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Fixed points for asymptotic contractions of integral Meir-Keeler type

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`This paper is dedicated to Professor Ljubomir Ćirić`

`Communicated by Professor V. Berinde`

Abstract

In this paper we introduce the notion of asymptotic contraction of integral Meir-Keeler type on a metric space and we prove a theorem which ensures existence and uniqueness of fixed points for such contractions. This result generalizes some recent results in the literature.

Keywords: Fixed points; Asymptotic contractions of integral type; Contractions of Meir-Keeler type.
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1. Introduction and preliminaries

Fixed point theory is an important and actual topic of nonlinear analysis. For the most important contributions on the metric and non-metric setting, see Goebel and Kirk [3], Kirk and Kang [4] and Kirk and Sims [5] (and the references therein). In 1969, Meir and Keeler [7] proved the following very interesting fixed point theorem, which is a generalization of the Banach contraction principle [1]. See also [8, 9, 10].

Theorem 1.1 (Meir and Keeler [7]). *Let (X, d) be a complete metric space and T be a mapping on X. Assume that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \le d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) < \varepsilon$ for $x, y \in X$. Then T has a unique fixed point.*

In 2002, Branciari [2] introduced a contraction of integral type and proved the following fixed point theorem, which is also a generalization of the Banach contraction principle.

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Theorem 1.2. *Let (X,d) be a complete metric space, $c \in]0,1[$, and $f : X \to X$ be a mapping such that for each $x,y \in X$,*

$$\int_0^{d(fx,fy)} \psi(t)dt \le c \int_0^{d(x,y)} \psi(t)dt,$$

where $\psi : [0,+\infty[\to [0,+\infty[$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty[$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \psi(t)dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to+\infty} f^n x = a$.

In 2003, Kirk [6] introduced the notion of asymptotic contraction on a metric space.

Definition 1.3. Let (X,d) be a metric space and let T be a mapping on X. Then T is called an asymptotic contraction on X if there exists a continuous function φ from $[0,+\infty[$ into itself and a sequence $\{\varphi_n\}$ of functions from $[0,+\infty[$ into itself such that

(i) $\varphi(0) = 0$,

(ii) $\varphi(r) < r$ for $r \in]0,+\infty[$,

(iii) $\{\varphi_n\}$ converges to φ uniformly on the range of d,

(iv) for $x,y \in X$ and $n \in \mathbb{N}$,
$$d(T^n x, T^n y) \le \varphi_n(d(x,y)).$$

For the class of asymptotic contractions, we have the following interesting result.

Theorem 1.4 (Kirk [6])**.** *Let (X,d) be a complete metric space and T be a continuous, asymptotic contraction on X with $\{\varphi_n\}$ and φ in Definition 1.3. Assume that there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x is bounded, and that φ_n is continuous for $n \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.*

Recently, Suzuki [11] introduced the notion of asymptotic contraction of Meir-Keeler type on a metric space, and proved a fixed point theorem for such class of contractions.

Definition 1.5. Let (X,d) be a metric space. Then a mapping T on X is said to be an asymptotic contraction of Meir-Keeler type (ACMK, for short) if there exists a sequence $\{\varphi_n\}$ of functions from $[0,+\infty[$ into itself satisfying the following:

(i) $\limsup_{n\to+\infty} \varphi_n(\varepsilon) \le \varepsilon$ for all $\varepsilon > 0$,

(ii) for each $\varepsilon > 0$ there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\varphi_\nu(t) \le \varepsilon$ for all $t \in [\varepsilon, \varepsilon+\delta]$,

(iii) $d(T^n x, T^n y) < \varphi_n(d(x,y))$ for all $n \in \mathbb{N}$ and $x,y \in X$ with $x \ne y$.

Theorem 1.6. *Let (X,d) be a complete metric space and T be an ACMK on X. Assume that T^m is continuous for some $m \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.*

Remark 1.7. Every contraction of Meir-Keeler type and each asymptotic contraction on a metric space is an asymptotic contraction of Meir-Keeler type (see Propositions 2 and 3 of [11]).

In this paper, we introduce the notion of asymptotic contraction of integral Meir-Keeler type, and prove a fixed point theorem for such contractions. Our result is a generalization of Theorem 1.6. Moreover, since Theorem 1.6 is a generalization of Theorems 1.1 and 1.4, our result generalizes also Theorems 1.1 and 1.4.

