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Existence and Ulam-Hyers stability results for coincidence problems

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Dedicated to the memory of Professor Viorel Radu

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Abstract

Let X, Y be two nonempty sets and $s, t : X \to Y$ be two single-valued operators.

By definition, a solution of the coincidence problem for s and t is a pair $(x^*, y^*) \in X \times Y$ such that

$$s(x^*) = t(x^*) = y^*.$$

It is well-known that a coincidence problem is, under appropriate conditions, equivalent to a fixed point problem for a single-valued operator generated by s and t. Using this approach, we will present some existence, uniqueness and Ulam - Hyers stability theorems for the coincidence problem mentioned above. Some examples illustrating the main results of the paper are also given.

Keywords: metric space, coincidence problem, singlevalued contraction, vector-valued metric, fixed point, Ulam-Hyers stability.
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1. Existence and Ulam-Hyers stability results for coincidence problems

Let (X, d), (Y, ρ) be two metric spaces and $s, t : X \to Y$ be two operators.
We denote by $Fix(s) := \{x \in X \mid s(x) = x\}$ the fixed point set of the operator s. Let us consider the following coincidence problem:

$$\text{find } (x, y) \in X \times Y \text{ such that } s(x) = t(x) = y. \tag{1.1}$$

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Definition 1.1. A solution of the coincidence problem (1.1) for s and t is a pair $(x^*, y^*) \in X \times Y$ such that

$$s(x^*) = t(x^*) = y^*.$$

Denote by $CP(s,t) \subset X \times Y$ the set of all solution for the coincidence problem (1.1).

Let (X,d), (Y,ρ) be two metric spaces and $s,t : X \to Y$ be two operators such that t is a injection. Then, t has a left inverse $t_l^{-1} : t(X) \to X$. Suppose also that $s(X) \subseteq t(X)$. Consider $f : X \times t(X) \to X \times t(X)$ defined by

$$f(x_1, x_2) = (t_l^{-1}(x_2), s(x_1)).$$

Lemma 1.2. *Under the above mentioned conditions, we have* $CP(s,t) = Fix(f)$.

Proof. We successively have $(x^*, y^*) \in Fix(f) \iff (x^*, y^*) = (t_l^{-1}(y^*), s(x^*)) \iff y^* = t(x^*)$ and $y^* = s(x^*) \iff t(x^*) = s(x^*) = y^* \iff (x^*, y^*) \in CP(s,t)$. Thus $CP(s,t) = Fix(f)$. □

Let (X,d), (Y,ρ) be two metric spaces, let d_Z be a metric (generated by d and ρ) on $Z := X \times Y$ and $s,t : X \to Y$ be two operators. Let us consider the coincidence problem (1.1).

Definition 1.3. The coincidence problem (1.1) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0,0) = 0$ such that for every $\varepsilon_1, \varepsilon_2 > 0$ and for each $w^* := (u^*, v^*) \in X \times Y$ an $(\varepsilon_1, \varepsilon_2)$-solution of the coincidence problem (1.1), i.e. $w^* := (u^*, v^*)$ satisfies the inequations $\begin{cases} \rho(s(u^*), v^*) \le \varepsilon_1 \\ \rho(t(u^*), v^*) \le \varepsilon_2, \end{cases}$ there exists a solution $z^* := (x^*, y^*)$ of (1.1) such that

$$d_Z(w^*, z^*) \le \psi(\varepsilon_1, \varepsilon_2). \tag{1.2}$$

If there exists $c_1, c_2 > 0$ such that $\psi(t_1, t_2) = c_1 t_1 + c_2 t_2$ for each $t_1, t_2 \in \mathbb{R}_+$ then the coincidence problem (1) is said to be Ulam-Hyers stable.

Definition 1.4. Let (X,d) be a metric space. An operator $f : X \to X$ is called contraction if there exists a constant $k \in [0,1[$ such that

$$d(f(x), f(y)) \le k \cdot d(x,y), \text{ for each } x, y \in X.$$

Definition 1.5. Let (X,d) be a metric space. An operator $f : X \to X$ is called dilatation if there exists a constant $k > 0$ such that

$$d(f(x), f(y)) \ge k \cdot d(x,y), \text{ for each } x, y \in X.$$

Our first result is the following.

