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Existence of solutions to the state dependent sweeping process with delay

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Abstract

In this paper, we prove, via new projection algorithm, the existence of solutions for functional differential inclusion governed by state dependent sweeping process with perturbation depending on all variables and with delay.

Keywords: functional differential inclusion, normal cone, sweeping process, perturbation, delay.
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1. Introduction

Functional differential inclusions with delay, express the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant. The class of differential inclusions with delay encompasses a large variety of differential inclusions and control systems. In particular, this class covers the differential inclusions, the differential inclusions with delay and the Volterra inclusions. A detailed discussion on this topic may be found in [1].
Let $\mathbb{H}$ be a real separable Hilbert space with the norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, I an interval of $\mathbb{R}$ and τ a positive scalar. Denote by $\mathcal{C}(I,\mathbb{H})$ the Banach space of continuous functions from I into $\mathbb{H}$. By $\mathcal{C}_\tau$ we mean the Banach space $\mathcal{C}([-\tau,0])$ with the norm $\|\varphi\|_\tau := \max_{s\in[-\tau,0]} \|\varphi(s)\|$.
For $T>0$, $x\in\mathcal{C}([\tau,T],\mathbb{H})$ and for any $t\in[0,T]$ we define the map $\mathcal{T}(t)$ from $\mathcal{C}([-\tau,T],\mathbb{H})$ into $\mathcal{C}_\tau$ as follows

$$\mathcal{T}(t)x := (x)_t,\ (x)_t(s) = x(t+s), \quad \text{for all } s\in[-\tau,0].$$

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$\mathcal{T}(t)x$ represents the history of the state from the time $t-\tau$ to the present time t.
In this paper, we present an existence result for functional differential inclusion governed by state dependent sweeping process

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + F(t, \mathcal{T}(t)u) \quad \text{a.e. on } [0,T], \\ u(t)) \in C(t,u(t)), \text{ for all } t \in [0,T], \\ u(s) = \mathcal{T}(0)u(s) = \varphi(s) \quad \text{for all} \quad s \in [-\tau, 0], \\ \varphi(0) = a \in C(0,a), \end{cases} \tag{1.1}$$

where $C : [0,T] \times \mathbb{H} \to 2^{\mathbb{H}}$ is a set-valued mapping taking values in a Hilbert space $\mathbb{H}$ and $F : [0,T] \times \mathcal{C}_\tau \to 2^{\mathbb{H}}$ is a set-valued mapping with convex weakly compact values. Here $N_{C(t,u(t))}(u(t))$ denotes the convex normal cone to $C(t,u(t))$ at $u(t)$.
Such problems with delay and convex compact valued upper semicontinuous perturbations have been studied [5, 6, 8] in the case when C does not depend on the state $u \in \mathbb{H}$.
Recently, in [4], the authors have obtained an existence result for (1.1) (i.e C depends on the time $t \in [0,T]$ and state $u \in \mathbb{H}$) via an implicit discretization technique based on the fixed point theorem. The main purpose of this paper is to give a new proof of the the existence of solutions for (1.1). Our proof is different from those given in [4], the main difference concerns with the convergence of a new explicit projection algorithm to a solution of (1.1). The problem (1.1) is motivated by some applications in mathematical economics, mechanics, control theory and viscosity, see ([2, 3, 10, 9, 11, 12]).

The paper is organized as follows. In section 2, we recall some definitions needed in the sequel of the paper. Section 3 is devoted to prove the existence of solutions of (1.1).

2. Notation and preliminaries

In the sequel, $\mathbb{H}$ denotes a real Hilbert space with the norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. Let S be a closed subset of $\mathbb{H}$. We denote by $\overline{\mathbb{B}}(x,r)$ the closed ball centered at x with radius $r > 0$ of $\mathbb{H}$ and by $d_S(.)$ the usual distance function associated with S, i. e. $d_S(x) := \inf\limits_{u \in S} \|x-u\|$ $(x \in \mathbb{H})$.We need first to recall some notations and definitions needed in the paper.
Let $\mathcal{L}([0,T])$ be a $\sigma-$ algebra of Lebesgue measurable subsets of $[0,T]$ and let $\mathcal{B}(X)$ be a Borel tribe of the topological space X. A set-valued mapping $F : [0,T] \to 2^{\mathbb{H}}$ is measurable if its graph belongs to $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbb{H})$.
Let $\varphi : \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous (l.s.c) function and let x be any point where φ is finite.

