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The elliptic sinh-Gordon equation in the half plane

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Abstract

Boundary value problems for the elliptic sinh-Gordon equation formulated in the half plane are studied by applying the so-called Fokas method. The method is a significant extension of the inverse scattering transform, based on the analysis of the Lax pair formulation and the global relation that involves all known and unknown boundary values. In this paper, we derive the formal representation of the solution in terms of the solution of the matrix Riemann-Hilbert problem uniquely defined by the spectral functions. We also present the global relation associated with the elliptic sinh-Gordon equation in the half plane. We in turn show that given appropriate initial and boundary conditions, the unique solution exists provided that the boundary values satisfy the global relation. Furthermore, we verify that the linear limit of the solution coincides with that of the linearized equation known as the modified Helmhotz equation.

Keywords: Boundary value problems, elliptic PDEs, sinh-Gordon equation, integrable equation.
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1. Introduction

A unified method introduced by A. S. Fokas [7, 8] (see also the monograph [12]), which can be considered a significant extension of the inverse scattering transform, is used for analyzing boundary value problems. This so-called Fokas method has been extensively applied to solve a large class of boundary value problems; for example, integrable nonlinear evolution equations such as the nonlinear Schrödinger, the Korteweg-deVries and the sine-Gordon equations [10] and linear and nonlinear elliptic partial differential equations [9, 19] as well as difference-differential equations [2, 3]. The rigorous result in the implementation of the Fokas method is that the solution can be expressed in terms of the solution for a Riemann-Hilbert (RH) problem uniquely defined by spectral functions that involve initial and boundary values. Importantly, the global relation that couples known and unknown boundary values plays a crucial role of analyzing the boundary value problems.

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Indeed, the global relation is not only able to allow the existence of the unique solution for the boundary value problem, but it is also able to to characterize the unknown boundary values that enter in the spectral functions. For example, for the typical Dirichlet boundary value problem, the Neumann boundary value is unknown. In this case, it is necessary to characterize the unknown Neumann boundary value.

The implementation of the Fokas method to the boundary value problems has remarkable advantages. In particular, we note that (i) it is possible to eliminate the unknown boundary values that appear in the representation for the solution. In general, this can be done by solving nonlinear Volterra integral equations for the unknown boundary values. However, there is a certain class of boundary conditions (BCs), called *linearizable*, which makes it possible to bypass the solution of the nonlinear Volterra integral equation. For such BCs, one can eliminate the unknown boundary values simply by using algebraic manipulation of the global relation [10]. Thus, the Fokas method is as effective as the classical inverse scattering transform. (ii) For evolution equations, the jump matrix of the RH problem defined by the spectral functions has an explicit (x,t)-dependence of exponential form. Thus, it is possible to study the appropriate asymptotics of the solution by using the Deift-Zhou method [5] for the long time behavior, or to study the small dispersion limit by using the Deift-Venakides-Zhou method [6].

It should be noted that the classical inverse scattering transform has been used in most cases of hyperbolic evolution equations, with some exceptions. For example, in [4, 18] the inverse scattering transform was applied to solve the initial value problem for the elliptic sinh-Gordon equation in the plane $\{(x,y)| -\infty < x < \infty, -\infty < y < \infty\}$. Recently, regarding boundary value problems of the nonlinear elliptic type, the Fokas method was applied to solve the elliptic sine-Gordon equation in the half plan $(y > 0)$. In [14, 19], it has been derived the formal representation of the solution and discussed the existence of the solution for the elliptic sine-Gordon equation by using the global relation. Hence, of particular interest appears in the study of the boundary value problem for the elliptic sinh-Gordon equation formulated in the half plane via the Fokas method.

In this paper we study the initial-boundary problem for the elliptic sinh-Gordon equation posed in the half plane

$$q_{xx} + q_{yy} = \sinh q, \qquad -\infty < x < \infty, \quad y > 0. \tag{1.1}$$

The elliptic sinh-Gordon equation arises in models of interacting charged particles in plasma physics [18]. Besides applications in physics, from a mathematical point of view, this equation is also interesting because it is completely integrable and is connected to the Toda lattice equations [1]. The purpose of this paper is to implement the Fokas method to the elliptic sinh-Gordon equation posed in the half plane (1.1). We derive the global relation based on the spectral analysis of the Lax pair formulation. We then show that given appropriate initial and boundary conditions, the unique solution for (1.1) exists provided that the boundary values satisfy this global relation. Moreover, we address the formal representation for the solution in terms of the solution of the RH problem defined by the spectral functions.