Theorem 1.6. *Let* (X,d) *and* (Y,ρ) *be two complete metric spaces. Suppose that the operator* $t : X \to Y$ *is a dilatation with constant* $k_t > 1$, *the operator* $s : X \to Y$ *is a contraction with constant* $k_s < 1$ *and* $s(X) \subseteq t(X)$. *Then the coincidence problem* (1.1) *for* s *and* t *has a unique solution.*

Proof. Since the operator $t : X \to Y$ is a dilatation with constant $k_t > 1$, we get that t is an injection and its left inverse $t_l^{-1} : \overline{t(X)} \to X$ is a contraction with constant $\frac{1}{k_t} < 1$, i.e.,

$$d(t_l^{-1}(y_1), t_l^{-1}(y_2)) \le \frac{1}{k_t} \cdot \rho(y_1, y_2), \text{ for each } y_1, y_2 \in \overline{t(X)}.$$

Let us consider $Z := X \times \overline{t(X)}$ and define $d^* : Z \times Z \to \mathbb{R}_+$ by

$$d^*((x_1, x_2), (u_1, u_2)) = d(x_1, u_1) + \rho(x_2, u_2),$$

for each $x = (x_1, x_2), u = (u_1, u_2) \in Z$. Then, (Z, d^*) is a complete metric space.

We prove that $f : Z \to Z,\quad f(x_1, x_2) := (t_l^{-1}(x_2), s(x_1))$ is a contraction on (Z, d^*). Indeed, we have:

$$d^*(f(x), f(u)) = d^*(f(x_1, x_2), f(u_1, u_2)) = d^*((t_l^{-1}(x_2), s(x_1)), (t_l^{-1}(u_2), s(u_1))) =$$

$$= d(t_l^{-1}(x_2), t_l^{-1}(u_2)) + \rho(s(x_1), s(u_1)) \le \frac{1}{k_t} \cdot \rho(x_2, u_2) + k_s \cdot d(x_1, u_1) \le$$

$$\le \max\left\{\frac{1}{k_t}, k_s\right\} \cdot (d(x_1, u_1) + \rho(x_2, u_2)) = \max\left\{\frac{1}{k_t}, k_s\right\} \cdot d^*((x_1, x_2), (u_1, u_2)) =$$

$$= \max\left\{\frac{1}{k_t}, k_s\right\} \cdot d^*(x, u).$$

Since $k := \max\left\{\frac{1}{k_t}, k_s\right\} < 1$, we deduce that

$$d^*(f(x), f(u)) \le k \cdot d^*(x, u), \text{ for each } (x, u) \in Z \times Z.$$

Hence f is a contraction with constant $k < 1$. By Banach's contraction principle we obtain that there exists a unique $x^* \in Z$ such that $x^* = f(x^*)$, i.e. $Fix(f) = \{x^*\}$. Thus, by Lemma 1.2 we obtain the conclusion. □

Remark 1.7. We also have the following estimation:

$$d^*((x_1, x_2), (x_1^*, x_2^*)) \le \frac{1}{1-k} \cdot d^*((x_1, x_2), (t_l^{-1}(x_2), s(x_1))),$$

for each $(x_1, x_2), (x_1^*, x_2^*) \in Z$.

Theorem 1.8. *Let $(X, d), (Y, \rho)$ be two complete metric spaces. Suppose that all the hypotheses of Theorem 1.6 hold and additionally suppose that for each $(u, v) \in X \times Y$ we have: $d(u, t_l^{-1}(v)) \le \rho(t(u), v)$. Then the coincidence problem* (1.1) *is Ulam-Hyers stable.*

Proof. Let $\varepsilon_1, \varepsilon_2 > 0$ and $w := (u, v) \in Z := X \times \overline{t(X)}$ be a solution of (2), i.e.,

$$\rho(s(u), v) \le \varepsilon_1 \text{ and } \rho(t(u), v) \le \varepsilon_2.$$

