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Some new results on complete U_n^*-metric space

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Abstract

In this paper, we give some new definitions of U_n^*-metric spaces and we prove a common fixed point theorem for two mappings under the condition of weakly compatible and establish common fixed point for sequence of generalized contraction mappings in complete U_n^*-metric space.

Keywords: U_n^*-metric space, complete U_n^*-metric space, sequence of contractive mapping.
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1. Introduction and Preliminaries

Recently Sedghi et. al. [11] introduced the concept of D^*-metric spaces and proved some common fixed point theorems (see also [3]–[12]).
In the present work, we introduce a new notion of generalized D^*-metric space called U^*-metric space of dimension n and study some fixed point results for two self-mappings f and g on U_n^*-metric spaces. Some fundamental properties of the proposed metric are studied.

Definition 1.1. [2] Let G be an ordered group. An ordered group metric (or OG-metric) on a nonempty set X is a symmetric nonnegative function d_G from $X \times X$ into G such that $d_G(x, y) = 0$ if and only if $x = y$ and such that the triangle inequality is satisfied; the pair (X, d_G) is an ordered group metric space (or OG-metric space).

For $n \geq 2$, let X^n denotes the cartesian product $X \times \ldots \times X$ and $\mathbb{R}^+ = [0, +\infty)$. We begin with the following definition.

Definition 1.2. Let X be a non-empty set. Let $U_n^* : X^n \longrightarrow G^+$ be a function that satisfies the following conditions:

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(U1) $U_n^*(x_1,\ldots,x_n)=0$ if $x_1=\ldots=x_n$,

(U2) $U_n^*(x_1,\ldots,x_n)>0$ for all $x_1,...,x_n$ with $x_i\neq x_j$, for some $i,j\in\{1,...,n\}$,

(U3) $U_n^*(x_1,\ldots,x_n)=U_n^*(x_{\pi_1},\ldots,x_{\pi_n})$, for every permutation $(\pi_{(1)},...,\pi_{(n)})$ of $(1,2,...,n)$,

(U4) $U_n^*(x_1,x_2,\ldots,x_n)\leq U_n^*(x_1,...,x_{n-1},a)+U_n^*(a,x_n,...,x_n)$, for all $x_1,\ldots,x_n,a\in X$.

The function U_n^* is called a universal ordered group metric of dimension n, or more specifically an OU_n^*-metric on X, and the pair (X,U_n^*) is called an OU_n^*-metric space.

For example we can place $G^+=\mathbb{Z}^+$ or $\mathbb{R}^+$. In the sequel, for simplicity we assume that $G^+=\mathbb{R}^+$.

Example 1.3. (a) Let (X,d) be a usual metric space, then (X,S_n) and (X,M_n) are U_n^*-metric spaces, where

$$\begin{aligned} S_n(x_1,\ldots,x_n) &= \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n} d(x_i,x_j),\\ M_n(x_1,\ldots,x_n) &= \max\{d(x_i,x_j):1\leq i<j\leq n\}. \end{aligned}$$

(b) Let ϕ be a non-decreasing and concave function with $\phi(0)=0$. If (X,d) is a usual metric space, then (X,ϕ_n) defined by

$$\phi_n(x_1,...,x_n)=\phi^{-1}\Bigg(\sum_{1\leq i<j\leq n}\phi(d(x_i,x_j)\Bigg)$$

is a U_n^*-metric.
(c) Let $X=C([0,T])$ be the set of all continuous functions defined on $[0,T]$. Defined $I_n:X^n\longrightarrow\mathbb{R}^+$ by

$$I_n(x_1,\ldots,x_n)=\sum_{1\leq i<j\leq n}\sup_{t\in[0,T]}|x_i(t)-x_j(t)|.$$

Then (X,I_n) is a U_n^*-metric space.
(d) Let $X=\mathbb{R}^n$ defined $L_n:\mathbb{R}^n\longrightarrow\mathbb{R}^+$ by

$$L_n(x_1,\ldots,x_n) = \sum_{1\leq i<j\leq n}\|x_i-x_j\|^{\frac{1}{r}}$$

For every $r\in\mathbb{R}^+$. Then (X,L_n) is a U_n^*-metric space.
(e) Let $X=\mathbb{R}$ defined $K_n:\mathbb{R}^n\longrightarrow\mathbb{R}$ by

$$K_n(x_1,...,x_n)=\begin{cases} 0 & \text{if } x_1=\cdots=x_n\\ \text{Mox}\{x_1,\cdots,x_n\} & \text{otherwise}\end{cases}$$

Then (X,K_n) is a U_n^*-metric space.