The outline of the paper is following. In section 2, we introduce the relevant notations, formulas and the regularity assumptions for the initial and boundary data. In section 3, we apply the Fokas method to solve the elliptic sinh-Gordon equation posed in the half plane. In particular, the existence of the solution is discussed by analyzing the the matrix RH problem as an inverse problem. In section 4, we verify that the linear limit of the solution for (1.1) coincides with that of the corresponding linearized equation known as the modified Helmhotz equation. We end with concluding remarks in section 5.

2. Preliminaries

It was shown in [4, 18] that the elliptic sinh-Gordon equation admits the following Lax pair formulation

$$\varphi_x + \omega_1(k)\sigma_3\varphi = Q(x,y,k)\varphi, \tag{2.1a}$$

$$\varphi_y + \omega_2(k)\sigma_3\varphi = iQ(x,y,-k)\varphi, \tag{2.1b}$$

where $k \in \mathbb{C}$ is a spectral parameter, φ is a 2×2 matrix-valued eigenfunction and

$$\omega_1(k) = -\frac{1}{2i}\left(k - \frac{1}{4k}\right), \qquad \omega_2(k) = -\frac{1}{2}\left(k + \frac{1}{4k}\right), \tag{2.2}$$

$$Q(x,y,k) = \frac{1}{4}\begin{pmatrix} \frac{i}{2k}(\cosh q - 1) & -\left(r + \frac{\sinh q}{2k}\right) \\ r - \frac{\sinh q}{2k} & -\frac{i}{2k}(\cosh q - 1)\end{pmatrix} \tag{2.3}$$

with

$$r(x,y) = iq_x(x,y) + q_y(x,y), \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

The Lax pair given in (2.1) is a slightly different formulation from that considered in [18], but they are equivalent. Using this Lax pair, we compare the linear limit of the elliptic sinh-Gordon equation to the solution of the linearized equation known as the modified Helmholtz equation, which will be discussed in Section 4. It should be remarked that the elliptic sinh-Gordon equation (1.1) is the compatible condition of the Lax pair (2.1) in the sense that $\phi_{xy} = \phi_{yx}$ from (2.1) implies that q solves (1.1) if the spectral parameter k is independent of x and y.

Note that

$$\operatorname{Re}\omega_1(k) = -\frac{1}{8}\operatorname{Im} k\left(4 + \frac{1}{|k|^2}\right), \qquad \operatorname{Re}\omega_2(k) = -\frac{1}{8}\operatorname{Re} k\left(4 + \frac{1}{|k|^2}\right), \tag{2.5}$$

which implies

$$\operatorname{Re}\omega_1(k) < 0 \quad \text{for } \operatorname{Im} k > 0, \qquad \operatorname{Re}\omega_2(k) < 0 \quad \text{for } \operatorname{Re} k > 0. \tag{2.6}$$

Also note that due to the symmetry of $Q(x,y,k)$, the eigenfunction has the same symmetry

$$\varphi_{11}(x,y,k) = \varphi_{22}(x,y,-k), \qquad \varphi_{21}(x,y,k) = -\varphi_{12}(x,y,-k), \tag{2.7}$$

where the subscripts denote the (i,j)-component of the matrix.

Throughout the paper, we denote the matrix commutator by $\hat{\sigma}A$, that is,

$$\hat{\sigma}A = [\sigma_3, A] = \sigma_3 A - A\sigma_3, \tag{2.8}$$

where A is a 2×2 matrix. This yields the notation

$$e^{\hat{\sigma}_3\xi}A = e^{\sigma_3\xi}Ae^{-\sigma_3\xi} = \begin{pmatrix} a_{11} & e^{2\xi}a_{12} \\ e^{-2\xi}a_{21} & a_{22}\end{pmatrix}.$$

We also denote

$$q(x,0) = g_0(x), \qquad q_y(x,0) = g_1(x), \tag{2.9}$$

where the functions g_0 and g_1 are assumed to be

$$g_0 + 2m\pi \in H^1(\mathbb{R}) \quad \text{for some } m \in \mathbb{Z}, \qquad g_1 \in H^1(\mathbb{R}). \tag{2.10}$$

Making use of the integrating factor $\mu(x,y,k) = \varphi(x,y,k)e^{(\omega_1(k)x + \omega_2(k)y)\sigma_3}$, it is convenient to derive the following modified Lax pair of the form

$$\mu_x + \omega_1(k)\,[\sigma_3, \mu] = Q(x,y,k)\mu, \tag{2.11a}$$

$$\mu_y + \omega_2(k)\,[\sigma_3, \mu] = iQ(x,y,-k)\mu. \tag{2.11b}$$

3. The elliptic sinh-Gordon equation in the half plane

In this section, we demonstrate the Fokas method to solve the elliptic sinh-Gordon equation in the half plane.