We recall that the subdifferential $\partial\varphi(x)$ (in the sense of convex analysis) is the set of all $\xi \in \mathbb{H}$ such that

$$\langle \xi, x' - x \rangle \le \varphi(x') - \varphi(x), \text{for all } x \in \mathbb{H}.$$

By convention we set $\partial\varphi(x) = \emptyset$ if $\varphi(x)$ is note finite. The support function of a convex $C \subset \mathbb{H}$ defined as

$$\delta^*(x, C) = \sup_{c \in C} < x, c >, \text{for } x \in \mathbb{H}.$$

Let S be a nonempty closed convex subset of $\mathbb{H}$ and x be a point in S. The convex normal cone of S at x is defined by (see for instance [1, 6])

$$N_S(x) = \{\xi \in \mathbb{H}, \langle \xi, x' - x \rangle \le 0 \text{ for all } x' \in S\}.$$

It is well known (see for example [6]) that $N_S(x)$ the normal cone of a closed convex set S at $x \in \mathbb{H}$ can be defined in terms projection operator $Proj_S(\cdot)$ as follows

$$N_S(x) = \{\xi \in \mathbb{H}, \text{ there exists } r > 0 \text{ such that } x \in Proj_S(x + r\xi)\}.$$

A convex weakly compact valued mapping $F : X \to 2^{\mathbb{H}}$ defined on a topological space X is scalarly upper semicontinuous if for every $x \in \mathbb{H}$, the scalar function $\delta^*(x, F(.))$ is upper semicontinuous on X (see [7]).

3. Main results

The following existence theorem establishes our main result in this paper.

Theorem 3.1. *Let $\mathbb{H}$ be a separable Hilbert space, and let $C : [0, T] \times \mathbb{H} \to 2^{\mathbb{H}}$ be a set-valued mapping with nonempty closed convex values satisfying the following assumptions:*
$(\mathcal{H}_1)$ C is lipschitz continuous with constants $L_1 \geq 0$ and $0 < L_2 < 1$, i.e for all $t, s \in [0, T]$ and $x, u, v \in \mathbb{H}$ we have

$$|d_{C(t,u)}(x) - d_{C(s,v)}(x)| \leq L_1|t - s| + L_2\|u - v\|;$$

$(\mathcal{H}_2)$ for any convergent sequence (t_n) in $[0, T]$ and for any bounded subset $A \subset \mathbb{H}$, $\bigcup\limits_n C(t_n, A)$ is ball-compact, that is the intersection of $\bigcup\limits_n C(t_n, A)$ with any closed ball of $\mathbb{H}$ is relatively compact in $\mathbb{H}$.

Let $F : [0, T] \times \mathcal{C}_\tau \longrightarrow 2^{\mathbb{H}}$ be a scalarly upper semicontinuous set-valued mapping with nonempty convex weakly compact values in $\mathbb{H}$ such that $F(t, \varphi) \subset \overline{\mathbb{B}}(0, L)$ for all $(t, \varphi) \in [0, T] \times \mathcal{C}_\tau$, for some $L > 0$.

Then, for any $\varphi \in \mathcal{C}_\tau$ with $\varphi(0) = u_0 \in C(0, u_0)$, there exists a continuous mapping $u : [-\tau, T] \to \mathbb{H}$ such that

$$(\mathcal{P}) \begin{cases} u(t) = \mathcal{T}(0)u(t) = \varphi(t) \quad on\ [-\tau, 0], \\ u(t) = u_0 + \int\limits_0^t \dot{u}(s)ds,\ \forall t \in [0, T], \\ with \quad \dot{u} \in L^\infty_{\mathbb{H}}([0, T]) \quad and \quad u(t) \in C(t, u(t)) \quad for\ all\ \ t \in [0, T], \\ -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + F(t, \mathcal{T}(t)u) \quad a.e.\ \ on\ [0, T]. \end{cases}$$