Remark 1.4. In a U_n^*-metric space, we prove that $U^*(x,...,x,y)=U^*(x,y,...y)$. For
(*i*) $U^*(x,...,x,y)\leq U^*(x,...,x)+U^*(x,y,...,y)=U^*(x,y,...,y)$ and similary
(*ii*) $U^*(y,...y,x)\leq U^*(y,...,y)+U^*(y,x,...,x)=U^*(y,x,...,x)$.

Hence by (*i*),(*ii*) we get $U^*(x,...,x,y)=U^*(x,y,...y)$.

Proposition 1.5. *Let (X,U) and (Y,V) be two U_n^*-metric spaces. Then (Z,W) is also a U_n^*-metric space, where $Z=X\times Y$ and $W(z_1,...,z_n)=max\{U(x_1,...,x_n),V(y_1,...,y_n)\}$ for $z_i=(x_i,y_i)\in Z$ with $x_i\in X,\ y_i\in Y,\ i=1,...,n$.*

Proof. Obviously (U1-U3) conditions are satisfied. To prove the (U4) inequality. Let $z_1, ..., z_n \in Z$, with $c = (a, b),\ z_i = (x_i, y_i),\ i = 1, ..., n,$

$$\begin{aligned} W(z_1, ..., z_n) = max\{U(x_1, ..., x_n), V(y_1, ..., y_n)\}) \ &\leq \ max\{U(x_1, ..., x_{n-1}, a) + U(a, x_n, ..., x_n), \\ &\qquad V(y_1, ..., y_{n-1}, b) + V(b, y_n, ..., y_n)\} \\ &\leq max\{U(x_1, ..., x_{n-1}, a), V(y_1, ..., y_{n-1}, b)\} \\ &+ max\{U(a, x_n, ..., x_n), V(b, y_n, ..., y_n)\} \\ &= W(z_1, ..., z_{n-1}, c) + W(c, z_n, ..., z_n). \end{aligned}$$

Hence (Z, W) is a U_n^*-metric space. □

Definition 1.6. A U_n^*-metric space X is said to be bounded if there exists a constant $M > 0$ such that $U_n^*(x_1, ..., x_n) \leq M$ for all $x_1, ..., x_n \in X$. A U_n^*-metric space X is said to be unbounded if it is not bounded.

Proposition 1.7. *Let (X, U_n^*) be a U_n^*-metric space and let $M > 0$ be a fixed positive real number. Then (X, V) is a bounded U_n^*-metric space with bound M, where the function V is given by*

$$V(x_1, ..., x_n) = \frac{MU^*(x_1, ..., x_n)}{(k + U^*(x_1, ..., x_n))}$$

for all $x_1, ..., x_n \in X$ and with $k > 0$.

Proof. Obviously (U1-U3) conditions are satisfied. We only prove the (U4) inequality. Let $x_1, ..., x_n \in X$,

$$\begin{aligned} V(x_1, ..., x_n) = \frac{MU^*(x_1, ..., x_n)}{(k + U^*(x_1, ..., x_n))} \ &= \ M - \frac{Mk}{(k + U^*(x_1, ..., x_n))} \\ &\leq \ M - \frac{Mk}{(k + U^*(x_1, ..., x_{n-1}, a) + U^*(a, x_n, ..., x_n))} \\ &= \ \frac{M(U^*(x_1, ..., x_{n-1}, a) + U^*(a, x_n, ..., x_n))}{(k + U^*(x_1, ..., x_{n-1}, a) + U^*(a, x_n, ..., x_n))} \\ &= \ \frac{M(U^*(x_1, ..., x_{n-1}, a))}{(k + U^*(x_1, ..., x_{n-1}, a) + U^*(a, x_n, ..., x_n))} \\ &+ \ \frac{M(U^*(a, x_n, ..., x_n))}{(k + U^*(x_1, ..., x_{n-1}, a) + U^*(a, x_n, ..., x_n))} \\ &\leq \ \frac{M(U^*(x_1, ..., x_{n-1}, a))}{(k + U^*(x_1, ..., x_{n-1}, a)} + \frac{M(U^*(a, x_n, ..., x_n))}{(k + U^*(a, x_n, ..., x_n))} \\ &= \ V(x_1, ..., x_{n-1}, a) + V(a, x_n, ..., x_n). \end{aligned}$$