By Theorem 1.6 there exists a unique $x^* := (x_1^*, x_2^*) \in CP(s, t) = Fix(f)$, where $f : Z \to Z,\ f(x_1, x_2) := (t_l^{-1}(x_2), s(x_1))$. From Remark 1.7, we have:

$$d^*((x_1, x_2), (x_1^*, x_2^*)) \le \frac{1}{1-k} \cdot d^*((x_1, x_2), (t_l^{-1}(x_2), s(x_1))), \text{ for each } x = (x_1, x_2) \in Z.$$

Then we obtain that:

$$d(x_1, x_1^*) + \rho(x_2, x_2^*) \le \frac{1}{1-k}[d(x_1, t_l^{-1}(x_2)) + \rho(x_2, s(x_1))].$$

Considering $x := (u, v) \in Z$ we get $d(u, t_l^{-1}(v)) \le \rho(t(u), v)$. Thus, we have:
$d(u, x_1^*) + \rho(v, x_2^*) \le \frac{1}{1-k}[d(u, t_l^{-1}(v)) + \rho(v, s(u))] \le \frac{1}{1-k}[\rho(t(u), v) + \varepsilon_1] \le \frac{1}{1-k}(\varepsilon_1 + \varepsilon_2)$.

Hence,

$$d^*(w, x^*) \le \frac{1}{1-k}(\varepsilon_1 + \varepsilon_2),$$

proving that the coincidence problem (1) is Ulam-Hyers stable. □

Similar proofs for Theorem 1.6 and Theorem 1.8 are possible if we consider on $Z := X \times \overline{t(X)}$ the metric $d^* : Z \times Z \to \mathbb{R}_+$ defined by

$$d^*((x_1, x_2), (u_1, u_2)) = \max\{d(x_1, u_1), \rho(x_2, u_2)\}.$$

We consider now the case of a vector-valued metric on $X \times X$. We will show the advantages of this approach, since the assumptions of the main result are weaker than that in the above theorems. Notice that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^2$.

We recall for some notations and concepts. Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(d1) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, and $c \in \mathbb{R}^m$, by $\alpha \leq \beta$ (respectively, $\alpha < \beta$) we mean that $\alpha_i \leq \beta_i$ (respectively, $\alpha_i < \beta_i$) for $i \in \{1, 2, \ldots, m\}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$ for $i \in \{1, 2, \ldots, m\}$.

A set X equipped with a vector-valued metric d is called a generalized metric space. We will denote such a space with (X, d). For the generalized metric spaces, the notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Theorem 1.9. *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following are equivalents:*
(i) $A^n \to 0$ as $n \to \infty$;
(ii)The eigen-values of A are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;
(iii) The matrix $I - A$ is non-singular and

$$(I - A)^{-1} = I + A + ... + A^n + ...;$$

(iv) The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnenegative elements.
(v) $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

We need, for the proof of our next result, the so-called Perov's fixed point theorem, see [6].

Theorem 1.10. *(A.I. Perov, [6]) Let (X, d) be a complete generalized metric space and the mapping $f : X \to X$ with the property that there exists a matrix $A \in M_{m,m}(\mathbb{R})$ such that*

$$d(f(x), f(y)) \leq Ad(x, y) \text{ for all } x, y \in X.$$

If A is a matrix convergent towards zero, then:

1) $Fix(f) = \{x^*\}$;

2) the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$;

3) one has the following estimation

$$d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1);$$

4) If $g : X \to X$ satisfies the condition $d(f(x), g(y)) \leq \eta$, for all $x \in X, \eta \in \mathbb{R}^m$ and considering the sequence $y_n = g^n(x_0)$ one has

$$d(y_n, x^*) \leq (I - A)^{-1}\eta + A^n(I - A)^{-1}d(x_0, x_1).$$

Theorem 1.11. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that the operator $t : X \to Y$ is a dilatation with constant $k_t > 0$, the operator $s : X \to Y$ is Lipschitz with the constant $k_s > 0$ and $s(X) \subset t(X)$. If $\frac{k_s}{k_t} \in [0, 1)$, then the coincidence problem (1.1) for s and t has a unique solution.*

