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ON THE STABILITY OF SOME QUADRATIC FUNCTIONAL EQUATION

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This paper is dedicated to the 60th Anniversary of Professor Themistocles M. Rassias

ABSTRACT. In this paper we establish the general solution of the functional equation which is closely associated with the quadratic functional equation and we investigate the Hyers-Ulam-Rassias stability of this equation in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The quadratic functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y) \tag{1.1}$$

and its generalizations has been studied by many authors in various classes of functions (see, e.g., [4, 6, 8]). For more general information on the stability of functional equations, refer to [3, 5, 7, 9, 10, 11, 12, 13, 15]. The quadratic functional equation was also used to characterize inner product spaces (see [1, 2, 14]). It is well known that a square norm on an inner product space X satisfies the important parallelogram equality

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2, \quad x,y\in X.$$

It is easily to check that a square norm also satisfies the equality

$$\|x-z\|^2+\|y-z\|^2=\frac{1}{2}\|x-y\|^2+2\left\|\frac{x+y}{2}-z\right\|^2, \quad x,y,z\in X.$$

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Motivated by this result we consider the following functional equation

$$f(x-z)+f(y-z)=\frac{1}{2}f(x-y)+2f\left(\frac{x+y}{2}-z\right) \tag{1.2}$$

and its pexiderized version

$$f(x-z)+g(y-z)=h(x-y)+k\left(\frac{x+y}{2}-z\right). \tag{1.3}$$

Clearly, the mapping $\mathbb{R}\ni x\to ax^2\in\mathbb{R}$, $a\in\mathbb{R}$, satisfies (1.2). Our purpose is to determine all solutions of equations (1.2), (1.3) and investigate the Hyers-Ulam-Rassias stability of equation (1.2).

2. General solutions of equations (1.2) and (1.3)

Throughout this section we assume that X and Y are uniquely 2-divisible abelian groups.

Theorem 2.1. *In the class of functions $f\colon X\to Y$ equations (1.1) and (1.2) are equivalent.*

Proof. Assume that $Q\colon X\to Y$ is a solution of equation (1.1). Then Q is even and $Q(0)=0$. Setting $y=x$ in (1.1) we get $Q(2x)=4Q(x)$, hence

$$Q(x)=4Q\left(\frac{x}{2}\right),\quad x\in X. \tag{2.1}$$

Replacing x and y by $x-z$ and $y-z$ in (1.1), respectively, we obtain

$$Q(x+y-2z)+Q(x-y)=2Q(x-z)+2Q(y-z),\quad x,y,z\in X.$$

Therefore on account of (2.1) one can easily check that Q is a solution of (1.2).

Assume that $f\colon X\to Y$ is a solution of equation (1.2). Putting $x=y=z=0$ in (1.2) we obtain $f(0)=0$. Setting $y=z=0$ in (1.2) we get

$$\frac{1}{2}f(x)=2f\left(\frac{x}{2}\right),\quad x\in X.$$

Replacing x by $x+y$ in the above equality we obtain

$$\frac{1}{2}f(x+y)=2f\left(\frac{x+y}{2}\right),\quad x,y\in X. \tag{2.2}$$

Setting $z=0$ in (1.2) we have

$$f(x)+f(y)=\frac{1}{2}f(x-y)+2f\left(\frac{x+y}{2}\right),\quad x,y\in X,$$

which means by virtue of (2.2) that f satisfies (1.1). □

Theorem 2.2. *Let functions $f,g,h,k\colon X\to Y$ satisfy (1.3). Then there exist a quadratic function $Q\colon X\to Y$, two additive functions $E,F\colon X\to Y$ and* *constants* C_1, C_2, C_3, C_4 *such that* $C_1 + C_2 = C_3 + C_4$ *and*

$$\begin{aligned} f(x) &= Q(x) + E(x) + C_1, \\ g(x) &= Q(x) + F(x) + C_2, \\ h(x) &= \frac{1}{2}Q(x) + \frac{1}{2}E(x) - \frac{1}{2}F(x) + C_3, \\ k(x) &= 2Q(x) + E(x) + F(x) + C_4 \end{aligned}$$

for all $x \in X$.