Hence (X, V) is a U_n^*-metric space.
Let $x_1, ..., x_n \in X$, Then we have,

$$V(x_1, ..., x_n) = \frac{MU^*(x_1, ..., x_n)}{(k + U^*(x_1, ..., x_n))} \leq \frac{MU^*(x_1, ..., x_n)}{(U^*(x_1, ..., x_n))} = M$$

This show that (X, V) is bounded with U_n^*-bound M. □

Definition 1.8. Let (X, U_n^*) be a U_n^*-metric space, then for $x_0 \in X$, $r > 0$, the U_n^*-ball with center x_0 and radius r is

$$B_{U^*}(x_0, r) = \{y \in X : U_n^*(x_0, y, ..., y) < r\}.$$

Definition 1.9. Let (X, U_n^*) be a U_n^*-metric space and $Y \subset X$.
(1) If for every $y \in Y$ there exist $r > 0$ such that $B_{U^*}(y, r) \subset Y$, then subset Y is called open subset of X.
(2) Subset Y of X is said to be U^*-bounded if there exists $r > 0$ such that $U^*(x, y, ..., y) < r$ for all $x, y \in Y$.
(3) A sequence $\{x_k\}$ in X converges to x if and only if

$$U^*(x_k, ..., x_k, x) = U^*(x, ..., x, x_k) \to 0 \quad \text{as} \quad k \to \infty.$$

That is for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall k \geq N \Longrightarrow U^*(x, ..., x, x_k) < \varepsilon \quad (\star).$$

This is equivalent with, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall l_1, ..., l_{n-1} \geq N \Longrightarrow U^*(x, x_{l_1}, ..., x_{l_{n-1}}) < \varepsilon \quad (\star\star).$$

(4) Let (X, U_n^*) be a U_n^*-metric space, then a sequence $\{x_k\} \subseteq X$ is said to be U_n^*-Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $U_n^*(x_k, x_m, ..., x_l) < \varepsilon$ for all $k, m, ..., l \geq N$. The U_n^*-metric space (X, U_n^*) is said to bo complete if every Cauchy sequence is convergent.

Remark 1.10. (i) Let τ be the set of all $Y \subset X$ with $y \in Y$ if and only if there exists $r > 0$ such that $B_{U^*}(y, r) \subset Y$. Then τ is a topology on X induced by the U_n^*-metric.
(ii) If have $(\star)$ of Definition 1.9, then for each $\varepsilon > 0$ there exists,
$N_1 \in \mathbb{N}$ such that for every $l_1 \geq N_1 \Longrightarrow U^*(x, ..., x, x_{l_1}) < \dfrac{\varepsilon}{n-1}$,
$N_2 \in \mathbb{N}$ such that for every $l_2 \geq N_2 \Longrightarrow U^*(x, ..., x, x_{l_2}) < \dfrac{\varepsilon}{n-1}$,
and similary there exist $N_{n-1} \in \mathbb{N}$ such that for every $l_{n-1} \geq N_{n-1} \Longrightarrow U^*(x, ..., x, x_{l_{n-1}}) < \dfrac{\varepsilon}{n-1}$.
Let $N_0 = max\{N_1, ..., N_{n-1}\}$ and $K_0 = min\{l_1, ..., l_{n-1}\}$. For $K_0 > N_0$ we have

$$\begin{aligned} U^*(x, x_{l_1}, ..., x_{l_{n-1}}) &\leq U^*(x, x_{l_1}, ..., x_{l_{n-2}}, x) + U^*(x, x_{l_{n-1}}, ..., x_{l_{n-1}}) \\ &\leq U^*(x, x, x_{l_1}, ..., x_{l_{n-3}}, x) + U^*(x, x_{l_{n-2}}, ..., x_{l_{n-2}}) \\ &+ U^*(x, x_{l_{n-1}}, ..., x_{l_{n-1}}) \\ &\leq \\ &\vdots \\ &\leq \sum_{i=1}^{n-1} U^*(x, x_{l_i}, ..., x_{l_i}) \\ &< \frac{(n-1)\varepsilon}{n-1} = \varepsilon. \end{aligned}$$

Conversely, set $l_1 = \cdots = l_{n-1} = k$ in $(\star\star)$ we have $U^*(x, ..., x, x_k) < \varepsilon$.