Proof. Since the operator $t : X \to Y$ is a dilatation with the $k_t > 0$, we have that t is injective and its left inverse $t_l^{-1} : \overline{t(X)} \to X$ is Lipschitz with constant $\frac{1}{k_t} > 0$, i.e.,

$$d(t_l^{-1}(y_1), t_l^{-1}(y_2)) \leq \frac{1}{k_t} \cdot \rho(y_1, y_2), \text{ for each } y_1, y_2 \in \overline{t(X)}.$$

Let us consider on $Z := X \times \overline{t(X)}$ a vectorial metric $d^V : Z \times Z \to \mathbb{R}_+^2$ defined by

$$d^V(x, u) = d^V((x_1, x_2), (u_1, u_2)) = (d(x_1, u_1), \rho(x_2, u_2)),$$

for each $(x, u) \in Z \times Z$.

We prove that $f : Z \to Z$, $f(x_1, x_2) := (t_l^{-1}(x_2), s(x_1))$ is an A-contraction on the space (Z, d^V). Indeed, we have:

$$d^V(f(x), f(u)) = d^V(f(x_1, x_2), f(u_1, u_2)) = d^V((t_l^{-1}(x_2), s(x_1)), (t_l^{-1}(u_2), s(u_1))) =$$
$$= (d(t_l^{-1}(x_2), t_l^{-1}(u_2)), \rho(s(x_1), s(u_1))) \leq \left(\frac{1}{k_t} \cdot \rho(x_2, u_2), k_s \cdot d(x_1, u_1)\right) =$$
$$= \begin{pmatrix} 0 & \frac{1}{k_t} \\ k_s & 0 \end{pmatrix} \cdot \begin{pmatrix} d(x_1, u_1) \\ \rho(x_2, u_2) \end{pmatrix}.$$

If we denote $A := \begin{pmatrix} 0 & \frac{1}{k_t} \\ k_s & 0 \end{pmatrix}$, then we got that $d^V(f(x), f(u)) \leq A \cdot d^V(x, u)$.

Since $\frac{k_s}{k_t} \in [0, 1)$, we deduce that A is a matrix convergent to zero.

We apply Perov's fixed point theorem for f and we deduce that there exists a unique fixed point for f, i.e., $Fix(f) = \{x^*\}$. □

Remark 1.12. We have the following estimation:

$$d^V((x_1, x_2), (x_1^*, x_2^*)) \leq (I - A)^{-1} \cdot d^V((x_1, x_2), (t_l^{-1}(x_2), s(x_1))),$$

for each $(x_1, x_2), (x_1^*, x_2^*) \in Z$.

Notice that, by Theorem 1.10 we also obtain an approximation and an error estimate for the solution of the coincidence problem, as well as a data dependence theorem.

Theorem 1.13. *Let $(X, d), (Y, \rho)$ be two complete metric spaces. Suppose that all the hypotheses of Theorem 1.11 hold and suppose additionally that for each $(u, v) \in X \times Y$ we have that $d(u, t_l^{-1}(v)) \leq \rho(t(u), v)$. Then the coincidence problem* (1.1) *is Ulam-Hyers stable, i.e., for each $\varepsilon_1, \varepsilon_2 > 0$ and for each $w := (u, v) \in X \times Y$ solution of (2), there exist a matrix $C \in \mathcal{M}_{22}(\mathbb{R}_+)$ and a solution x^* of (1) such that $d^V(w, x^*) \leq C \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$.*

Proof. Let $\varepsilon_1, \varepsilon_2 > 0$ and $w := (u, v) \in X \times Y$ be a solution of (2), i.e.,

$$\rho(t(u), v) \leq \varepsilon_1 \text{ and } \rho(s(u), v) \leq \varepsilon_2.$$

Let $f : Z \to Z$, $f(x_1, x_2) := (t_l^{-1}(x_2), s(x_1))$.