Eigenfunctions. In order to analyze the eigenfunctions associated with the Lax pair formulation (2.11), we introduce a differential 1-form W given by

$$W(x,t,k) = [Q(x,y,k)dx + iQ(x,y,-k)dy]\,\mu(x,y,k), \tag{3.1}$$

and hence, equations (2.11) can be written in the form

$$d\left[e^{(\omega_1(k)x+\omega_2(k)y)\hat{\sigma}_3}\mu(x,y,k)\right] = e^{(\omega_1(k)x+\omega_2(k)y)\hat{\sigma}_3}W(x,y,k). \tag{3.2}$$

We now define eigenfunctions that satisfy both parts of the Lax pair (2.11) as

$$\mu_j(x,y,k) = I + \int_{(x_j,y_j)}^{(x,y)} e^{-(\omega_1(k)(x-\xi)+\omega_2(k)(y-\eta))\hat{\sigma}_3}W_j(\xi,\eta,k), \tag{3.3}$$

where $(x,y),(x_j,y_j) \in D = \{-\infty < x < \infty,\ 0 < y < \infty\}$, W_j is the differential form defined in (3.1) with μ_j and (x_j,y_j) is a fixed point in D. If D is the interior of the convex polygon, it has been shown in [12] that the all vertices of the polygon can be taken for (x_j,y_j).

We choose three distinct points (x_j,y_j) in D, $j=1,2,3$ (see fig. 1),

$$(x_1,y_1) = (-\infty,y), \quad (x_2,y_2) = (\infty,y), \quad (x_3,y_3) = (x,\infty)$$

and the corresponding eigenfunctions satisfy the following integral equations:

$$\mu_1(x,y,k) = I + \int_{-\infty}^{x} e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3}\left(Q\mu_1\right)(\xi,y,k)d\xi, \tag{3.4a}$$

$$\mu_2(x,y,k) = I - \int_{x}^{\infty} e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3}\left(Q\mu_2\right)(\xi,y,k)d\xi, \tag{3.4b}$$

$$\mu_3(x,y,k) = I - i\int_{y}^{\infty} e^{-\omega_2(k)(y-\eta)\hat{\sigma}_3}\left(Q(x,\eta,-k)\mu_3(x,\eta,k)\right)d\eta. \tag{3.4c}$$

Note that the eigenfunctions μ_j, $j=1,2,3$, also enjoy the same symmetry as in the form of (2.7). Since the off-diagonal components of the matrix-valued eigenfunctions μ involve the explicit exponential terms, the regions of analyticity of the eigenfunctions can be determined:

- $[\mu_1]_1$ and $[\mu_2]_2$ are analytic for $\operatorname{Im} k > 0$
- $[\mu_1]_2$ and $[\mu_2]_1$ are analytic for $\operatorname{Im} k < 0$
- $[\mu_3]_1$ is analytic for $\operatorname{Re} k < 0$, while $[\mu_3]_2$ is analytic for $\operatorname{Re} k > 0$,

where $[\cdot]_j$ denotes the j-th column of the matrix. Hereafter, we then write each column of μ_j as the following notations:

$$\mu_1 = \left(\mu_1^+,\mu_1^-\right), \quad k \in \left(\mathbb{C}^+,\mathbb{C}^-\right), \qquad \mu_2 = \left(\mu_2^-,\mu_2^+\right), \quad k \in \left(\mathbb{C}^-,\mathbb{C}^+\right), \tag{3.5a}$$

$$\mu_3 = \left(\mu_3^-,\mu_3^+\right), \quad k \in \left(\operatorname{Re} k \le 0, \operatorname{Re} k \ge 0\right), \tag{3.5b}$$

where $\mathbb{C}^\pm$ denote the upper/lower half plane of the complex k-plane, respectively.