Proposition 1.11. *In a U_n^*-metric space, (X, U_n^*), the following are equivalent.*
(i) *The sequence $\{x_k\}$ is U_n^*-Cauchy.*
(ii) *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $U_n^*(x_k, ..., x_k, x_l) < \varepsilon$, for all $k, l \geq N$.*

Lemma 1.12. *Let (X, U^*) be a U_n^*-metric space.*
(1) *If $r > 0$, then the ball $B_{U^*}(x, r)$ with center $x \in X$ and radius r is the open ball.*
(2) *If sequence $\{x_k\}$ in X converges to x, then x is unique.*
(3) *If sequence $\{x_k\}$ in X converges to x, then sequence $\{x_k\}$ is a Cauchy sequence.*
(4) *The function of U_n^* is continuous on X^n.*

Proof. proof 1)
Let $w \in B_{U^*}(x,r)$ so that $U^*(x,w,...,w) < r$. If set $U^*(x,w,...,w) = \delta$ and $r' = r - \delta$ then we prove that $B_{U^*}(w,r') \subseteq B_{U^*}(x,r)$. Let $y \in B_{U^*}(w,r')$, by (U_4) we have $U^*(x,y,...,y) = U^*(y,...,y,x) \leq U^*(y,...,y,w) + U^*(w,x,...,x) < r' + \delta = r$.
proof 2)
Let $x_k \longrightarrow y$ and $y \neq x$. Since $\{x_k\}$ converges to x and y, for each $\varepsilon > 0$ there exists,
$N_1 \in \mathbb{N}$ such that for every $k \geq N_1 \Longrightarrow U^*(x,...,x,x_k) < \dfrac{\varepsilon}{2}$
and
$N_2 \in \mathbb{N}$ such that for every $k \geq N_2 \Longrightarrow U^*(y,...,y,x_k) < \dfrac{\varepsilon}{2}$.
If set $N_0 = mox\{N_1, N_2\}$, then for every $k \geq N_0$ by (U_4) we have

$$U^*(x,...,x,y) \leq U^*(x,...,x,x_k) + U^*(x_k,y,....,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

then $U^*(x,...,x,y) = 0$ is a contradiction. So $x = y$.
proof 3)
Since $x_k \longrightarrow x$ for each $\varepsilon > 0$ there exists,
$N_1 \in \mathbb{N}$ such that for every $k \geq N_1 \Longrightarrow U^*(x_k,...,x_k,x) < \dfrac{\varepsilon}{2}$
and
$N_2 \in \mathbb{N}$ such that for every $l \geq N_1 \Longrightarrow U^*(x,x_l...,x_l) < \dfrac{\varepsilon}{2}$.
If set $N_0 = mox\{N_1, N_2\}$, then for every $k, l \geq N_0$ by (U_4) we have

$$U^*(x_k,...,x_k,x_l) \leq U^*(x_k,...,x_k,x) + U^*(x,x_l,...,x_l) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence sequence $\{x_k\}$ is a Cauchy sequence.
proof 4)
Let the sequence $\{((x_1)_k,...,(x_n)_k)\}$ in X^n converges to a point $(z_1,...,z_n)$ i.e.

$$\lim_{k\to\infty} (x_i)_k = z_i \quad i = 1,...,n$$

for each $\varepsilon > 0$ there exists,
$N_1 \in \mathbb{N}$ such that for every $k > N_1 \Longrightarrow U^*\big(z_1,...,z_1,(x_1)_k\big) < \dfrac{\varepsilon}{n}$
$N_2 \in \mathbb{N}$ such that for every $k > N_2 \Longrightarrow U^*\big(z_2,...,z_2,(x_2)_k\big) < \dfrac{\varepsilon}{n}$
$\vdots$
$N_n \in \mathbb{N}$ such that for every $k > N_n \Longrightarrow U^*\big(z_n,...,z_n,(x_n)_k\big) < \dfrac{\varepsilon}{n}$.
If set $N_0 = mox\{N_1,...,N_n\}$, then for every $k \geq N_0$ we have

$$\begin{aligned}
U^*\big((x_1)_k,...,(x_n)_k\big) &\leq U^*\big((x_1)_k,...,(x_{n-1})_k,z_n\big) + U^*\big(z_n,(x_n)_k,...,(x_n)_k\big)\\
&\leq U^*\big((x_1)_k,...,(x_{n-2})_k,z_n,z_{n-1}\big) + U^*\big(z_{n-1},(x_{n-1})_k,...,(x_{n-1})_k\big)\\
&+ U^*\big(z_n,(x_n)_k,...,(x_n)_k\big)\\
&\leq\\
&\vdots\\
&\leq U^*(z_1,...,z_n) + \sum_{i=1}^{n} U^*\big(z_i,(x_i)_k,...,(x_i)_k\big)\\
&\leq U^*(z_1,...,z_n) + \frac{n\varepsilon}{n} = U^*(z_1,...,z_n) + \varepsilon.
\end{aligned}$$