Proof. Since the group Y is uniquely divisible by 2 (i.e. the map $X \ni x \to x+x \in Y$ is bijective), then we may split f into its even and odd parts $f_e, f_o \colon X \to Y$ by

$$f_e(x) := \frac{f(x) + f(-x)}{2}, \qquad f_o(x) := \frac{f(x) - f(-x)}{2}, \quad x \in X.$$

Clearly, f_e is even, f_o is odd and $f = f_e + f_o$. Similarly we define g_e, g_o, h_e, h_o, k_e, k_o. Obviously $f_o(0) = g_o(0) = h_o(0) = k_o(0) = 0$. Since functions f, g, h, k satisfy (1.3), then

$$f_e(x-z) + g_e(y-z) = h_e(x-y) + k_e\left(\frac{x+y}{2} - z\right), \quad x, y, z \in X, \tag{2.3}$$

$$f_o(x-z) + g_o(y-z) = h_o(x-y) + k_o\left(\frac{x+y}{2} - z\right), \quad x, y, z \in X. \tag{2.4}$$

Let $C_1 := f_e(0)$, $C_2 := g_e(0)$, $C_3 := h_e(0)$, $C_4 := k_e(0)$. Setting $x = y = z = 0$ in (2.3) we get $C_1 + C_2 = C_3 + C_4$. Let

$$\begin{aligned} f_1(x) &:= f_e(x) - C_1, \\ g_1(x) &:= g_e(x) - C_2, \\ h_1(x) &:= h_e(x) - C_3, \\ k_1(x) &:= k_e(x) - C_4 \end{aligned}$$

for all $x \in X$. Then f_1, g_1, h_1, k_1 are also even and $f_1(0) = g_1(0) = h_1(0) = k_1(0) = 0$. Moreover

$$f_1(x-z) + g_1(y-z) = h_1(x-y) + k_1\left(\frac{x+y}{2} - z\right), \quad x, y, z \in X. \tag{2.5}$$

Setting, successively, $y = x$, $z = 0$ and $x = z = 0$ and $y = z = 0$ in (2.5), we get

$$f_1(x) + g_1(x) = k_1(x), \tag{2.6}$$

$$g_1(x) = h_1(x) + k_1\left(\frac{x}{2}\right), \tag{2.7}$$

$$f_1(x) = h_1(x) + k_1\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. Comparing (2.7) and (2.8) we arrive at

$$f_1(x) = g_1(x), \quad x \in X. \tag{2.9}$$

Applying (2.9) to (2.6) one gets

$$k_1(x) = 2f_1(x), \quad x \in X. \tag{2.10}$$

Replacing x by $\frac{x}{2}$ in (2.10) we obtain

$$k_1\left(\frac{x}{2}\right) = 2f_1\left(\frac{x}{2}\right), \quad x \in X. \tag{2.11}$$

From (2.8) and (2.11) we have

$$f_1(x) = h_1(x) + 2f_1\left(\frac{x}{2}\right), \quad x \in X. \tag{2.12}$$

Setting $y = 0$ and $z = \frac{x}{2}$ in (2.5) we get

$$f_1\left(\frac{x}{2}\right) + g_1\left(\frac{x}{2}\right) = h_1(x), \quad x \in X. \tag{2.13}$$

Applying (2.9) to (2.13) one gets

$$h_1(x) = 2f_1\left(\frac{x}{2}\right), \quad x \in X. \tag{2.14}$$

Comparing (2.12) and (2.14) we arrive at

$$h_1(x) = \frac{1}{2}f_1(x), \quad x \in X.$$

Therefore

$$f_1(x) = g_1(x) = 2h_1(x) = \frac{1}{2}k_1(x), \quad x \in X.$$

Hence f_1 satisfies (1.2) and on account of Theorem 2.1 we define $f_1(x) := Q(x)$ for all $x \in X$, where $Q\colon X \to Y$ is a quadratic function. Thus

$$\begin{aligned} f_e(x) &= Q(x) + C_1, \\ g_e(x) &= Q(x) + C_2, \\ h_e(x) &= \frac{1}{2}Q(x) + C_3, \\ k_e(x) &= 2Q(x) + C_4 \end{aligned}$$

for all $x \in X$.