Spectral functions. The matrix eigenfunctions μ_1 and μ_2 are both fundamental solutions of the Lax pair (2.11). Hence, they are related by the so-called spectral function $S(k)$:

$$\mu_2(x,y,k) = \mu_1(x,y,k)e^{-(\omega_1(k)x+\omega_2(k)y)\hat{\sigma}_3}S(k), \quad k \in \mathbb{R},\ x \in \mathbb{R},\ 0 \le y < \infty. \tag{3.6}$$

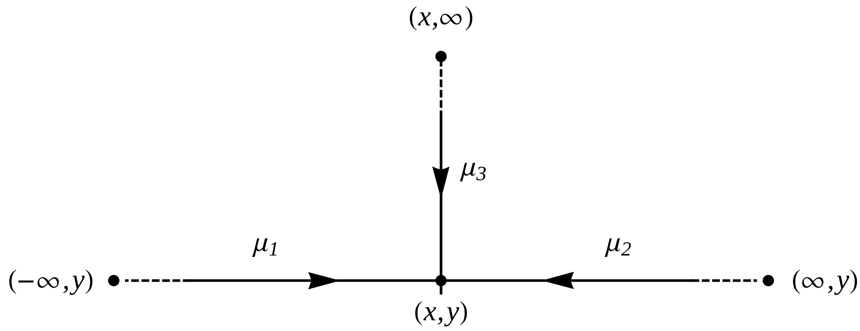

Figure 1: The eigenfunctions μ_1, μ_2 and μ_3.

Letting $y = 0$ and $x \to -\infty$ in (3.6), we find the spectral function given by

$$S(k) = I - \int_{-\infty}^{\infty} e^{\omega_1(k)\xi\hat{\sigma}_3} \left(Q\mu_2\right)(\xi, 0, k) d\xi. \tag{3.7}$$

From the symmetry of the eigenfunctions μ_2, we write the spectral function $S(k)$ in the form

$$S(k) = \begin{pmatrix} a(k) & -b(-k) \\ b(k) & a(-k) \end{pmatrix}. \tag{3.8}$$

We define $\Phi(x, k) = \mu_2(x, 0, k)$. More specifically, the function Φ satisfies

$$\Phi(x, k) = I - \int_{x}^{\infty} e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q_0\Phi)(\xi, k) d\xi, \quad k \in \left(\mathbb{C}^-, \mathbb{C}^+\right), \quad x \in \mathbb{R}, \tag{3.9}$$

where $Q_0(x, k) = Q(x, 0, k)$, that is,

$$Q_0(x, k) = \frac{1}{4} \begin{pmatrix} \frac{i}{2k}\left(\cosh q(x, 0) - 1\right) & -\left(r(x, 0) + \frac{\sinh q(x,0)}{2k}\right) \\ r(x, 0) - \frac{\sinh q(x,0)}{2k} & -\frac{i}{2k}\left(\cosh q(x, 0) - 1\right) \end{pmatrix}. \tag{3.10}$$

From (3.7), it follows that the spectral function $S(k)$ can be determined by the function Φ

$$S(k) = \lim_{x \to -\infty} \left[e^{\omega_1(k)x\hat{\sigma}_3} \Phi(x, k) \right]. \tag{3.11}$$

This justifies that $a(k)$ has an analytic continuation for $\operatorname{Im} k < 0$ and $b(k)$ is defined only for $k \in \mathbb{R}$. Moreover, due to the symmetry of the eigenfunction μ_2, Φ can be written as

$$\Phi(x, k) = \begin{pmatrix} \Phi_1(x, k) & -\Phi_2(x, -k) \\ \Phi_2(x, k) & \Phi_1(x, -k) \end{pmatrix}. \tag{3.12}$$

Recalling the domain of analyticity of the eigenfunction μ_2, we also denote

$$\Phi = \left(\Phi^-, \Phi^+\right), \qquad k \in \left(\mathbb{C}^-, \mathbb{C}^+\right). \tag{3.13}$$

Thus, we obtain the integral representations for the spectral functions $a(k)$ and $b(k)$ and we summarize these representations below.

