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A coincident point and common fixed point theorem for weakly compatible mappings in partial metric spaces

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Abstract

In this paper, we prove the existence of a coincident point and a common fixed point for two self mappings defined on a complete partial metric space X. We will consider generalized cyclic representation of the set X with respect to the two self maps defined on X and a contractive condition involving a generalized distance altering function. Our results generalizes several corresponding results in the existing literature.

Keywords: Coincidence point, fixed point, contraction, partial metric space.
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1. Introduction and Preliminaries

The partial metric spaces were introduced by Matthews in [22] as a part of the study of denotational semantics of dataflow networks. He introduced this notion to solve some difficulties of the domain theory and showed the Banach's contraction principle [7] can be generalized in context of partial metric spaces for applications in program verifications (see for example [13, 21, 24, 27, 31, 32, 33, 36].

Now, we recall definition and properties of partial metric spaces.

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Definition 1.1. ([22, 23]) A partial metric "p" on X is a function from $X \times X$ to R^+ such that for every element x, y and z of X it satisfies following axioms.
$p_1 : 0 \leq p(x,x) \leq p(x,y)$.
$p_2 : p(x,x) = p(x,y) = p(y,y)$ if and only if $x = y$.
$p_3 : p(x,y) = p(y,x)$. (symmetry)
$p_4 : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$. (triangular inequality)
If "p" is a partial metric on X then (X,p) is called a partial metric space.

Remark 1.2. It is clear that if $p(x,y) = 0$, then from p_1 and p_2 $x = y$. But if $x = y$ then $p(x,y)$ may not be zero.

Each partial metric "p" on X generates a T_0 topology τ_p on X which has the collection of all open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ as a base. Where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for each $\varepsilon > 0$ and $x \in X$. Notice that for a partial metric p on X, the function $d_p : X \times X \to R^+$ defined by $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x, y, z \in X$ is a metric on X.

Definition 1.3. ([4, 22, 23])

1. A sequence $\{x_n\}$ in a partial metric space (X,p) converges to the limit $x \in X$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.
2. A sequence $\{x_n\}$ in a partial metric space (X,p) is called Cauchy if and only if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists and is finite.
3. A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $\lim_{m,n\to\infty} p(x_m,x_n) = p(x,x)$.
4. The mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ If for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0,\delta)) \subset B_p(f(x_0),\varepsilon)$.

The following lemmas will be frequently used in the proofs of the main results.

Lemma 1.4. ([4, 22, 23, 30])

1. *A sequence $\{x_n\}$ is a Cauchy sequence in a partial metric space (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*
2. *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x,x_n) = 0$, if and only if*
$$p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$
Where x is the limit of $\{x_n\}$ in (X,d_p).
3. *Let (X,p) be a complete partial metric space. Then*
   (a) If $p(x,y) = 0$, then $x = y$.
   (b) If $x \neq y$, then $p(x,y) > 0$.
4. *Let (X,p) be a partial metric space. Assume that the sequence $\{x_n\}$ is converging to z as $n \to \infty$. such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for all elements y of X.*

Khan *et al.* [19] initiated the use of control function in the fixed point theory of metric spaces, which they called an altering distance function. Generalizations of distance altering function have been used in fixed point theory in metric and probabilistic metric spaces in works like [11, 12, 25, 26, 35]. In [12] Chaudhury and Dutta presented the following definition of the generalized distance altering function of two variables.

Definition 1.5. ([12]) A function $\psi : [0,\infty)^2 \to [0,\infty)$ is said to be generalized altering distance function of two variables if :

1. ψ is continuous,
2. ψ is monotone increasing in both variables, and
3. $\psi(x,y) = 0$ only if $x = y = 0$.

The class of all such functions are denoted by Ω . We define $\alpha(x) = \psi(x,x)$ for $x \in [0,\infty)$. Clearly $\alpha(x) = 0$ if and only if $x = 0$.

In 2003, Kirk *et al.* [20] introduced the notion of cyclic representation of a non empty set X and characterized the Banach's contraction principle in the context of cyclic mapping. After the distinguished result of Kirk *et al.* [20], a number of fixed point theorems have been proved, for details see [9, 16, 17, 28, 29, 34]. In 2005, Rus [34] introduced the following definition as the generalization of cyclic mapping defined in [20].