Setting, successively, $y = x$, $z = 0$ and $x = z = 0$ and $y = z = 0$ in (2.4), we get

$$f_o(x) + g_o(x) = k_o(x), \tag{2.15}$$

$$g_o(x) = -h_o(x) + k_o\left(\frac{x}{2}\right), \tag{2.16}$$

$$f_o(x) = h_o(x) + k_o\left(\frac{x}{2}\right) \tag{2.17}$$

for all $x \in X$. Comparing (2.15), (2.16) and (2.17) we arrive at

$$k_o(x) = 2k_o\left(\frac{x}{2}\right), \quad x \in X. \tag{2.18}$$

Putting $y = 0$ and $z = \frac{x}{2}$ in (2.4) we have

$$f_o\left(\frac{x}{2}\right) - g_o\left(\frac{x}{2}\right) = h_o(x), \quad x \in X. \tag{2.19}$$

From (2.16) and (2.17) we obtain

$$f_o(x) - g_o(x) = 2h_o(x), \quad x \in X, \tag{2.20}$$

hence

$$f_o\left(\frac{x}{2}\right) - g_o\left(\frac{x}{2}\right) = 2h_o\left(\frac{x}{2}\right), \quad x \in X. \tag{2.21}$$

Comparing (2.19) and (2.21) we see that

$$h_o(x) = 2h_o\left(\frac{x}{2}\right), \quad x \in X. \tag{2.22}$$

Setting $z = 0$ in (2.4) we get

$$f_o(x) + g_o(y) = h_o(x-y) + k_o\left(\frac{x+y}{2}\right), \quad x, y \in X. \tag{2.23}$$

Interchanging the roles of variables in (2.23) we obtain

$$f_o(y) + g_o(x) = -h_o(x-y) + k_o\left(\frac{x+y}{2}\right), \quad x, y \in X. \tag{2.24}$$

Adding (2.23) and (2.24), and applying (2.15) and (2.18) we get

$$\begin{aligned} k_o(x) + k_o(y) &= f_o(x) + g_o(x) + f_o(y) + g_o(y) \\ &= 2k_o\left(\frac{x+y}{2}\right) \\ &= k_o(x+y), \quad x, y \in X, \end{aligned}$$

i.e. k_o is an additive function. Subtracting (2.24) from (2.23) and applying (2.20) we have

$$\begin{aligned} 2h_o(x) - 2h_o(y) &= f_o(x) - g_o(x) - f_o(y) + g_o(y) \\ &= 2h_o(x-y), \quad x, y \in X, \end{aligned}$$

hence replacing y by $-y$ in the above equation we see that h_o is also an additive function. Since the functions h_o and k_o are additive, then (2.17) and (2.16) immediately imply that the functions f_o and g_o are also additive. Let

$$f_o(x) := E(x), \qquad g_o(x) := F(x), \quad x \in X,$$

where $E, F\colon X \to Y$ are additive functions. Therefore from (2.20) and (2.15) we have

$$\begin{aligned} h_o(x) &= \frac{1}{2}E(x) - \frac{1}{2}F(x), \quad x \in X, \\ k_o(x) &= E(x) + F(x), \quad x \in X. \end{aligned}$$

Finally, since $f = f_e + f_o$, then

$$f(x) = Q(x) + E(x) + C_1, \quad x \in X.$$

Similarly

$$\begin{aligned} g(x) &= Q(x) + F(x) + C_2, \\ h(x) &= \frac{1}{2}Q(x) + \frac{1}{2}E(x) - \frac{1}{2}F(x) + C_3, \\ k(x) &= 2Q(x) + E(x) + F(x) + C_4 \end{aligned}$$

for all $x \in X$, which completes the proof. □

3. Stability of equation (1.2)

Throughout this section we assume that X is a uniquely 2-divisible abelian group and Y is a Banach space. By $\mathbb{N}$ we denote the set of positive integers. Theorem 2.2 allows us to prove the Hyers-Ulam-Rassias stability of equation (1.3). However, in this paper we will only prove the stability of equation (1.2).

