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SEVERAL DISCRETE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper, we establish some interesting discrete inequalities involving convex functions and pose an open problem.

1. Introduction

The following problem was posed by Qi in his article [13]: *"Under what condition does the inequality*

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1} \tag{1.1}$$

hold for $t > 1$?".

There are numerous answers and extension results to this open problem [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 15, 16]. These results were obtained by different approaches, such as, e.g. Jensen's inequality, the convexity method [16]; functional inequalities in abstract spaces [1, 2]; probability measures view [4, 7]; Hölder inequality and its reversed variants [2, 12]; analytical methods [11, 15]; Cauchy's mean value theorem [3, 14].

In [9], the authors introduced the following discrete version of (1.1) as follows, *"Under what condition does the inequality*

$$\sum_{i=1}^n x_i^\alpha a_i \geq \left(\sum_{i=1}^n x_i a_i \right)^\beta \tag{1.2}$$

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hold for $\alpha, \beta > 0$?". (For the infinite series, the same method in the above finite series can be discussed.) Very recently, some similar discrete inequalities were developed (for instance, the reference [10]). In the paper, based on the results in [5], we will establish some discrete type inequalities and pose an open problem.

2. Main results

Before starting the results for convex function, we firstly show the following results.

Theorem 2.1. *Let $\{x_i, i = 1, \ldots, n\}$, $\{y_i, i = 1, \ldots, n\}$ be two sequences of nonnegative real numbers such that $x_i \le y_i$ for all $1 \le i \le n$,*

$$\frac{x_1}{y_1} \ge \frac{x_2}{y_2} \ge \cdots \ge \frac{x_n}{y_n} \text{ and } x_1 \le x_2 \le \cdots \le x_n.$$

Then we have

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \ge \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n y_i^p} \tag{2.1}$$

for all $p \ge 1$. If

$$\frac{x_1}{y_1} \le \frac{x_2}{y_2} \le \cdots \le \frac{x_n}{y_n} \quad \text{and} \quad x_i \ge y_i$$

for all $1 \le i \le n$, then the inequality in (2.1) reverses.

Proof. Let $z_i = x_i^{p-1}$, then $z_1 \le z_2 \le \cdots \le z_n$ by $p \ge 1$. From the assumptions of Theorem 2.1, we have

$$(z_i - z_j)\left(\frac{x_j}{y_j} - \frac{x_i}{y_i}\right) \ge 0, \quad \text{for all } 1 \le i, j \le n. \tag{2.2}$$

Firstly we need to prove

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \ge \frac{\sum_{i=1}^n x_i z_i}{\sum_{i=1}^n y_i z_i}. \tag{2.3}$$

This is to say

$$\sum_{i=1}^n x_i \sum_{i=1}^n y_i z_i \ge \sum_{i=1}^n y_i \sum_{i=1}^n x_i z_i$$

which is equivalent to

$$D := \sum_{i=1}^n \sum_{j=1}^n z_j (x_i y_j - y_i x_j) \ge 0.$$

Noting

$$D = \sum_{i=1}^n \sum_{j=1}^n z_i (x_j y_i - y_j x_i)$$

then we have

$$\begin{aligned} 2D &= \sum_{i=1}^n \sum_{j=1}^n (z_i - z_j)(x_j y_i - y_j x_i) \\ &= \sum_{i=1}^n \sum_{j=1}^n y_i y_j (z_i - z_j) \left(\frac{x_j}{y_j} - \frac{y_i}{x_i}\right) \end{aligned}$$

which yields the inequality (2.3) by the condition (2.2). Since $x_i \le y_i$ for all $1 \le i \le n$, then

$$\begin{aligned}\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \geq & \frac{\sum_{i=1}^n x_i z_i}{\sum_{i=1}^n y_i z_i} \\ = & \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n y_i x_i^{p-1}} \geq \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n y_i^p}\end{aligned}$$

which is the first result. The proof of the other result is similar to (2.1). □

Next, we give some inequalities involving convex function.

Theorem 2.2. *Let $\{x_i, i = 1, \dots, n\}$, $\{y_i, i = 1, \dots, n\}$ and be two sequences of nonnegative real numbers such that $x_i \le y_i$ for all $1 \le i \le n$,*

$$\frac{x_1}{y_1} \geq \frac{x_2}{y_2} \geq \cdots \geq \frac{x_n}{y_n} \text{ and } x_1 \leq x_2 \leq \cdots \leq x_n.$$

Assume that $\phi(x)$ is a convex function with $\phi(0) = 0$. Then we have

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \geq \frac{\sum_{i=1}^n \phi(x_i)}{\sum_{i=1}^n \phi(y_i)}. \tag{2.4}$$

Proof. Since $\phi(x)$ is convex with $\phi(0) = 0$, then $\frac{\phi(x)}{x}$ is increasing. Hence from $x_i \le y_i$ for all $1 \le i \le n$, we have

$$\frac{\phi(x_i)}{x_i} \leq \frac{\phi(y_i)}{y_i}, \quad \text{for all } 1 \leq i \leq n.$$

Let $g(x) = \frac{\phi(x)}{x}$, then $g(x)$ is also increasing. So we have

$$\begin{aligned}\frac{\sum_{i=1}^n \phi(x_i)}{\sum_{i=1}^n \phi(y_i)} = & \frac{\sum_{i=1}^n x_i g(x_i)}{\sum_{i=1}^n y_i g(y_i)} \\ \leq & \frac{\sum_{i=1}^n x_i g(x_i)}{\sum_{i=1}^n y_i g(x_i)} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i}.\end{aligned}$$

Here the last inequality stems from the similar proof of Theorem 2.1. □

Theorem 2.3. *Let $\{x_i, i = 1, \dots, n\}$, $\{y_i, i = 1, \dots, n\}$ and $\{z_i, i = 1, \dots, n\}$ be three sequences of nonnegative real numbers such that $x_i \le y_i$ for all $1 \le i \le n$,*

$$\frac{x_1}{y_1} \geq \frac{x_2}{y_2} \geq \cdots \geq \frac{x_n}{y_n}, \; x_1 \leq x_2 \leq \cdots \leq x_n \text{ and } z_1 \leq z_2 \leq \cdots \leq z_n.$$

Assume that $\phi(x)$ is a convex function with $\phi(0) = 0$. Then we have

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \geq \frac{\sum_{i=1}^n \phi(x_i) z_i}{\sum_{i=1}^n \phi(y_i) z_i}. \tag{2.5}$$

Proof. The proof is similar to Theorem 2.3. We have

$$\begin{aligned}\frac{\sum_{i=1}^n \phi(x_i) z_i}{\sum_{i=1}^n \phi(y_i) z_i} = & \frac{\sum_{i=1}^n \frac{\phi(x_i)}{x_i} x_i z_i}{\sum_{i=1}^n \frac{\phi(y_i)}{y_i} y_i z_i} \\ \leq & \frac{\sum_{i=1}^n \frac{\phi(x_i)}{x_i} x_i z_i}{\sum_{i=1}^n \frac{\phi(x_i)}{x_i} y_i z_i} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i}.\end{aligned}$$

□

At last, we give an open problem as follows.

Open Problem 1. Suppose that $\phi(x)$ is a convex function with $\phi(0) = 0$. Under what conditions does the inequality

$$\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \geq \frac{\left(\sum_{i=1}^{n} \phi(x_i) z_i\right)^{\delta}}{\left(\sum_{i=1}^{n} \phi(y_i) z_i\right)^{\lambda}}$$

hold for δ, λ?

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