Proposition 3.1. *Given $q(x,0)=g_0(x)$, the spectral functions $\{a(k),\ b(k)\}$ are defined by*

$$\begin{aligned} a(k) =& 1-\frac{1}{4}\int_{-\infty}^{\infty}\left\{\frac{i}{2k}\left(\cosh g_0(\xi)-1\right)\Phi_1(\xi,k)\right.\\ &\left.-\left(i\dot{g}_0(\xi)+q_y(\xi,0)+\frac{\sinh g_0(\xi)}{2k}\right)\Phi_2(\xi,k)\right\}d\xi, \qquad \operatorname{Im}k<0, \end{aligned} \tag{3.14a}$$

$$\begin{aligned} b(k) =& -\frac{1}{4}\int_{-\infty}^{\infty}e^{-2\omega_1(k)\xi}\left\{\left(i\dot{g}_0(\xi)+q_y(\xi,0)-\frac{\sinh g_0(\xi)}{2k}\right)\Phi_1(\xi,k)\right.\\ &\left.-\frac{i}{2k}\left(\cosh g_0(\xi)-1\right)\Phi_2(\xi,k)\right\}d\xi, \qquad k\in\mathbb{R}, \end{aligned} \tag{3.14b}$$

where the functions Φ_1 and Φ_2 are solutions of the x-part of the Lax pair (2.11a) with $y=0$, that is, Φ_1 and Φ_2 solve the following system of ordinary differential equations:

$$\Phi_{1x}=\frac{1}{4}\left[\frac{i}{2k}\left(\cosh g_0(x)-1\right)\Phi_1-\left(i\dot{g}_0(x)+q_y(x,0)+\frac{\sinh g_0(x)}{2k}\right)\Phi_2\right], \tag{3.15a}$$

$$\Phi_{2x}-2\omega_1(k)\Phi_2=\frac{1}{4}\left[\left(i\dot{g}_0(x)+q_y(x,0)-\frac{\sinh g_0(x)}{2k}\right)\Phi_1-\frac{i}{2k}\left(\cosh g_0(x)-1\right)\Phi_2\right] \tag{3.15b}$$

with $\lim_{x\to\infty}(\Phi_1,\Phi_2)=(1,0)$.

Global relation. Note that the eigenfunctions μ_1 and μ_3 solve the same differential equations (2.11) with the same boundary condition at infinity. Similarly, μ_2 and μ_3 solve the same differential equations with the same boundary condition at infinity. Thus, we know

$$\mu_3^-(x,y,k)=\mu_2^-(x,y,k), \qquad k\in\mathbb{C}_{\text{III}}, \tag{3.16a}$$

$$\mu_3^-(x,y,k)=\mu_1^+(x,y,k), \qquad k\in\mathbb{C}_{\text{II}}, \tag{3.16b}$$

where $\mathbb{C}_{\text{II}}$ and $\mathbb{C}_{\text{III}}$ denote the second and third quadrants of the complex k-plane, respectively. Since $\Phi^-(x,k)=\mu_2^-(x,0,k)$, we find the following global relation

$$\Phi^-(x,k)=\mu_3^-(x,0,k), \qquad x\in\mathbb{R}, \quad k\in\mathbb{C}_{\text{III}}. \tag{3.17}$$

In particular, letting $x\to-\infty$ in (3.17) yields $a(k)=1$ for $k\in\mathbb{C}_{\text{III}}$ and $b(k)=0$ for $k\le 0$. From the analytic continuation, we find the global relation in terms of the spectral functions $a(k)$ and $b(k)$

$$a(k)=1, \quad \operatorname{Im}k\le 0, \qquad b(k)=0, \quad k\le 0. \tag{3.18}$$

We also obtain an alternative global relation by analyzing eigenfunctions μ_1 and μ_3. Indeed, let $\Psi(x,k)=\mu_1(x,0,k)$, that is,

$$\Psi(x,k)=I+\int_{-\infty}^{\infty}e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3}Q_0(\xi,k)\Psi(\xi,k)d\xi, \qquad x\in\mathbb{R}, \quad k\in\left(\mathbb{C}^+,\mathbb{C}^-\right). \tag{3.19}$$

According to the domain of analyticity, we denote

$$\Psi=\left(\Psi^+,\Psi^-\right), \qquad k\in\left(\mathbb{C}^+,\mathbb{C}^-\right). \tag{3.20}$$

Then, we also find the following global relation

$$\Psi^+(x,k)=\mu_3^-(x,0,k), \qquad x\in\mathbb{R}, \quad k\in\mathbb{C}_{\text{II}}. \tag{3.21}$$