From Remark 1.12, for $x^* := (x_1^*, x_2^*) \in CP(s, t) = Fix(f)$, we have:

$$d^V((u, v), (x_1^*, x_2^*)) \leq (I - A)^{-1} \cdot d^V((u, v), (t_l^{-1}(v), s(u))) =$$
$$= (I - A)^{-1} \cdot \begin{pmatrix} d(u, t_l^{-1}(v)) \\ \rho(v, s(u)) \end{pmatrix} \leq (I - A)^{-1} \cdot \begin{pmatrix} \rho(t(u), v) \\ \rho(v, s(u)) \end{pmatrix} \leq (I - A)^{-1} \cdot \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

Thus, we obtain that $d^V(w, x^*) \leq (I - A)^{-1} \cdot \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$. Hence, the coincidence problem (1) is Ulam-Hyers stable. □

Theorem 1.14. *Let (X, d) and (Y, ρ) be two metric spaces and $f_i, g_i : X \to Y$, $i \in \{1, 2\}$ be four operators. Consider the following coincidence equations:*

$$f_1(x) = g_1(x), \quad x \in X \tag{1.3}$$

$$f_2(x) = g_2(x), \quad x \in X. \tag{1.4}$$

Let us consider the sets:

$$C_{i\varepsilon} := \{x \in X | \rho(f_i(x), g_i(x)) \leq \varepsilon\}, i \in \{1, 2\}.$$

If the following conditions are satisfied:
(i) $C(f_2, g_2) \subseteq C(f_1, g_1)$;
(ii) the coincidence equation (1.4) *is Ulam-Hyers stable;*
(iii) $C_{1\varepsilon} \subseteq C_{2\varepsilon}$, for each $\varepsilon > 0$;
then, the coincidence equation (1.3) *is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $y_1^* \in X$ such that $\rho(f_1(y_1^*), g_1(y_1^*)) \leq \varepsilon$. We deduce that $y_1^* \in C_{1\varepsilon}$. Now by (iii) we obtain that $y_1^* \in C_{2\varepsilon}$ and, thus, we have $\rho(f_2(y_1^*), g_2(y_1^*)) \leq \varepsilon$. Condition (ii) implies that there exists a solution, $x_2^* \in X$, for the coincidence equation (1.4), such that $d(y_1^*, x_2^*) \leq c_2\varepsilon$, for some $c_2 > 0$.

Since $x_2^* \in C(f_2, g_2)$, taking into account (i), we get that $x_2^* \in C(f_1, g_1)$. Hence, $x_2^* \in X$ is a solution for the coincidence equation (1.3).

Thus, we have obtained that $d(y_1^*, x_2^*) \leq c_2\varepsilon$, showing that the coincidence equation (1.3) is Ulam-Hyers stable. □

In the particular case $Y := X$ and $g_1 = g_2 := 1_X$, we get the following Ulam-Hyers stability result for a fixed point equation.

Theorem 1.15. *Let (X, d) be a metric space and $f_1, f_2 : X \to X$ be two operators. Consider the following fixed point equations:*

$$f_1(x) = x, \quad x \in X \tag{1.5}$$

$$f_2(x) = x, \quad x \in X. \tag{1.6}$$

Let us consider the sets:

$$F_{i\varepsilon} := \{x \in X | d(f_i(x), x) \leq \varepsilon\}, i = \{1, 2\}.$$

If the following conditions are satisfied:
(i) $Fix(f_1) = Fix(f_2)$;
(ii) the fixed point equation (1.6) *is Ulam-Hyers stable;*
(iii) $F_{1\varepsilon} \subseteq F_{2\varepsilon}$, for each $\varepsilon > 0$;
then, the fixed point equation (1.5) *is Ulam-Hyers stable.*

Definition 1.16. Given a set X and two metrics, ρ and d, we say that ρ and d are strongly equivalent metrics on X if there exist $h, k > 0$ such that

$$h \cdot d(x, y) \leq \rho(x, y) \leq k \cdot d(x, y), \text{ for any } x, y \in X. \tag{1.7}$$

If we suppose that $X = Y$ and ρ, d are two strongly equivalent metrics on X, then we can obtain a Ulam-Hyers stability result for the coincidence equation (1.3).