Proof. We proceed by approximation: a sequence of continuous mappings (u_n) in $\mathcal{C}([-\tau, T], \mathbb{H})$ will be defined such that a subsequence of it converges uniformly in $[-\tau, T]$ to a solution of $(\mathcal{P})$. The sequence is defined via a new projection algorithm. We give the proof in three steps.
Step 1. Construction of approximants. For each $n \in \mathbb{N}^*$, we consider the following partition of the interval $[0, T]$ by points

$$t^n_i = i\mu_n;\ 0 \leq i \leq n - 1, \mu_n := \frac{T}{n}.$$

We put

$$T(0)u_n := \varphi \quad \text{on} \quad [-\tau, 0]. \tag{3.1}$$

Let $f : [0, T] \times \mathcal{C}_\tau \longrightarrow \mathbb{H}$ be a scalarly $\mathcal{B}([0, T] \times \mathcal{C}_\tau)$-measurable selection of F. Let $u^n_0 = u_0 = \varphi(0) \in C(t^n_0, u_0), t \in [t^n_0, t^n_1]$ and let us set

$$u^n_1 = Proj_{C(t^n_1,u^n_0)}\Big(u^n_0 - \mu_n f(t^n_0, \mathcal{T}(t^n_0)u_n)\Big),$$

the existence of projection is ensured since C has closed convex values. Here $\mathcal{T}(t^n_0)u_n := (u_n)_{t^n_0}$ where $(u_n)_{t^n_0}(s) = u_n(t^n_0 + s) = \varphi(s)$ for all $s \in [-\tau, 0]$. Then $u^n_1 \in \overline{\mathbb{B}}(u_0, \frac{L_1+2L}{1-L_2}\mu_n)$. Indeed, we have

$$\begin{aligned} \|u^n_1 - u^n_0\| &\leq \|u^n_1 - (u^n_0 - \mu_n f(t^n_0, \mathcal{T}(t^n_0)u_n)\| + \|\mu_n f(t^n_0, \mathcal{T}(t^n_0)u_n)\| \\ &= d_{C(t^n_1,u^n_0)}\Big(u^n_0 - \mu_n f(t^n_0, \mathcal{T}(t^n_0)u_n)\Big) + \|\mu_n f(t^n_0, \mathcal{T}(t^n_0)u_n)\| \\ &= d_{C(t^n_1,u^n_0)}(u^n_0) + 2\mu_n\|f(t^n_0, \mathcal{T}(t^n_0)u_n)\|. \end{aligned}$$

The initial condition $u^n_0 \in C(t^n_1, u^n_0)$ and $(\mathcal{H}_1)$, imply that

$$\begin{aligned} \|u^n_1 - u^n_0\| &\leq d_{C(t^n_1,u^n_0)}(u^n_0) - d_{C(t^n_0,u^n_0)}(u^n_0) + 2\mu_n L \\ &\leq L_1\mu_n + 2\mu_n L \\ &\leq \frac{L_1 + 2L}{1 - L_2}\mu_n. \end{aligned}$$

For each $t \in [t_0^n, t_1^n]$, set

$$u_n(t) = u_0^n + \Big(\frac{u_1^n - u_0^n}{\mu_n}\Big)(t - t_0^n),$$

where $u_0^n = u_0 = \varphi(0) \in C(0, u_0)$. By construction we have $u_1^n \in C(t_1^n, u_n(t_0^n))$ and for a. e. $t \in [t_0^n, t_1^n[$,

$$-\dot{u}_n(t) = -\frac{u_1^n - u_0^n}{\mu_n} \in N_{C(t_1^n, u_n(t_0^n))}(u_n(t_1^n)) + f(t_0^n, \mathcal{T}(t_0^n)u_n) \tag{3.2}$$

with

$$\|u_1^n - u_0^n\| \leq \frac{L_1 + 2L}{1 - L_2}\mu_n. \tag{3.3}$$

By induction for $0 \leq i \leq n-1$, we assert that there exists $u_{i+1}^n \in \overline{\mathbb{B}}(u_i^n, \frac{L_1+2L_2}{1-L_2}\mu_n)$ such that

$$u_{i+1}^n = Proj_{C(t_{i+1}^n, u_i^n)}\Big(u_i^n - \mu_n f(t_i^n, \mathcal{T}(t_i^n)u_n)\Big).$$