Theorem 3.1. *Let $f\colon X \to Y$ be a function satisfying the inequality*

$$\left\| f(x-z) + f(y-z) - \frac{1}{2} f(x-y) - 2f\left(\frac{x+y}{2} - z\right) \right\| \le \varphi(x,y,z) \tag{3.1}$$

for all $x, y, z \in X$, where $\varphi\colon X \times X \times X \to [0,\infty)$ is a function fulfilling the following conditions

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{4^n} = 0, \quad x, y, z \in X,$$

$$\psi(x) := 2\sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k} < \infty, \quad x \in X.$$

Then there exists a unique quadratic function $Q\colon X \to Y$ such that

$$\|f(x) - Q(x)\| \le \psi(x) + 2\psi(0), \quad x \in X.$$

Proof. Putting $x = y = z = 0$ in (3.1) we obtain

$$\|f(0)\| \le 2\varphi(0,0,0). \tag{3.2}$$

Replacing x by $4x$ and setting $y = z = 2x$ in (3.1) we get

$$\left\| \frac{1}{2} f(2x) + f(0) - 2f(x) \right\| \le \varphi(4x, 2x, 2x), \quad x \in X.$$

Defining a new function $f_1\colon X \to Y$ by $f_1(x) := f(x) - \frac{2}{3} f(0)$ for all $x \in X$ and dividing the above inequality by 2 we have

$$\left\| f_1(x) - \frac{1}{4} f_1(2x) \right\| \le \frac{1}{2}\varphi(4x, 2x, 2x), \quad x \in X. \tag{3.3}$$

Now we show by induction that

$$\left\| f_1(x) - \frac{1}{4^n} f_1(2^n x) \right\| \le 2\sum_{k=1}^{n} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k}, \quad x \in X. \tag{3.4}$$

For $n = 1$ we have (3.3). Assume the validity of the inequality (3.4) for some $n \in \mathbb{N}$ and for all $x \in X$. We will prove it for $n+1$. Thus

$$\begin{aligned}\left\| f_1(x) - \frac{1}{4^{n+1}} f_1(2^{n+1}x) \right\| &\le \left\| f_1(x) - \frac{1}{4} f_1(2x) \right\| + \left\| \frac{1}{4} f_1(2x) - \frac{1}{4^{n+1}} f_1(2^n \cdot 2x) \right\| \\ &\le \frac{1}{2}\varphi(4x, 2x, 2x) + \frac{1}{2} \sum_{k=1}^{n} \frac{\varphi(2^{k+2}x, 2^{k+1}x, 2^{k+1}x)}{4^k} \\ &= 2 \sum_{k=1}^{n+1} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k}, \quad x \in X,\end{aligned}$$

which proves (3.4) for all $n \in \mathbb{N}$. Hence by (3.4) we obtain that

$$\begin{aligned}\left\| \frac{f_1(2^n x)}{4^n} - \frac{f_1(2^m x)}{4^m} \right\| &= \frac{1}{4^m} \left\| \frac{f_1(2^n x)}{4^{n-m}} - f_1(2^m x) \right\| \\ &\le \frac{2}{4^m} \sum_{k=1}^{n-m} \frac{\varphi(2^{k+m+1}x, 2^{k+m}x, 2^{k+m}x)}{4^k} \\ &= 2 \sum_{k=m+1}^{n} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k}\end{aligned}$$

for all $x \in X$ and $m, n \in \mathbb{N}$ with $n > m$. Since the right-hand side of the above inequality tends to zero as $m \to \infty$, then $\left\{ \frac{f_1(2^n x)}{4^n} \right\}_{n\in\mathbb{N}}$ is a Cauchy sequence for all $x \in X$ and thus converges by the completeness of Y. Therefore we can define a function $Q\colon X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{4^n}, \quad x \in X.$$

Note that $Q(0) = 0$ and Q is even. Replacing x, y, z by $2^n x$, $2^n y$, $2^n z$ in (3.1) and dividing both sides by 4^n, and after then taking the limit in the resulting inequality as $n \to \infty$, we have

$$Q(x - z) + Q(y - z) - \frac{1}{2}Q(x - y) - 2Q\left(\frac{x + y}{2} - z\right) = 0, \quad x, y, z \in X.$$

Therefore on account of Theorem 2.1 a function Q is quadratic.