Theorem 1.17. *Let X be a nonempty set, ρ and d two strongly equivalent metrics. If the coincidence equation* (1.3) *is Ulam-Hyers stable with respect to metric d, then it is Ulam-Hyers stable with respect to metric ρ.*

Proof. Suppose that the coincidence equation (1.3) is Ulam-Hyers stable with respect to metric d, i.e., exists $c_1 > 0$ such that for each $\varepsilon > 0$ and each solution $y^* \in X$ of the inequation

$$d(f_1(y), g_1(y)) \leq \varepsilon \tag{1.8}$$

there exists a solution $x^* \in X$ of (1.3) such that

$$d(y^*, x^*) \leq c_1\varepsilon.$$

We prove that the coincidence equation (1.3) is Ulam-Hyers stable with respect to metric ρ. Let $\varepsilon > 0$ and $y^* \in X$ such that $\rho(f_1(y^*), g_1(y^*)) \leq \varepsilon$. Taking into account of (1.7) we have $d(f_1(y^*), g_1(y^*)) \leq \frac{1}{h}\rho(f_1(y^*), g_1(y^*)) \leq \frac{\varepsilon}{h} := \varepsilon'$. Because the coincidence equation (1.3) is Ulam-Hyers stable with respect to metric d, we get that exists $x^* \in X$ solution of (1.3) such that $d(y^*, x^*) \leq c_1\varepsilon'$, so we have $d(y^*, x^*) \leq c_1\frac{\varepsilon}{h}$.

Using again the condition (1.7) we deduce that $\rho(y^*, x^*) \leq k \cdot d(y^*, x^*) \leq \frac{kc_1}{h}\varepsilon$. Denote by $\frac{kc_1}{h}\varepsilon := c$, we have $\rho(y^*, x^*) \leq c\varepsilon$. Hence, the coincidence equation (1.3) is Ulam-Hyers stable with respect to metric ρ. □

Example 1.18. Let us consider on $\mathbb{R}$ a metric d ($d(x, y) \in \mathbb{R}_+, d(x, y) = |x - y|$) and the operators $f_i, g_i : \mathbb{R} \to \mathbb{R}, i \in \{1, 2\}$ defined by:
$f_1(x) = \arctan(x) - 7x$, $g_1(x) = \begin{cases} \sin(x) & , x \leq 0 \\ -6x - 1, & x > 0 \end{cases}$, $f_2(x) = -4x$, $g_2(x) = \frac{1}{3}x$.
Consider the following coincidence equations:

$$\arctan(x) - 7x = \begin{cases} \sin(x) & , x \leq 0 \\ -6x - 1, & x > 0 \end{cases} \tag{1.9}$$

$$-4x = \frac{1}{3}x, \ x \in \mathbb{R}. \tag{1.10}$$

We have $C(f_1, g_1) = \{0\}$ and $C(f_2, g_2) = \{0\}$, hence $C(f_2, g_2) \subseteq C(f_1, g_1)$.

We prove that the coincidence equation (1.10) is Ulam-Hyers stable. Let $\varepsilon_1, \varepsilon_2 > 0$ and $(u, v) \in \mathbb{R} \times \mathbb{R}$ a solution of the approximative coincidence problem

$$|-4u - v| \leq \varepsilon_1 \quad \text{and} \quad \left|\frac{1}{3}u - v\right| \leq \varepsilon_2. \tag{1.11}$$

We have:

$$|-4u - v| \leq \varepsilon_1 \Longleftrightarrow -\varepsilon_1 \leq -4u - v \leq \varepsilon_1 \Longleftrightarrow -\varepsilon_1 - 4u \leq v \leq \varepsilon_1 + 4u. \tag{1.12}$$