Here $T(t_i^n)u_n := (u_n)_{t_i^n}$ with $(u_n)_{t_i^n}(s) = u_n(t_i^n + s)$ for all $s \in [-\tau, 0]$. Indeed, the Lipschitz property of C ensures that

$$\begin{aligned}
\|u_{i+1}^n - u_i^n\| &\leq \|u_{i+1}^n - u_i^n + \mu_n f(t_i^n, \mathcal{T}(t_i^n)u_n\| + \|\mu_n f(t_i^n, \mathcal{T}(t_i^n)u_n)\| \\
&= d_{C(t_{i+1}^n, u_i^n)}\Big(u_i^n - \mu_n f(t_i^n, \mathcal{T}(t_i^n)u_n)\Big) + \mu_n\|f(t_i^n, \mathcal{T}(t_i^n)u_n)\| \\
&\leq d_{C(t_{i+1}^n, u_i^n)}(u_i^n) - d_{C(t_i^n, u_{i-1}^n)}(u_i^n) + 2\mu_n\|f(t_i^n, \mathcal{T}(t_i^n)u_n)\|.
\end{aligned}$$

$$\leq L_1|t_{i+1}^n - t_i^n| + L_2\|u_i^n - u_{i-1}^n\| + 2\mu_n L. \tag{3.4}$$

We have also

$$\begin{aligned}
\|u_i^n - u_{i-1}^n\| &\leq \|u_i^n - u_{i-1}^n + \mu_n f(t_{i-1}^n, \mathcal{T}(t_{i-1}^n)u_n\| + \|\mu_n f(t_{i-1}^n, \mathcal{T}(t_{i-1}^n)u_n)\| \\
&= d_{C(t_i^n, u_{i-1}^n)}\Big(u_{i-1}^n - \mu_n f(t_{i-1}^n, \mathcal{T}(t_{i-1}^n)u_n)\Big) + \mu_n\|f(t_{i-1}^n, \mathcal{T}(t_{i-1}^n)u_n)\| \\
&\leq d_{C(t_i^n, u_{i-1}^n)}(u_{i-1}^n) - d_{C(t_{i-1}^n, u_{i-2}^n)}(u_{i-1}^n) + 2\mu_n\|f(t_{i-1}^n, \mathcal{T}(t_{i-1}^n)u_n)\| \\
&\leq L_1|t_i^n - t_{i-1}^n| + L_2\|u_{i-1}^n - u_{i-2}^n\| + 2\mu_n L.
\end{aligned}$$

By induction we obtain

$$\begin{aligned}
\|u_i^n - u_{i-1}^n\| &\leq L_1 u_n + 2\mu_n L + L_2\Big(2\mu_n L + L_1\mu_n + L_2\|u_{i-2}^n - u_{i-3}^n\|\Big) \\
&= L_1\mu_n\big(1 + L_2\big) + 2\mu_n L\big(1 + L_2\big) + L_2^2\|u_{i-2}^n - u_{i-3}^n\| \\
&\qquad \vdots \\
&\leq L_1\mu_n\big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2}\big) \\
&\quad + 2\mu_n L\big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2}\big) + L_2^{i-1}\|u_1^n - u_0^n\|.
\end{aligned}$$

(3.3) entails

$$\begin{aligned}
\|u_i^n - u_{i-1}^n\| &\leq L_1\mu_n\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big) \\
&\quad + 2\mu_n L\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big) \\
&\leq \mu_n\big(L_1 + 2L\big)\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big).
\end{aligned}$$

$$\leq \mu_n\big(L_1 + 2L\big)\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big). \tag{3.5}$$

Hence (3.4) and (3.5) imply that

$$\begin{aligned}\|u^n_{i+1}-u^n_i\| &\le \mu_n\big(L_1+2L\big)\\ &+ \mu_n\big(L_1+2L\big)\Big(L_2+L_2^2+\cdots+L_2^{i-1}+L_2^i\Big)\\ &= \mu_n\big(L_1+2L\big)\Big(1+L_2+L_2^2+\cdots+L_2^{i-1}+L_2^i\Big).\end{aligned}$$

Using the fact that $L_2<1$, we get

$$\begin{aligned}\|u^n_{i+1}-u^n_i\| &\le \mu_n\big(L_1+2L\big)\Big(\tfrac{1-L_2^{i+1}}{1-L_2}\Big).\\ &\le \Big(\tfrac{L_1+2L}{1-L_2}\Big)\mu_n.\end{aligned}$$