Riemann-Hilbert problem. As an inverse problem, we formulate a matrix Riemann-Hilbert (RH) problem. First, note that from the global relation (3.18), for $k \in \mathbb{R}$ the spectral function $S(k)$ simply becomes

$$S(k) = \begin{pmatrix} 1 & -b(-k) \\ b(k) & 1 \end{pmatrix}. \tag{3.22}$$

Recalling equation (3.6), we then formulate the following matrix RH problem

$$M^-(x,y,k) = M^+(x,y,k)J(x,y,k), \qquad k \in \mathbb{R}, \tag{3.23}$$

where the sectionally analytic functions $M^\pm$ are defined by

$$M^+ = \left(\mu_2^+, \mu_1^+\right), \quad k \in \mathbb{C}^+, \qquad M^- = \left(\mu_1^-, \mu_2^-\right), \quad k \in \mathbb{C}^- \tag{3.24}$$

and the jump matrix is given by

$$J = \begin{pmatrix} 1 & -b(-k)e^{-2\theta(x,y,k)} \\ b(k)e^{2\theta(x,y,k)} & 1 \end{pmatrix}, \tag{3.25}$$

where

$$\theta(x,y,k) = \omega_1(k)x + \omega_2(k)y. \tag{3.26}$$

Note that

$$\det M^\pm = 1, \qquad M^\pm = I + O(1/k), \quad k \to \infty. \tag{3.27}$$

The solution of the elliptic sinh-Gordon equation can be obtained from the unique solution of the RH problem. In this respect, we expand the solution of the RH problem M as

$$M(x,y,k) = I + \frac{M^{(1)}(x,y)}{k} + \frac{M^{(2)}(x,y)}{k^2} + O(1/k^2), \qquad k \to \infty. \tag{3.28}$$

Substituting this expansion into the x-part of the Lax pair (2.11a), from the $(2,1)$-component at $O(1)$, we find

$$iq_x(x,y) + q_y(x,y) = -4iM_{21}^{(1)}(x,y) \tag{3.29}$$

and the $(1,1)$-component at $O(1/k)$ yields

$$M_{11x}^{(1)}(x,y) = -\frac{1}{8i}\left(\cosh q(x,y) - 1\right) - \frac{1}{4}\left(iq_x(x,y) + q_y(x,y)\right) M_{21}^{(1)}(x,y). \tag{3.30}$$

Simplifying the above equation with (3.29), we obtain the reconstruction formula for the solution of (1.1) given by

$$\cosh q(x,y) = 1 - 8iM_{11x}^{(1)}(x,y) - 8\left(M_{21}^{(1)}\right)^2. \tag{3.31}$$

Note that the RH problem also can be solved via the Cauchy projector, which results in the integral representation for the solution of (1.1) in terms of the solution M. Indeed, let $\tilde{J} = I - J$. Then, the matrix RH problem can be written as

$$M^+ - M^- = M^+\tilde{J}, \qquad k \in \mathbb{R}, \tag{3.32}$$

where the jump matrix $\tilde{J}$ is given by

$$\tilde{J}(x,y,k) = \begin{pmatrix} 0 & b(-k)e^{-2\theta(x,y,k)} \\ -b(k)e^{2\theta(x,y,k)} & 0 \end{pmatrix}, \qquad k \in \mathbb{R}. \tag{3.33}$$

Employing the Cauchy projector, the solution of the RH problem is found in terms of the Cauchy type of the integral equation:

$$M^+(x,y,k) = I + \frac{1}{2i\pi}\int_0^\infty M^+(x,y,l)\tilde{J}(x,y,l)\frac{dl}{l-k}. \tag{3.34}$$

In particular, the first column of (3.34) yields

$$\begin{pmatrix} M_{11}^{+}(x,y,k) \\ M_{21}^{+}(x,y,k) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi}\int_0^\infty \begin{pmatrix} M_{12}^{+}(x,y,l)b(l)e^{2\theta(x,y,l)} \\ M_{22}^{+}(x,y,l)b(l)e^{2\theta(x,y,l)} \end{pmatrix} \frac{dl}{l-k}. \tag{3.35}$$

Substituting equation (3.35) into the Lax pair and collecting the $(2,1)$-component of the resulting expression, finally, we obtain the integral representation of the solution for the elliptic sinh-Gordon equation in the half plane given by

$$iq_x(x,y) + q_y(x,y) = -\frac{2}{\pi}\int_0^\infty \left(\mu_1^+\right)_{22}(x,y,k)b(k)e^{\omega_1(k)x+\omega_2(k)y}dk. \tag{3.36}$$

In summary, we now state the existence theorem of the solution for the elliptic sinh-Gordon equation in the half plane.