Definition 1.6. ([34])
Let X be a non-empty set, m be a positive integer and $F : X \to X$ be a mapping. $X = \bigcup_{i=1}^{m} A_i$ is said to be the cyclic representation of X with respect to F if

1. $A_i, i = 1, 2, ..., m$ are nonempty sets.
2. $F(A_1) \subset A_2, F(A_2) \subset A_3, ..., F(A_{m-1}) \subset A_m, F(A_m) \subset A_1$.

After this many authors focused on the fixed point theorems for metric spaces as well as on complete partial metric spaces with generalized cyclic representation, for details see [2, 3, 5, 6, 8, 10, 15, 18]. In [18] the following notion of generalized cyclic representation of non empty set with respect to two self maps is defined.

Definition 1.7. ([18])
Let X be a non-empty set, m be a positive integer and $S, T : X \to X$ be two mappings. $X = \bigcup_{i=1}^{m} A_i$ is said to be cyclic representation of X with respect to S and T if

1. $A_i, i = 1, 2, ..., m$ are nonempty sets,
2. $T(A_1) \subset S(A_2), T(A_2) \subset S(A_3), ..., T(A_{m-1}) \subset S(A_m), T(A_m) \subset S(A_1)$.

Definition 1.8. ([14])
Let T and S be two self-maps on X. If $Sw = Tw = z$, for some $w \in X$, then w is called a coincidence point of S and T, and z is called a point of coincidence of S and T. If $w = z$, then z is called the common fixed point of S and T.

Definition 1.9. ([14])
Consider the two self-maps S and T defined on a non-empty set X. If $STx = TSx$, for all $x \in X$, then S and T are said to be commuting maps. If they commute at their coincidence points only then they are said to be weakly compatible that is, if $STx = TSx$ whenever, $Tx = Sx$.

Lemma 1.10. *([1])*
Let S and T be weakly compatible self-maps on X. If w is the unique point of coincidence of S and T and $Sw = Tw = z$, then z will be unique common fixed point of S and T.

2. Main Result

Now, we state the main result of this paper which includes the generalizations given in Definitions 1.5 and 1.7. Lastly we discuss an example to illustrate the usability of the main result.

Theorem 2.1. *Let (X,p) be a complete partial metric space, m be a positive integer, $A_1, A_2, ..., A_m$ be non-empty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. Let $S,T : X \to X$ be two self-mappings such that $X = \bigcup_{i=1}^{m} A_i$ is the cyclic representation of X with respect to S and T, for any $x \in A_i, y \in A_{i+1}$ and $i \in \{1,2,...,m\}$,*

$$\alpha\left(p\left(Tx,Ty\right)\right) \leq \psi\left(p\left(Sx,Sy\right),p\left(Sx,Tx\right)\right) - \phi\left(p\left(Sx,Sy\right),p\left(Sx,Tx\right)\right) \tag{2.1}$$

is satisfied, where $A_{m+1} = A_1$, $\psi, \phi \in \Omega$ and $\alpha(x) = \psi(x,x)$ for $x \in [0,\infty)$. Suppose that $S(A_i)$, for all i are closed subsets of X. If S is one to one then there exists $z \in \bigcap_{i=1}^{m} A_i$ such that $Sz = Tz$. In particular if the pair $\{S,T\}$ is weakly compatible then they have a unique common fixed point.

Proof. Let $x_1 \in A_1$ be an arbitrary element then by the cyclic representation of X we can find an element $x_2 \in A_2$ such that $Tx_1 = Sx_2$. For $x_2 \in A_2$, we can find $x_3 \in A_3$ such that $Tx_2 = Sx_3$. Continuing in this way, we can construct a sequence $\{x_n\}$ as follows $Tx_n = Sx_{n+1}$, for all $n \in N$. If for some $k \in N$, we have $Sx_{k+1} = Sx_k$, then $Sx_{k+1} = Sx_k = Tx_k$, which shows that x_k is the point of coincidence of S and T. Now, suppose that $Sx_{n+1} \neq Sx_n$ for all $n \in N$. Then by definition of X there exists $i_n \in \{1,2,...,m\}$ such that $x_n \in A_{i_{n+1}}$ and $x_{n-1} \in A_{i_n}$. For $x = x_n$ and $y = x_{n-1}$ we have

$$\begin{aligned}\alpha\left(p\left(Tx_n,Tx_{n-1}\right)\right) \leq\ & \psi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_n,Tx_n\right)\right)\\ & -\phi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_n,Tx_n\right)\right)\end{aligned}$$

$$\begin{aligned}\alpha\left(p\left(Sx_{n+1},Sx_n\right)\right) &\leq \psi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_n,Sx_{n+1}\right)\right)\\ &\quad -\phi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_n,Sx_{n+1}\right)\right)\\ &\leq \psi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_n,Sx_{n+1}\right)\right).\end{aligned} \tag{2.2}$$