Now, for $t\in[t^n_i,t^n_{i+1}[$, $0\le i\le n-1$, we define

$$u_n(t)=u^n_i+\frac{u^n_{i+1}-u^n_i}{\mu_n}(t-t^n_i).$$

Then for $t\in[t^n_i,t^n_{i+1}[$, we have

$$-\dot{u}_n(t)=-\frac{u^n_{i+1}-u^n_i}{\mu_n}\in N_{C\left(t^n_{i+1},u_n(t^n_i)\right)}\big(u_n(t^n_{i+1})\big)+f(t^n_i,\mathcal{T}(t^n_i)u_n),$$

with the estimate

$$\|\tfrac{u^n_{i+1}-u^n_i}{\mu_n}\| \le \big(\tfrac{L_1+2L}{1-L_2}\big).$$

Now, let us define the step function from $[0,T]$ to $[0,T]$ by

$$\begin{array}{ll}\theta_n(t)=t^n_i; & t\in]t^n_i,t^n_{i+1}],\\ \delta_n(t)=t^n_{i+1}; & t\in]t^n_i,t^n_{i+1}],\\ \theta_n(0)=\delta_n(0)=0. & \end{array}$$

We set $f_n(t)=f(\theta_n(t),\mathcal{T}(\theta_n(t))u_n)$ for all $t\in[0,T]$. Then we have

$$-\dot{u}_n(t)\in N_{C\left(\delta_n(t),u_n(\theta_n(t)\right)}\big(u_n(\delta_n(t))\big)+f_n(t)\quad \text{a. e. on}\,[0,T] \tag{3.6}$$

and

$$\|\dot{u}_n(t)\| \le \big(\tfrac{L_1+2L}{1-L_2}\big)\quad \text{a. e. on}\,[0,T]. \tag{3.7}$$

Step 2. The convergence of (u_n) and (f_n).

By (3.7), the sequence $(u_n(t))$ is equi-lipschitz with constant $\dfrac{L_1+2L}{1-L_2}$.

We iterate the estimate

$$\|u^n_i-u^n_{i-1}\| \le \big(\tfrac{L_1+2L}{1-L_2}\big)\mu_n,$$

for $i=1,2,\cdots,n$, we get

$$\|u^n_i-u^n_0\| \le \big(\tfrac{L_1+2L}{1-L_2}\big)i\mu_n,$$

and so

$$\|u^n_i\| \le \|u_0\|+\big(\tfrac{L_1+2L}{1-L_2}\big)T:=\alpha. \tag{3.8}$$

By construction

$$u_n(\delta_n(t))\in C\big(\delta_n(t),u_n(\theta_n(t))\big)\cap\overline{\mathbb{B}}(0,\alpha),$$

so that $u_n(\delta_n(t))$ is relatively compact for every $t \in [0,T]$ in $\mathbb{H}$, because

$$\mathcal{K}(t) := \bigcup_n C\big(\delta_n(t), u_n(\theta_n(t))\big)$$

is ball-compact thanks to $(\mathcal{H}_2)$ and (3.8). Furthermore, (3.7) gives

$$\|u_n(\delta_n(t)) - u_n(t)\| \le \frac{L_1 + 2L}{1 - L_2}|\delta_n(t) - t|,$$

or

$$\lim_{n\to\infty} \|u_n(\delta_n(t)) - u_n(t)\| = 0. \tag{3.9}$$

So by (3.9) $(u_n(t))$ is relatively compact too. Therefore $(u_n(\cdot))$ is relatively compact in $C([0,T],\mathbb{H})$. Hence we may suppose that $(\dot{u}_n)$ $\sigma(L^\infty([0,T],\mathbb{H}), L^1([0,T],\mathbb{H}))$ converges in $L^\infty([0,T],\mathbb{H})$ to a function v with $\|v(t)\| \le \dfrac{L_1 + 2L}{1 - L_2}$ for a. e. $t \in [0,T]$, and (u_n) converges in $\mathcal{C}([-r,T],\mathbb{H})$ to a continuous function u such that

$$u(t) = u_0 + \int_0^t \dot{u}(s)ds, \quad \forall t \in [0,T]$$

with $\dot{u} = v$ and $u = \varphi$ on $[-r,0]$.