Hence we have

$$U^*\big((x_1)_k,...,(x_n)_k\big) - U^*(z_1,...,z_n) < \varepsilon$$

$$\begin{aligned}
U^*(z_1,...,z_n) &\leq U^*\big(z_1,...,z_{n-1},(x_n)_k\big)+U^*\big((x_n)_k,z_n,...,z_n\big)\\
&\leq U^*\big(z_1,...,z_{n-2},(x_n)_k,(x_{n-1})_k\big)+U^*\big((x_{n-1})_k,z_{n-1},...,z_{n-1}\big)\\
&+ U^*\big((x_n)_k,z_n,...,z_n\big)\\
&\leq\\
&\vdots\\
&\leq U^*\big((x_1)_k,...,(x_n)_k\big)+\sum_{i=1}^{n}U^*\big((x_i)_k,z_i,...,z_i\big)\\
&\leq U^*\big((x_1)_k,...,(x_n)_k\big)+\frac{n\varepsilon}{n}=U^*\big((x_1)_k,...,(x_n)_k\big)+\varepsilon.
\end{aligned}$$

That is,

$$U^*(z_1,...,z_n)-U^*\big((x_1)_k,...,(x_n)_k\big)<\varepsilon.$$

Therefore we have $|U^*\big((x_1)_k,...,(x_n)_k\big)-U^*(z_1,...,z_n)|<\varepsilon$, that is

$$\lim_{k\to\infty}U^*\big((x_1)_k,...,(x_n)_k\big)=U^*(z_1,...,z_n).$$

□

Definition 1.13. ([6]) Let f and g be mappings from a U_n^*-metric space (X,U_n^*) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is $fx=gx$ implies that $fgx=gfx$.

Definition 1.14. Let (X,U_n^*) be a U_n^*-metric space, for $A_1,...,A_n\subseteq X$, define

$$\Delta_{U^*}(A_1,...,A_n)=sup\{U^*(a_1,...,a_n)|\ a_i\in A_i,\ i=1,...,n\}.$$

Remark 1.15. It follows immediately from the definition that
(i) If A_i consists of a single point a_i we write

$$\Delta_U^*(A_1,...,A_{i-1},A_i,A_{i+1},...,A_n)=\Delta_U^*(A_1,...,A_{i-1},a_i,A_{i+1},...,A_n).$$

If $A_1,...,A_n$ also consists of a single point $a_1,...,a_n$ respectively, we write

$$\Delta_U^*(A_1,...,A_n)=\Delta_U^*(a_1,...,a_n).$$

Also we have

$$\Delta_{U^*}(A_1,...,A_n)=0\Longleftrightarrow A_1=\cdots=A_n=\{a\},$$
$$\Delta_{U^*}(A_1,...,A_n)=\Delta_{U^*}(A_{\pi_1},...,A_{\pi_n}),$$

for for every permutation $(\pi_{(1)},...,\pi_{(n)})$ of $(1,2,...,n)$.
In particular for $\varnothing\neq A_1=\cdots=A_n\subseteq X$,

$$\Delta_{U^*}(A_1)=sup\{U^*(b_1,...,b_n)|b_1,...,b_n\in A_1\}.$$

(ii) If $A\subseteq B$, then $\Delta_{U^*}(A)\leq\Delta_{U^*}(B)$.
(iii) For a sequence $A_k=\{x_k,x_{k+1},x_{k+2},\cdots\}$ in U_n^*-metric space (X,U_n^*), let $a_k=\Delta_{U^*}(A_k)$ for $k\in\mathbb{N}$. Then
(a) : Since $A_{k+1}\subseteq A_k$ hence $\Delta_{U^*}(A_{k+1})\leq\Delta_{U^*}(A_k)$, for every $k\geq 1$.
(b) : $U^*(x_{l_1},...,x_{l_n})\leq\Delta_{U^*}(A_k)=a_k$ for every $l_1,...,l_n\geq k$,
(c) : $0\leq\Delta_{U^*}(A_k)=a_k$.
Therefore, $\{a_k\}$ is decreasing and bounded for all $k\in\mathbb{N}$, and so there exists an $0\leq a$ such that $\lim_{k\to\infty}a_k=a$.