Taking the limit in (3.4) as $n \to \infty$, we obtain

$$\|f_1(x) - Q(x)\| \le 2 \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k}, \quad x \in X,$$

i.e. from (3.2) and the definition of f_1 we get

$$\begin{aligned}\|f(x) - Q(x)\| &\le 2 \sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k} + \frac{2}{3}\|f(0)\| \\ &\le \psi(x) + \frac{4}{3}\varphi(0, 0, 0) \\ &= \psi(x) + 2\psi(0), \quad x \in X. \end{aligned} \tag{3.5}$$

To prove the uniqueness, let Q_1 be another quadratic function satisfying (3.5). Thus we have

$$\begin{aligned}\|Q(x)-Q_1(x)\| &\le \left\|\frac{Q(2^n x)}{4^n}-\frac{f_1(2^n x)}{4^n}\right\|+\left\|\frac{Q_1(2^n x)}{4^n}-\frac{f_1(2^n x)}{4^n}\right\| \\ &= \frac{1}{4^n}\Big[\|Q(2^n x)-f_1(2^n x)\|+\|Q_1(2^n x)-f_1(2^n x)\|\Big] \\ &\le \frac{2}{4^n}\big[\psi(2^n x)+4\psi(0)\big], \quad x\in X.\end{aligned}$$

Taking the limit as $n\to\infty$, we conclude that $Q(x)=Q_1(x)$ for all $x\in X$, which completes the proof. ☐

Theorem 3.2. *Let $f\colon X\to Y$ be a function satisfying the inequality*

$$\left\|f(x-z)+f(y-z)-\frac{1}{2}f(x-y)-2f\left(\frac{x+y}{2}-z\right)\right\|\le\varphi^*(x,y,z) \tag{3.6}$$

for all $x,y,z\in X$, where $\varphi^*\colon X\times X\times X\to[0,\infty)$ is a function fulfilling the following conditions

$$\lim_{n\to\infty}4^n\varphi^*\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)=0, \quad x,y,z\in X,$$

$$\psi^*(x):=\frac{1}{2}\sum_{k=1}^{\infty}4^k\varphi^*\left(\frac{x}{2^{k-2}},\frac{x}{2^{k-1}},\frac{x}{2^{k-1}}\right)<\infty, \quad x\in X. \tag{3.7}$$

Then there exists a unique quadratic function $Q\colon X\to Y$ such that

$$\|f(x)-Q(x)\|\le\psi^*(x), \quad x\in X.$$

Proof. Setting $x=0$ in (3.7) we get $\sum\limits_{k=1}^{\infty}4^k\varphi^*(0,0,0)<\infty$, hence $\varphi^*(0,0,0)=0$. Putting $x=y=z=0$ in (3.6) we obtain $\|f(0)\|\le 2\varphi^*(0,0,0)=0$, i.e. $f(0)=0$. Replacing x by $2x$ and setting $y=z=x$ in (3.6), and multiplying both sides of the resulting inequality by 2 we get

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\|\le 2\varphi^*(2x,x,x), \quad x\in X. \tag{3.8}$$

An induction argument implies easily that

$$\left\|f(x)-4^n f\left(\frac{x}{2^n}\right)\right\|\le\frac{1}{2}\sum_{k=1}^{n}4^k\varphi^*\left(\frac{x}{2^{k-2}},\frac{x}{2^{k-1}},\frac{x}{2^{k-1}}\right), \quad x\in X. \tag{3.9}$$

Proceeding similarly as in the proof of Theorem 3.1, we easily have that $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence for all $x\in X$ and we can define a function $Q\colon X\to Y$ by

$$Q(x)=\lim_{n\to\infty}4^n f\left(\frac{x}{2^n}\right), \quad x\in X.$$

Note that $Q(0)=0$ and Q is even. Taking the limit in (3.9) as $n\to\infty$, we obtain

$$\|f(x)-Q(x)\|\le\frac{1}{2}\sum_{k=1}^{\infty}4^k\varphi^*\left(\frac{x}{2^{k-2}},\frac{x}{2^{k-1}},\frac{x}{2^{k-1}}\right)=\psi^*(x), \quad x\in X. \tag{3.10}$$