As $\|f_n(t)\| \le L$ for all $n \in \mathbb{N}^*$ and for all $t \in [0,T]$, we may assume that the sequence $f_n(t) = f(\theta_n(t), \mathcal{T}(\theta_n(t))u_n)$ $\sigma(L^\infty([0,T],\mathbb{H}), L^1([0,T],\mathbb{H}))$ converges to a function $f \in L^\infty([0,T],\mathbb{H})$ with $\|f(t)\| \le L$ a. e. $t \in [0,T]$.
Since $\lim_{n\to+\infty} \theta_n(t) = \lim_{n\to+\infty} \delta_n(t) = t$, we have

$$\begin{aligned} \lim_{n\to+\infty} u_n(\theta_n(t)) &= \lim_{n\to+\infty} u_n(\delta_n(t)) \\ = \lim_{n\to+\infty} u_n(t) &= u(t) \end{aligned} \tag{3.10}$$

uniformly on $[0,T]$.
Step 3. Existence of solutions.
By construction we have

$$u_n(\delta_n(t)) \in C(\delta_n(t), u_n(\theta_n(t))), \qquad \text{for all } t \in [0,T] \text{ and for all } n \in \mathbb{N}^*.$$

Then $(\mathcal{H}_1)$, (3.2) and (3.10) entail

$$\begin{aligned} d_{C(t,u(t))}(u(t)) &= d_{C(t,u(t))}(u(t)) - d_{C(\delta_n(t),u_n(\theta_n(t)))}(u_n(\delta_n(t))) \\ &= d_{C(t,u(t))}(u(t)) - d_{C(\delta_n(t),u_n(\theta_n(t)))}(u_n(\delta_n(t)) - u(t) + u(t)) \\ &\le \|u_n(\delta_n(t)) - u(t)\| + L_1|t - \delta_n(t)| \\ &+ L_2\|u_n(\theta_n(t)) - u(t)\|_\infty \to 0 \text{ as } n \to \infty, \end{aligned}$$

and so the closedness of the set $C(t,u(t))$ ensures that

$$u(t) \in C(t,u(t)) \text{ for all } t \in [0,T].$$

Now, we prove that $f(t) \in F(t, \mathcal{T}(t)u)$ a.e. $t \in [0,T]$. Let $t \in [0,T]$. We have

$$
\begin{aligned}
\|\mathcal{T}(\theta_n(t))u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} &\le \|\mathcal{T}(t)u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} + \sup_{s\in[-r,0]} \|u_n(\theta_n(t)+s) - u_n(t+s)\| \\
&\le \|\mathcal{T}(t)u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} + \sup_{\{s_1,s_2\in[-r,1],|s_1-s_2|\le\mu_n\}} \|u_n(s_1) - u_n(s_2)\| \\
&\le \|\mathcal{T}(t)u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} + \sup_{\{s_1,s_2\in[-r,0],|s_1-s_2|\le\mu_n\}} \|u_n(s_1) - u_n(s_2)\| \\
&\quad + \sup_{\{s_1,s_2\in[0,1],|s_1-s_2|\le\mu_n\}} \|u_n(s_1) - u_n(s_2)\| \\
&\le \|\mathcal{T}(t)u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} + \sup_{\{s_1,s_2\in[-r,0],|s_1-s_2|\le\mu_n\}} \|\varphi(s_1) - \varphi(s_2)\| \\
&\quad + \sup_{\{s_1,s_2\in[0,1],|s_1-s_2|\le\mu_n\}} \|u_n(s_1) - u_n(s_2)\| \\
&\le \|\mathcal{T}(t)u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} + \sup_{\{s_1,s_2\in[-r,0],|s_1-s_2|\le\mu_n\}} \|\varphi(s_1) - \varphi(s_2)\| + \frac{L_1+2L}{1-L_2}\mu_n.
\end{aligned}
$$

The continuity of φ and the uniform convergence of u_n to u give

$$\lim_{n\to\infty} \|\mathcal{T}(\theta_n(t))u_n - \mathcal{T}(t)u\|_{\mathcal{C}_0} = 0. \tag{3.11}$$