2. Asymptotic contraction of integral Meir-Keeler type

In this section we introduce the notion of asymptotic contraction of Meir-Keeler type, and prove a fixed point result for such class of contractions.

Let Ψ be the class of functions $\psi : [0, +\infty[\to [0, +\infty[$ with the following properties:

(j) ψ is Lebesgue-integrable on each interval $[0, a[$, with $a > 0$,

(jj) $\int_0^{\varepsilon} \psi(t)dt > 0$ for each $\varepsilon > 0$.

Definition 2.1. Let (X, d) be a metric space. Then a mapping T on X is said to be an asymptotic contraction of integral Meir-Keeler type (ACIMK, for short) if there exists a sequence $\{\varphi_n\}$ of functions from $[0, +\infty[$ into itself satisfying the following:

(i) $\limsup_{n\to+\infty} \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon > 0$,

(ii) for each $\varepsilon > 0$ there exist $\delta > 0$ and $s \in \mathbb{N}$ such that $\varphi_s(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$,

(iii) $\int_0^{d(T^n x, T^n y)} \psi(t)dt < \varphi_n(\int_0^{d(x,y)} \psi(t)dt)$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$, where $\psi \in \Psi$.

Lemma 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. Assume that there exists a sequence $\{\varphi_n\}$ of functions from $[0, +\infty[$ into itself satisfying the following:*

(a) for each $\varepsilon > 0$ there exist $\delta > 0$ and $s \in \mathbb{N}$ such that $\varphi_s(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$,

(b) $\int_0^{d(T^n x, T^n y)} \psi(t)dt < \varphi_n(\int_0^{d(x,y)} \psi(t)dt)$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$, where $\psi \in \Psi$.

If $d(T^n u, T^{n+1} u) \to 0$ for some $u \in X$, then $\{T^n u\}$ is a Cauchy sequence.

Proof. For fixed $\varepsilon > 0$, let $\sigma = \int_0^{\varepsilon} \psi(t)dt$. By (a), there exist $\delta > 0$ and $s \in \mathbb{N}$ such that $\varphi_s(t) \leq \sigma$ for each $t \in [\sigma, \sigma + \delta]$. Now, we choose $\nu \in]0, \varepsilon[$ such that

$$\int_{\varepsilon}^{\varepsilon+\nu} \psi(t)dt < \delta.$$

In correspondence of ν, there exists $n(\nu) \in \mathbb{N}$ such that $d(u_n, u_{n+1}) < \frac{\nu}{s}$ for all $n \geq n(\nu)$, where $u_n = T^n u$. Suppose that there exist $m, p \in \mathbb{N}$, with $m > p \geq n(\nu)$ such that $d(u_m, u_p) > 2\varepsilon$ and define

$$k = \min\{j \in \mathbb{N} : p < j \text{ and } \varepsilon + \nu \leq d(u_p, u_j)\} \leq m.$$

From

$$2\nu < \varepsilon + \nu \leq d(u_p, u_k) \leq \sum_{j=p}^{k-1} d(u_j, u_{j+1}) \leq \sum_{j=p}^{k-1} \frac{\nu}{s} = (k - p)\frac{\nu}{s},$$

we deduce that $2s < k - p$ and hence $p < k - 2s < k - s$. It implies that $d(u_p, u_{k-s}) < \varepsilon + \nu$. Then

$$\begin{aligned} d(u_p, u_{k-s}) &\geq d(u_p, u_k) - d(u_{k-s}, u_k) \\ &\geq d(u_p, u_k) - \sum_{j=0}^{s-1} d(u_{k-j-1}, u_{k-j}) \\ &\geq \varepsilon + \nu - s\frac{\nu}{s} = \varepsilon. \end{aligned}$$

Consequently,

$$\sigma = \int_0^{\varepsilon} \psi(t)dt \leq \int_0^{d(u_p, u_{k-s})} \psi(t)dt \leq \int_0^{\varepsilon+\nu} \psi(t)dt < \sigma + \delta.$$