I.) If $u \geq 0$, we deduce that $-5u \leq -4u \leq 5u$ and taking into account (1.12), we have

$$|v| \leq \varepsilon_1 + 5u. \tag{1.13}$$

On the other hand we have:
$\left|\frac{1}{3}u - v\right| \leq \varepsilon_2 \Longleftrightarrow -\varepsilon_2 \leq \frac{1}{3}u - v \leq \varepsilon_2 \Longleftrightarrow -3\varepsilon_2 + 3v \leq u \leq 3\varepsilon_2 + 3v$. Using the relation (1.12) we obtain:

$$|u| \leq \frac{3\varepsilon_1 + 3\varepsilon_2}{13.} \tag{1.14}$$

Taking into account (1.13) and (1.14) we get:

$$|v| \leq \frac{28\varepsilon_1 + 15\varepsilon_2}{13} \tag{1.15}$$

From relations (1.14) and (1.15), we obtain

$$|u|+|v| \leq \frac{31\varepsilon_1+18\varepsilon_2}{13} := \psi_1(\varepsilon_1,\varepsilon_2).$$

II.) If $u<0$, we deduce that $5u \leq -4u \leq -5u$ and taking into account (1.12), we have

$$|v| \leq \varepsilon_1 - 5u. \tag{1.16}$$

From relations (1.14) and (1.16) we obtain

$$|v| \leq \frac{-2\varepsilon_1-15\varepsilon_2}{13}. \tag{1.17}$$

Using the relations (1.14) and (1.17), we have

$$|u|+|v| \leq \frac{\varepsilon_1-12\varepsilon_2}{13} := \psi_2(\varepsilon_1,\varepsilon_2).$$

If we denote $\psi(\varepsilon_1,\varepsilon_2) := \max\{\psi_1(\varepsilon_1,\varepsilon_2),\psi_2(\varepsilon_1,\varepsilon_2)\}$, we have $|u|+|v| \leq \psi(\varepsilon_1,\varepsilon_2)$, (when $\psi(\varepsilon_1,\varepsilon_2)$ satisfy the conditions of Definition 1.3), hence the coincidence equation (1.10) is Ulam-Hyers stable.

Let us consider the sets:

$$C_{1\varepsilon} := \{x \in (-\infty,0] \,|\, |\arctan(x)+7x-\sin(x)| \leq \varepsilon\} \cup \{x \in (0,\infty) \,|\, |\arctan(x)+6x+1| \leq \varepsilon\},$$

$$C_{2\varepsilon} := \{x \in \mathbb{R} \,|\, \left|-4x-\frac{1}{3}x\right| \leq \varepsilon\}.$$

We prove that $C_{1\varepsilon} \subseteq C_{2\varepsilon}$. Let $x \in C_{1\varepsilon}$.

I) Let $x \in (-\infty,0]$ such that $|\arctan(x)+7x-\sin(x)| \leq \varepsilon$. We proof:

$$\left|-4x-\frac{1}{3}x\right| \leq \varepsilon, \text{ i.e. } |x| \leq \frac{3\varepsilon}{13}. \tag{1.18}$$

On the other hand we have:

$$|x| \leq |7x| \leq |7x+\arctan(x)-\sin(x)| + |\arctan(x)-\sin(x)| \leq \varepsilon + 2|x| \Longrightarrow |x| \leq \frac{\varepsilon}{5}. \tag{1.19}$$

Taking into account (1.18) and (1.19) we get: $|x| \leq \frac{\varepsilon}{5} \leq \frac{3\varepsilon}{13}$. Hence $x \in C_{2\varepsilon}$.

II) Let $x \in (0,\infty)$, such that $|\arctan(x)+6x+1| \leq \varepsilon$. We have:
$|x| \leq |6x| \leq |6x+1| \leq |6x+1+\arctan(x)| + |\arctan(x)| \leq \varepsilon + |x| \Longrightarrow |x| \leq \frac{\varepsilon}{5}$.
Using (1.18) we obtain that:
$|x| \leq \frac{\varepsilon}{5} \leq \frac{3\varepsilon}{13}$. Hence $x \in C_{2\varepsilon}$.

So, we deduce that $C_{1\varepsilon} \subseteq C_{2\varepsilon}$. Since all the conditions of Theorem 1.14 hold, then the coincidence equation (1.9) is Ulam-Hyers stable.

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