Lemma 1.16. *Let (X, U_n^*) be an U_n^*-metric space. If $\lim_{k\to\infty} a_k = 0$, then sequence $\{x_k\}$ is a Cauchy sequence.*

Proof. Since $\lim_{k\to\infty} a_k = 0$, we have that for every $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ such that for every $k > N_0$, $|a_k - 0| < \varepsilon$. That is $a_k = \Delta_{U^*}(A_k) < \varepsilon$. Then for $l_1, ..., l_n \geq k > N_0$ by (b) of Remark 1.15 we have

$$U^*(x_{l_1}, ..., x_{l_n}) \leq sup\{U^*(x_i, ..., x_j) \mid x_i, ..., x_j \in A_k\} = a_k < \varepsilon.$$

Therefore, $\{x_k\}$ is a Cauchy sequence in X. □

2. Main results

Theorem 2.1. *Let X be a U_n^*-complete metric space*
I) *If f and g be self-mappings of a complete U_n^*-metric space (X, U_n^*) satisfying:*
i) $g(X) \subseteq f(X)$, and $f(X)$ is closed subset of X,
ii) the pair (f, g) is weakly compatible,
iii) $U^*(gz_1, ..., gz_n) \leq \psi(U^*(fz_1, ..., fz_n))$, for every $z_1, ..., z_n \in X$, where $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.
Then f and g have a unique common fixed point in X.
II) *If $f_k : X \longrightarrow X$ be a sequence maps such that*

$$U^*(f_i z_1, f_j z_2, ..., f_l z_{n-1}, z_n) \leq \beta U^*(z_1, ..., z_n)$$

for all $i \neq j$ and $z_1, ..., z_n \in X$ with $0 \leq \beta < \dfrac{1}{2}$. Then $\{f_k\}$ have a unique common fixed point.

Proof. proof **I)**
Let x_0 be an arbitrary point in X. By (i), we can choose a point x_1 in X such that $y_0 = gx_0 = fx_1$ and $y_1 = gx_1 = fx_2$. In general, there exists a sequence $\{y_k\}$ such that, $y_k = gx_k = fx_{k+1}$, for $k = 0, 1, 2, \cdots$. We prove that sequence $\{y_k\}$ is a Cauchy sequence. Let $A_k = \{y_k, y_{k+1}, y_{k+2}, \cdots\}$ and $a_k = \Delta_{U^*}(A_k)$, $k \in \mathbb{N}$. Then we know $\lim_{k\to\infty} a_k = a$ for some $a \geq 0$.
Taking $z_i = x_{l_i+l}$ in (iii) for $l \geq 1$ and $l_1, ..., l_n \geq 0$

$$\begin{aligned} U^*(y_{l_1+l}, ..., y_{l_n+l}) &= U^*(gx_{l_1+l}, ..., gx_{l_n+l}) \\ &\leq \psi(U^*(fx_{l_1+l}, ..., fx_{l_n+l})) \\ &= \psi(U^*(y_{l_1+l-1}, ..., y_{l_n+l-1})) \end{aligned}$$

Since $U^*(y_{l_1+l-1}, ..., y_{l_n+l-1}) \leq a_{l-1}$, for every $l_1, ..., l_n \geq 0$ and ψ is increasing in t, we get

$$U^*(y_{l_1+l}, ..., y_{l_n+l}) \leq \psi(U^*(y_{l_1+l-1}, ..., y_{l_n+l-1})).$$

Therefore

$$\sup_{l_1,...,l_n\geq 0} \{U^*(y_{l_1+l}, ..., y_{l_n+l}) \leq \psi(a_{l-1}).$$

Hence, we have $a_l \leq \psi(a_{l-1})$. Letting $l \to \infty$, we get $a \leq \psi(a)$. If $a \neq 0$, then $a \leq \psi(a) < a$, which is a contradiction. Thus $a = 0$ and hence $\lim_{k\to\infty} a_k = 0$. Thus Lemma 1.16 $\{y_k\}$ is a Cauchy sequence in X. By the completeness of X, there exists a $v \in X$ such that

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} gx_k = \lim_{k\to\infty} = fx_{k+1} = v.$$