As we did in the proof of Theorem 3.1, we can similarly show that Q is a unique quadratic function satisfying (3.10). The proof is completed. □

Corollary 3.3. *Let $\varepsilon \geq 0$ and $p \neq 2$ be fixed real numbers. Assume that a function $f\colon X \to Y$ satisfies the inequality*

$$\left\| f(x-z) + f(y-z) - \frac{1}{2}f(x-y) - 2f\left(\frac{x+y}{2} - z\right)\right\| \leq \varepsilon\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \tag{3.11}$$

for all $x, y, z \in X$ ($x, y, z \in X\backslash\{0\}$ if $p < 0$). Then there exists a unique quadratic function $Q\colon X \to Y$ such that

$$\big\|f(x) - Q(x)\big\| \leq \frac{2^{p+1}(2^p+2)\varepsilon\|x\|^p}{|4-2^p|}, \quad x \in X.$$

Proof. We apply Theorems 3.1 and 3.2 with $\varphi(x,y,z) = \varphi^*(x,y,z) := \varepsilon\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in X$ ($x, y, z \in X\backslash\{0\}$ if $p < 0$). It is not hard to check that these Theorems can be applied to the above function with $p < 2$ and $p > 2$, respectively. If $p < 2$, we have

$$\begin{aligned}\psi(x) &= 2\sum_{k=1}^{\infty} \frac{\varphi(2^{k+1}x, 2^k x, 2^k x)}{4^k} \\ &= 2(2^p+2)\sum_{k=1}^{\infty} 2^{k(p-2)}\varepsilon\|x\|^p \\ &= \frac{2^{p+1}(2^p+2)\varepsilon\|x\|^p}{4-2^p}\end{aligned}$$

for all $x \in X$ ($x \in X\backslash\{0\}$ if $p < 0$). If $p > 2$, we have

$$\begin{aligned}\psi^*(x) &= \frac{1}{2}\sum_{k=1}^{\infty} 4^k \varphi^*\left(\frac{x}{2^{k-2}}, \frac{x}{2^{k-1}}, \frac{x}{2^{k-1}}\right) \\ &= 2^{p-1}(2^p+2)\sum_{k=1}^{\infty} 2^{k(2-p)}\varepsilon\|x\|^p \\ &= \frac{2^{p+1}(2^p+2)\varepsilon\|x\|^p}{2^p-4}\end{aligned}$$

for all $x \in X$. Thus applying Theorems 3.1 and 3.2 to the two cases $p < 2$ and $p > 2$, respectively, we obtain easily the result. □

Corollary 3.4. *Let $\varepsilon \geq 0$ be fixed real number. Assume that a function $f\colon X \to Y$ satisfies the inequality*

$$\left\| f(x-z) + f(y-z) - \frac{1}{2}f(x-y) - 2f\left(\frac{x+y}{2} - z\right)\right\| \leq \varepsilon \tag{3.12}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q\colon X \to Y$ such that

$$\big\|f(x) - Q(x)\big\| \leq 2\varepsilon, \quad x \in X. \tag{3.13}$$

Proof. Putting $\varphi(x, y, z) := \varepsilon$ in Theorem 3.1, we get immediately the result. □

Remark 3.5. Observe that the estimation (3.13) in Corollary 3.4 cannot be sharpened. To see that, fix a vector $e \in Y$ from the unit ball, and define a function $f\colon X \to Y$ by the formula $f(x) = 2\varepsilon e$ for all $x \in X$. Then inequality (3.12) is satisfied, so there exists a quadratic function $Q\colon X \to Y$ such that the condition (3.13) holds. Since the function f is bounded, then $Q = 0$. Thus $\|f(x) - Q(x)\| = \|2\varepsilon e\| = 2\varepsilon$ for all $x \in X$.

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