We show that $d(u_{p+s}, u_k) \leq \varepsilon$. If $d(u_{p+s}, u_k) > \varepsilon$, by (b), we have

$$\begin{aligned}\int_0^{\varepsilon} \psi(t)dt &\leq \int_0^{d(u_{p+s},u_k)} \psi(t)dt = \int_0^{d(T^s u_p, T^s u_{k-s})} \psi(t)dt \\ &< \varphi_s(\int_0^{d(u_p,u_{k-s})} \psi(t)dt) \\ &\leq \int_0^{\varepsilon} \psi(t)dt = \sigma,\end{aligned}$$

which is a contradiction. Then

$$d(u_p, u_k) \leq \sum_{j=1}^{s} d(u_{p+j-1}, u_{p+j}) + d(u_{p+s}, u_k) < s\frac{\nu}{s} + \varepsilon = \nu + \varepsilon,$$

that is a contradiction with the definition of k. Therefore $d(u_n, u_m) < 2\varepsilon$ for all $m > n \geq n(\nu)$ and so $\{u_n\}$ is a Cauchy sequence.

□

Theorem 2.3. *Let (X, d) be a complete metric space and T be an ACIMK on X. Assume that T^m is continuous for some $m \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.*

Proof. Let $\{\varphi_n\}$ be as in Definition 2.1. We first show that

$$\lim_{n\to+\infty} d(T^n x, T^n y) = 0 \quad \text{for all } x, y \in X. \tag{2.1}$$

Fix $x, y \in X$ with $x \neq y$. If $T^m x = T^m y$ for some $m \in \mathbb{N}$, clearly (2.1) holds. We assume that $T^m x \neq T^m y$ for all $m \in \mathbb{N}$ and define

$$\alpha := \limsup_{n\to+\infty} \int_0^{d(T^n x, T^n y)} \psi(t)dt > 0.$$

Now, (ii) of Definition 2.1 ensures that there is $s \in \mathbb{N}$ such that

$$\int_0^{d(T^s x, T^s y)} \psi(t)dt < \varphi_s(\int_0^{d(x,y)} \psi(t)dt) \leq \int_0^{d(x,y)} \psi(t)dt.$$

By (i) of Definition 2.1, we have

$$\begin{aligned}\alpha &:= \limsup_{n\to+\infty} \int_0^{d(T^{n+s}x, T^{n+s}y)} \psi(t)dt \\ &\leq \limsup_{n\to+\infty} \varphi_n(\int_0^{d(T^s x, T^s y)} \psi(t)dt) \\ &\leq \int_0^{d(T^s x, T^s y)} \psi(t)dt \\ &< \varphi_s(\int_0^{d(x,y)} \psi(t)dt) \leq \int_0^{d(x,y)} \psi(t)dt.\end{aligned}$$

Consequently, we deduce that $\alpha < \int_0^{d(T^p x, T^p y)} \psi(t)dt$ for all $p \in \mathbb{N}$ and hence

$$\lim_{n\to+\infty} \int_0^{d(T^n x, T^n y)} \psi(t)dt = \alpha. \tag{2.2}$$

By (ii) of Definition 2.1, there exist $\delta > 0$ and $m \in \mathbb{N}$ such that $\varphi_m(t) \leq \alpha$ for every $t \in [\alpha, \alpha + \delta]$. Now, we choose $p \in \mathbb{N}$ such that

$$\int_0^{d(T^p x, T^p y)} \psi(t)dt \leq \alpha + \delta.$$

From

$$\int_0^{d(T^{m+p}x, T^{m+p}y)} \psi(t)dt < \varphi_m(\int_0^{d(T^p x, T^p y)} \psi(t)dt) \leq \alpha,$$

which is a contradiction, we deduce that $\alpha = 0$. Therefore, we obtain (2.1) as consequence of the property $\int_0^\varepsilon \psi(t)dt > 0$ for all $\varepsilon > 0$ and (2.2), with $\alpha = 0$.