Let $f(X)$ is closed, there exist $w \in X$ such that $fw = v$, Now we show that $gw = v$ For this it is enough set $x_k, ..., x_k, w$ replacing $z_1, ..., z_n$ respectively, in inequality (iii) we get

$$U^*(gx_k, ..., gx_k, gw) \leq \psi(U^*(fx_k, ..., fx_k, fw))$$

Taking $k \to \infty$, we get

$$U^*(v, ..., v, gw) \leq \psi(U^*(0)) = 0,$$

it implies $gw = v$.
Since the pair (f, g) are weakly compatible, hence we get, $gfw = fgw$. Thus $fv = gv$. Now we prove that $gv = v$. If we substitute $z_1, ..., z_n$ in (iii) by $x_k, ..., x_k$ and v respectively, we get

$$U^*(gx_k, ..., gx_k, gu) \leq \psi(U^*(fx_k, ..., fx_k, fv))$$

Taking $k \to \infty$, we get

$$U^*(v, ..., v, gv) \leq \psi(U^*(v, ..., v, gv)).$$

If $gv \neq v$, then $U^*(v, ..., v, gv) < U^*(v, ..., v, gv)$, is contradiction. Therefore,

$$fv = gv = v.$$

For the uniqueness, let v and v' be fixed points of f, g. Taking $z_1 = ... = z_{n-1} = v$ and $z_n = v'$ in (iii), we have

$$\begin{aligned} U^*(v, ..., v, v') &= U^*(gv, ..., gv, gv') \\ &\leq \psi(U^*(fv, ..., fv, fv')) \\ &= \psi(U^*(v, ..., v, v')) \\ &< U^*(v, ..., v, v'), \end{aligned}$$

which is a contradiction. Thus we have $v = v'$.
proof **II)**
Let $x_0 \in X$ be any fixed arbitrary element define a sequence $\{x_k\}$ in X as. $x_{k+1} = f_{k+1}x_k$ for all $k = 0, 1, 2, \cdots$.
Let $d_k = U^*(x_k, x_{k+1}, ..., x_{k+1})$ for all $k = 0, 1, 2, \cdots$.
Now

$$\begin{aligned} d_{k+1} &= U^*(x_{k+1}, x_{k+2}, ..., x_{k+2}) \\ &= U^*(f_{k+1}x_k, f_{k+2}x_{k+1}, ..., f_{k+2}x_{k+1}, x_{k+2}) \\ &\leq \beta U^*(x_k, x_{k+1}, ..., x_{k+1}, x_{k+2}) \\ &\leq \beta U^*(x_k, x_{k+1}, ..., x_{k+1}, x_{k+1}) + \beta U^*(x_{k+1}, x_{k+2}, ..., x_{k+2}) \\ &= \beta d_k + \beta d_{k+1}. \end{aligned}$$

Hence
$d_{k+1} \leq \dfrac{\beta}{1-\beta} d_k$,
$d_k \leq \dfrac{\beta}{1-\beta} d_{k-1}$ for all $n = 1, 2, \cdots$. Let $\alpha = \dfrac{\beta}{1-\beta}$, we have
$d_k \leq \alpha\ d_{k-1} \leq \alpha^k d_0 \to 0$ as $k \to \infty$. Therefore
$\lim_{k \to \beta} d_k = 0$. Thus
$\lim_{k \to \beta} U^*(x_k, x_{k+1}, ..., x_{k+1}) = 0$.
Now we shall prove that $\{x_k\}$ is a U_n^*-Cauchy sequence in X.
Let $l > k > N_0$ for some $N_0 \in \mathbb{N}$. Now

$$\begin{aligned} U^*(x_k, ..., x_k, x_l) &\leq U^*(x_k, ..., x_k, x_{k+1}) + U^*(x_{k+1}, ..., x_{k+1}, x_l) \\ &\leq \sum_{t=\infty}^{l-1} U^*(x_t, ..., x_t, x_{t+1}) \to 0 \text{ as } k, l \to \infty \end{aligned}$$

Hence $\lim_{k,l\to\infty} U^*(x_k, ..., x_k, x_l) = 0$.
Thus $\{x_k\}$ is U_n^*-Cauchy sequence in X.
Since X is U_n^*-complete $x_k \to x$ in X. We prove that x is a fixed point of f_k for all k suppose there exist a k' such that $f_{k'}x \neq x$. Then

$$\begin{aligned} U^*(f_{k'}, x, ..., x) &= \lim_{k\to\infty} U^*(f_{k'}x, x_{k+1}, ..., x_{k+1}, x) \\ &= \lim_{k\to\infty} U^*(f_{k'}x, f_{k+1}x_k, ..., f_{k+1}x_k, x) \\ &\leq \beta \lim_{k\to\infty} U^*(x, x_{k+1}, ..., x_{k+1}, x) = 0. \end{aligned}$$