As

$$f_n(t) = f(\theta_n(t), \mathcal{T}(\theta_n(t))u_n) \in F(\theta_n(t), \mathcal{T}(\theta_n(t))u_n).$$

Using(3.11) and invoking the scalarly upper semicontinuity of F on $[0,1]\times\mathcal{C}_\tau$ and a closure type lemma in ([7], Theorem VI-14) we get $f(t) \in F(t, \mathcal{T}(t)u)$ a.e. $t \in [0,T]$.
Recall that the inclusion

$$-\dot{u}_n(t) \in N_{C(\delta_n(t), u_n(\theta_n(t)))} u_n(\delta_n(t)) + f_n(t)$$

is equivalent to

$$\delta^*\big(-\dot{u}_n(t) - f_n(t), C(\delta_n(t), u_n(\theta_n(t)))\big) + \langle \dot{u}_n(t) + f_n(t), u_n(\delta_n(t))\rangle \le 0 \text{ a. e.}$$

By integrating on $[0,T]$ we get

$$\int_0^T \delta^*\big(-\dot{u}_n(t) - f_n(t), C(\delta_n(t), u_n(\theta_n(t)))\big)dt + \int_0^T \langle \dot{u}_n(t) + f_n(t), u_n(\delta_n(t))\rangle dt \le 0.$$

Since the sequences (f_n), $(\dot{u}_n)$ $\sigma(L^\infty([0,T],\mathbb{H}), L^1([0,T],\mathbb{H}))$ converge to f and $\dot{u}$ respectively, then

$$\int_0^T \langle \dot{u}_n(t) + f_n(t), u_n(\delta_n(t))\rangle dt = \int_0^T \langle \dot{u}(t) + f(t), u(t)\rangle dt.$$

Furthermore, from $(\mathcal{H}_1)$, we get the estimate

$$
\begin{aligned}
&\int_0^T |\delta^*\big(-\dot{u}_n(t) - f_n(t), C(\delta_n(t), u_n(\theta_n(t)))\big) - \delta^*\big(-\dot{u}_n(t) - f_n(t), C(t, u(t))\big)|dt \\
&\le \int_0^T \|-\dot{u}_n(t) - f_n(t)\|\big(L_1|\delta_n(t) - t| + L_2\|u_n(\theta_n(t)) - u(t)\|\big)dt \\
&\le \Big(\frac{L_1+2L}{1-L_2} + \alpha\Big) \int_0^T \big(L_1|\delta_n(t) - t| + L_2\|u_n(\theta_n(t)) - u(t)\|\big)dt.
\end{aligned}
$$

By passing to the limit when n goes to ∞ in the preceding estimate and taking account into (3.9) and (3.10), we see that the integral

$$\int_0^T |\delta^*\big(-\dot{u}_n(t) - f_n(t), C(\delta_n(t), u_n(\theta_n(t)))\big) - \delta^*\big(-\dot{u}_n(t) - f_n(t), C(t, u(t))\big)|dt$$

goes to 0 when n goes to ∞. Using the classical lower semicontinuity of convex integral functionals (see e.g. [2] Theorem 8.1.6), we have

$$\liminf_{n\to\infty} \int_0^T \delta^*\big(-\dot{u}_n(t) - f_n(t), C(t, u(t))\big)dt \geq \int_0^T \delta^*\big(-\dot{u}(t) - f(t), C(t, u(t))\big)dt.$$

Hence we deduce that

$$\liminf_{n\to\infty} \int_0^T \delta^*\big(-\dot{u}_n(t) - f_n(t), C(\delta_n(t), u_n(\theta_n(t)))\big)dt \geq \int_0^T \delta^*\big(-\dot{u}(t) - f(t), C(t, u(t))\big)dt$$

so that

$$\int_0^T \delta^*\big(-\dot{u}(t) - f(t), C(t, u(t))\big)dt + \int_0^T \langle \dot{u}(t) + f(t), u(t)\rangle dt \leq 0.$$

As $u(t) \in C(t, u(t))$ for all $t \in [0, T]$, the last inequality is equivalent to

$$-\dot{u}(t) \in N_{C(t,u(t))}u(t) + f(t), \qquad \text{a. e. t} \in [0, T].$$

The proof is therefore complete since $f(t) \in F(t, \mathcal{T}(t)u)$ for a. e. $t \in [0, T]$.

□

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