Let $x \in X$ and consider the sequence $\{T^n x\}$, which is a Cauchy sequence by Lemma 2.2. Since X is complete, there exists $z \in X$ such that $T^n x \to z$. Then, from the continuity of T^m, we have

$$z = \lim_{n \to +\infty} T^{n+m} x = \lim_{n \to +\infty} T^m(T^n x) = T^m z,$$

that is, z is a fixed point of T^m. Since

$$\lim_{n \to +\infty} d(T^{nm+1}x, Tz) = \lim_{n \to +\infty} d(T^{nm+1}x, T^{nm+1}z) = 0$$

by (2.1), we have

$$Tz = \lim_{n \to +\infty} T^{nm+1} x = z,$$

that is, z is a fixed point of T. If $Tx = x$, then

$$d(z, x) = \lim_{n \to +\infty} d(T^n z, T^n x) = 0$$

by (2.1), and hence $x = z$. Therefore the fixed point of T is unique. Finally, since x is arbitrary, $\lim_{n \to +\infty} T^n x = z$ for every $x \in X$. This completes the proof.

□

Remark 2.4. Every asymptotic contraction of Meir-Keeler is an asymptotic contraction of integral Meir-Keeler type and so Theorem 2.3 is a generalization of Theorem 1.6. Moreover, since each contraction of Branciari is an asymptotic contraction of integral Meir-Keeler type, we deduce that Theorem 2.3 is a generalization of Theorem 1.2.

The following example shows that Theorem 2.3 is a proper generalization of Theorem 1.2.

Example 2.5. Let $X = [0, +\infty[$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Define $T : X \to X$ and $\psi, \varphi : [0, +\infty[\to [0, +\infty[$ by

$$T(x) = \frac{x}{1+x}, \quad \forall\, x \in X, \quad \psi(t) = 2t \text{ and } \varphi(t) = \frac{t}{1+t}, \ \forall\, t \in [0, +\infty[.$$

We have

$$\begin{aligned} \int_0^{d(Tx,Ty)} \psi(t)dt &= \frac{|x-y|^2}{[(1+x)(1+y)]^2} \\ &< \frac{|x-y|^2}{1+|x-y|^2} \\ &= \varphi(|x-y|^2) \\ &= \varphi(\int_0^{d(x,y)} \psi(t)dt). \end{aligned}$$

This implies that T is an asymptotic contraction of integral Meir-Keeler type with respect to the sequence $\{\varphi_n\}$, where $\varphi_n = \varphi$ for all $n \in \mathbb{N}$. Therefore all the conditions of Theorem 2.3 are fulfilled. Consequently, it follows from Theorem 2.3 that T has a unique fixed point $0 \in X$.
In this case Theorem 1.2 cannot be used to have the existence of a fixed point of T in X because its assumptions are not satisfied. In fact, assume that there exists some constant $c \in]0, 1[$ such that

$$\int_0^{d(Tx,Ty)} \psi(t)dt \leq c \int_0^{d(x,y)} \psi(t)dt,$$

that is

$$\frac{|x-y|^2}{[(1+x)(1+y)]^2} \leq c|x-y|^2$$

for all $x, y \in X$ with $x \neq y$. This yields that $1 \leq c < 1$, which is a contradiction.

Now, we give an example of an asymptotic contraction of integral Meir-Keeler type that is not an asymptotic contraction of Meir-Keeler type.

Example 2.6. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Define $T : X \to X$ and $\psi, \varphi_n : [0, +\infty[\to [0, +\infty[$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n}, \end{cases} \qquad \psi(t) = \begin{cases} 0 & \text{if } t = 0 \\ t^{1/t-2}[1 - \ln t] & \text{if } t \in]0, 1/2] \\ 1/4 & \text{if } t > 1/2, \end{cases}$$

$$\varphi_n(t) = \begin{cases} t & \text{if } n \text{ is odd} \\ t/2 & \text{if } n \text{ is even.} \end{cases}$$

Since

$$\int_0^{d(Tx,Ty)} \psi(t)dt \leq \frac{1}{2} \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$ with $x \neq y$ (see Example 3.6 of [2]), we deduce that T is an asymptotic contraction of integral Meir-Keeler type with respect to the sequence $\{\varphi_n\}$.
We note that for every even $n \in \mathbb{N}$, one can choose $p \in \mathbb{N}$ such that $\frac{p}{n+p} > k$ for every $k \in]0, 1[$. Then, for $x = 0$ and $y = 1/p$, we have

$$d(T^n x, T^n y) = \frac{1}{n+p} > \frac{k}{p} = k\, d(x, y).$$

It follows that T is not an asymptotic contraction of Meir-Keeler type with respect to the sequence $\{\varphi_n\}$.

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