Therefore $U^*(f_{k'}, x, ..., x) = 0$, Therefore $f_k x = x$ for all k. Thus x is common fixed point of $\{f_k\}$ for all k. For the uniqueness, suppose $x \neq y$ such that $f_k y = y$ for all k. Then

$$\begin{aligned} U^*(x, y, ..., y) &= U^*(f_i x, f_j y, ..., f_j y, y) \\ &\leq \beta\, U^*(x, y, ..., y) \end{aligned}$$

This implies $(1-\beta)U^*(x, y, ..., y) \leq 0$.
Since $x \neq y$ we have $U^*(x, y..., y) > 0$ her $(1-\beta) < 0$.
This implies $\beta > 1$ which contraction to $\beta < \frac{1}{2}$.
Thus $\{f_k\}$ have a unique common fixed point. □

Corollary 2.2. *Let f be self-mapping of a complete U_n^*-metric space (X, U_n^*) satisfying:*

$$U^*(z_1, ..., z_n) \leq \psi(U^*(f^m z_1, ..., f^m z_n)),$$

for every $z_1, ..., z_n \in X$, f is surjective and $m \in \mathbb{N}$, where $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.
Then f have a unique fixed point in X.

Proof. If we define $g = I$ identity map in Theorem 2.1. There exists a unique $v \in X$ such that $f^m v = v$. Thus

$$f^m(fv) = f(f^m v) = fv.$$

Since v is unique, we have $fv = v$. □

Corollary 2.3. *Let g be self-mapping of a complete U_n^*-metric space (X, U_n^*) satisfying:*

$$U^*(g^m z_1, ..., g^m z_n) \leq \psi(U^*(z_1, ..., z_n)),$$

for every $z_1, ..., z_n \in X$ and $m \in \mathbb{N}$, where $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.
Then g have a unique fixed point in X.

Proof. If we define $f = I$ identity map in Theorem 2.1. There exists a unique $v \in X$ such that $g^m v = v$. Thus

$$g^m(gv) = g(g^m v) = gv.$$

Since v is unique, we have $gv = v$. □

Corollary 2.4. *Let f and g be self-mappings of a complete U_n^*-metric space (X, U_n^*) satisfying:*
(i) $g^r(X) \subseteq f^s(X)$, and $f^s(X)$ is closed subset of X,
(ii) the pair (f^s, g^r) is weakly compatible and $f^s g = g f^s$, $g^r f = f g^r$,
(iii) $U^*(g^r z_1, ..., g^r z_n) \leq \psi(U^*(f^s z_1, ..., f^s z_n))$, for every $z_1, ..., z_n \in X$ and $r, s \in \mathbb{N}$ where $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is

a nondecreasing continuous function with $\psi(t) < t$ *for every* $t > 0$.
Then f *and* g *have a unique common fixed point in* X.

Proof. By Theorem 2.1 there exists a fixed point $v \in X$ such that $f^s v = g^r v = v$. On the other hand, we have

$$gv = g(g^r v) = g^r(gv) \text{ and } gv = g(f^s v) = f^s(gv).$$

Since v is unique, we have $gv = v$. Similarly, we have $fv = v$. □

Corollary 2.5. *Let* f, g *and* h *be self-mappings of a complete* U_n^**-metric space* (X, U_n^*) *satisfying:*
(i) $g(X) \subseteq fh(X)$, *and* $fh(X)$ *is closed subset of* X,
(ii) the pair (fh, g) *is weakly compatible and* $fh = hf$, $gh = hg$,
(iii) $U^*(gz_1, ..., gz_n) \leq \psi(U^*(fhz_1, ..., fhz_n))$, *for every* $z_1, ..., z_n \in X$, *where* $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a nondecreasing continuous function with* $\psi(t) < t$ *for every* $t > 0$.
Then f, g *and* h *have a unique common fixed point in* X.

Proof. By Theorem 2.1 there exists a fixed point $v \in X$ such that $fhv = gv = v$.
Now, we prove that $hv = v$. If $hv \neq v$ in (iii), then we have

$$\begin{aligned} U^*(hv, v, ..., v) &= U^*(hgv, gv, ..., gv) \\ &= U^*(ghv, gv, .., gv) \\ &\leq \psi(U^*(fhhv, fhv, ..., fhv)) \\ &= \psi(U^*(hv, v, ..., v)) \\ &< U^*(hv, v, ..., v), \end{aligned}$$

which is a contradiction. Thus we have $hv = v$. Therefore,

$$fv = fhv = v = hv = gv.$$

□

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