Theorem 3.2. *Assume the functions $g_0 + 2m\pi \in H^1(\mathbb{R})$ for some $m \in \mathbb{Z}$, and $g_1 \in H^1(\mathbb{R})$ with the sufficiently small H^1 norms of g_0 and g_1. Let the functions $a(k)$ and $b(k)$ be given by (3.14) in Proposition 3.1. Suppose that given $g_0(x)$, there exists a function $g_1(x)$ such that the global relation is satisfied*

$$a(k) = 1, \quad \operatorname{Im} k \le 0, \qquad b(k) = 0, \quad k \le 0. \tag{3.37}$$

Let $M(x,y,k)$ be the solution of the following matrix Riemann-Hilbert (RH) problem

$$M^-(x,y,k) = M^+(x,y,k)J(x,y,k), \qquad k \in \mathbb{R}, \tag{3.38}$$

where $\det(M^\pm) = 1$, *$M^\pm = I + O(1/k)$ as $k \to \infty$ and the jump matrix is given in (3.25).*

Then the Reimann-Hilbert problem is uniquely solvable and the function $q(x,y)$ defined by

$$iq_x + q_y = -4i\lim_{k\to\infty} kM_{21}, \qquad \cosh q(x,y) = 1 - 8i\lim_{k\to\infty} kM_{11x} - 8\lim_{k\to\infty}(kM_{21})^2 \tag{3.39}$$

solves the elliptic sinh-Gordon equation (1.1) in the half plane satisfying

$$q(x,0) = g_0(x), \qquad q_y(x,0) = g_1(x). \tag{3.40}$$

Proof. First , note that the vanishing lemma [12] verifies the unique solvability of the RH problem (3.42), that is, if $M = O(1/k)$ as $k \to \infty$, the RH problem has only the trivial solution. The proof that $q(x,y)$ defined in (3.39) solves the elliptic sinh-Gordon equation (1.1) is based on the so-called dressing method [12] which was also discussed before the theorem.

To prove (3.40), define the map $\{a(k), b(k)\} \to \{g_0(x), g_1(x)\}$ given by

$$\cosh g_0(x) = 1 - 8i\lim_{k\to\infty} kM_{11x}^{(x)} - 8\lim_{k\to\infty}\left(kM_{21}^{(x)}\right)^2, \tag{3.41a}$$

$$i\dot{g}_0(x) + g_1(x) = -4i\lim_{k\to\infty} kM_{21}^{(x)}, \tag{3.41b}$$

where $M^{(x)}$ is the solution of the following matrix Riemann-Hilbert problem

$$M_-^{(x)}(x,k) = M_+^{(x)}(x,k)J^{(x)}(x,k), \qquad k \in \mathbb{R}, \tag{3.42}$$

with $M^{(x)} = I + O(1/k)$ as $k \to \infty$ and the jump matrix $J^{(x)}$ defined by

$$J^{(x)}(x,k) = \begin{pmatrix} 1 & -\frac{b(-k)}{a(-k)}e^{-2\omega_1(k)x} \\ \frac{b(k)}{a(k)}e^{2\omega_1(k)x} & 1 \end{pmatrix}, \qquad k \in \mathbb{R}. \tag{3.43}$$

Note that if the global relation (3.37) is satisfied, then

$$J^{(x)}(x,k) = J(x,0,k), \tag{3.44}$$

which implies that $M^{(x)}(x,k) = M(x,0,k)$. Hence, letting $y = 0$ in (3.39), equation (3.40) follows. $\square$

4. Linear limit

We next show that the linear limit of the solution of (1.1) coincides with the solution of the linearized equation for the elliptic sinh-Gordon equation known as the modified Helmholtz equation in the half plane

$$q_{xx} + q_{yy} = q, \qquad -\infty < x < \infty, \quad y > 0, \tag{4.1}$$
$$q(x,0) = g_0(x), \qquad -\infty < x < \infty. \tag{4.2}$$

It has been shown in [19] that the Fokas method was applied to solve (4.1) and the solution of (4.1) was found in the form

$$q(x,y) = \frac{1}{2\pi}\int_0^\infty e^{-\tilde{\omega}_1(k)x-\tilde{\omega}_2(k)y}\tilde{\omega}_2(k)\hat{g}_0(k)\frac{dk}{k}, \tag{4.3}$$

where the linear dispersion relations are defined by

$$\tilde{\omega}_1(k) = \frac{1}{2i}\left(k-\frac{1}{k}\right), \qquad \tilde{\omega}_2(k) = \frac{1}{2}\left(k+\frac{1}{k}\right), \tag{4.4}$$

and the spectral function $\hat{g}_0$ is given by

$$\hat{g}_0(k) = \int_{-\infty}^\infty e^{\tilde{\omega}_1(k)x}q(x,0)dx. \tag{4.5}$$

