n\to\infty} \|w_{n-1} - v_{n-1}\| \\
&= 0.
\end{aligned}$$

With Lemma 2.3 again, we get $p(t) \leq 0$, which means $v(t) \geq w(t)$. Considering $v(t) \leq w(t)$, we have $v(t) = w(t) := \phi(t)$. We have got the uniqueness of solution to (1.1) with $\lambda = -1$. The proof of Theorem 3.3 is complete. □

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