Regarding the linear limit, we consider equation (3.36). First, we note that in the linear limit,

$$\left(\mu_1^+\right)_{22} \to 1, \qquad \Phi_1 \to 1, \qquad \Phi_2 \to 0. \tag{4.6}$$

Hence, the spectral function $b(k)$ becomes

$$b(k) = -\frac{1}{4}\int_{-\infty}^\infty e^{-2\omega_1(k)\xi}\left(r(\xi)-\frac{g_0(\xi)}{2k}\right)d\xi, \tag{4.7}$$

where $r(x) = i\dot{g}_0(x) + q_y(x,0)$ to suppress the notation for brevity, and then equation (3.36) leads to

$$iq_x(x,y) + q_y(x,y) = -\frac{2}{\pi}\int_0^\infty b(k)e^{2\theta(x,y,k)}dk. \tag{4.8}$$

Noting $i\omega_1(k) + \omega_2(k) = -k$, we find the linear limit of the solution for the elliptic sinh-Gordon equation in the half plane given by

$$q(x,y) = \frac{1}{\pi}\int_0^\infty b(k)e^{2\theta(x,y,k)}\frac{dk}{k}. \tag{4.9}$$

It should be now required to characterize the spectral function $b(k)$ appearing in (4.9). Note that $b(-k) = 0$ for $k \geq 0$ thanks to the global relation (3.18) and hence, we find

$$\frac{1}{4}\int_{-\infty}^\infty e^{2\omega_1(k)\xi}\left(r(\xi)+\frac{g_0(\xi)}{2k}\right)d\xi = 0, \qquad k \geq 0. \tag{4.10}$$

Employing the change of variable $k \to \frac{1}{4k}$ in (4.10), we then obtain

$$\frac{1}{4}\int_{-\infty}^\infty e^{-2\omega_1(k)\xi}\left(r(\xi)+2kg_0(\xi)\right)d\xi = 0. \tag{4.11}$$

Adding the above equation with (4.7), the function $b(k)$ can be written as

$$b(k) = -\omega_2(k)\int_{-\infty}^\infty e^{-2\omega_1(k)\xi}g_0(\xi)d\xi. \tag{4.12}$$

Thus, substituting (4.12) into (4.9) yields the linear limit given by

$$q(x,y) = -\frac{1}{\pi}\int_0^\infty e^{2\theta(x,y,k)}\omega_2(k)\int_{-\infty}^\infty e^{-2\omega_1(k)\xi}g_0(\xi)d\xi\frac{dk}{k}. \tag{4.13}$$

Finally, performing the change of variable $2k \to k$, we find the linear limit of the solution for the elliptic sinh-Gordon equation in the half plane, which coincides with (4.3).

5. Concluding remarks

In conclusion, we have studied the boundary value problems for the elliptic sinh-Gordon equation posed in the half plane. We have done so by applying the Fokas method, which is a significant extension of the inverse scattering transform. We have derived the global relation for the elliptic sinh-Gordon equation that involves all known and unknown boundary values. Also, we have presented the existence theorem for the unique solution provided that the boundary values satisfy the global relation. Moreover, we have derived the representation of the solution in terms of the unique solution of the Riemann-Hilbert problem whose jump matrix are uniquely defined by the spectral functions. In addition to solving the elliptic sinh-Gordon equation, we have verified that the linear limit coincides with the solution of the linearized equation known as the modified Helmhotz equation.

We remark that one of the most difficult steps in the implementation of the Fokas method is to characterize unknown boundary values, known as the Dirichlet-to-Neumann map [11]. For example, for the Dirichlet boundary value problem of the elliptic sinh-Gordon equation in the half plane, the Dirichlet boundary datum $q(x, 0)$ is given, whereas the Neumann boundary value $q_y(x, 0)$ is unknown. This unknown boundary value appears in the representation of the solution through the spectral functions and hence, it is required to characterize the unknown Neumann boundary value for the explicit solution. This can be done by analyzing the global relation as was done in [14]. It is noted that in [14] the Dirichlet-to-Neumann map for a nonlinear elliptic partial differential equation, namely the elliptic sine-Gordon equation, was reported for the first time. Moreover, we expect that it could be possible to study the effective characterization of the Dirichlet-to-Neumann map by employing the perturbative scheme as in [13, 15, 16, 17]. We will discuss regarding these issues in the future.

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