Since $\alpha(x) = \psi(x,x)$, so

$$\psi\left(p\left(Sx_{n+1},Sx_n\right),p\left(Sx_{n+1},Sx_n\right)\right) \leq \psi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_{n+1},Sx_n\right)\right).$$

If $p(Sx_n,Sx_{n-1}) < p(Sx_{n+1},Sx_n)$, then

$$\begin{aligned}\alpha\left(p\left(Sx_{n+1},Sx_n\right)\right) &\leq \psi\left(p\left(Sx_n,Sx_{n-1}\right),p\left(Sx_{n+1},Sx_n\right)\right)\\ &< \psi\left(p\left(Sx_{n+1},Sx_n\right),p\left(Sx_{n+1},Sx_n\right)\right) = \alpha\left(p\left(Sx_{n+1},Sx_n\right)\right),\end{aligned}$$

which is a contradiction, since ψ is monotone increasing in both variables and $\alpha(p(Sx_{n+1},Sx_n)) \neq 0$ whenever $p(Sx_{n+1},Sx_n) \neq 0$. Thus

$$p\left(Sx_{n+1},Sx_n\right) \leq p\left(Sx_n,Sx_{n-1}\right)$$

for all $n \in N$ and $\{p(Sx_{n+1},Sx_n)\}$ is a decreasing sequence of non-negative real numbers, so there must exists some $r \geq 0$, such that

$$p\left(Sx_{n+1},Sx_n\right) \to r \text{ as } n \to \infty \tag{2.3}$$

Taking limit as $n \to \infty$ in (2.2) then utilizing (2.3) and the definition of ψ and ϕ we have $\alpha(r) \leq \psi(r,r) - \phi(r,r) < \psi(r,r) = \alpha(r)$. Therefore, $\alpha(r) = 0$. Which forces $r = 0$. Hence

$$\lim_{n\to\infty} p\left(Sx_n,Sx_{n+1}\right) = 0. \tag{2.4}$$

As $p(Sx_n,Sx_n) \leq p(Sx_{n+1},Sx_n)$ and $p(Sx_{n+1},Sx_{n+1}) \leq p(Sx_{n+1},Sx_n)$ which implies

$$\lim_{n\to\infty} p\left(Sx_n,Sx_n\right) = \lim_{n\to\infty} p\left(Sx_{n+1},Sx_{n+1}\right) = \lim_{n\to\infty} p\left(Sx_{n+1},Sx_n\right) = 0. \tag{2.5}$$

Also by using the definition of d_p we have $d_p(Sx_{n+1},Sx_n) = 2p(Sx_{n+1},Sx_n) - p(Sx_{n+1},Sx_{n+1}) - p(Sx_n,Sx_n)$, and

$$\lim_{n\to\infty} d_p\left(Sx_{n+1},Sx_n\right) = 0. \tag{2.6}$$

To show that $\{Sx_n\}$ is a Cauchy sequence in (X, d_p). Assume that $\{Sx_n\}$ is not Cauchy sequence then there exists some $\varepsilon > 0$ for which we can find the subsequences $\{Sx_{m(k)}\}$ and $\{Sx_{n(k)}\}$ of $\{Sx_n\}$ with $n(k) > m(k) > k$ such that

$$d_p\left(Sx_{m(k)}, Sx_{n(k)}\right) \geq \varepsilon. \tag{2.7}$$

Further, we can choose $n(k)$ corresponding to $m(k)$, in such a way that it is the smallest integer satisfying inequality (2.7), hence

$$d_p\left(Sx_{m(k)}, Sx_{n(k)-1}\right) < \varepsilon. \tag{2.8}$$

From inequality (2.7),

$$\begin{aligned}\varepsilon \leq d_p\left(Sx_{m(k)}, Sx_{n(k)}\right) &\leq d_p\left(Sx_{m(k)}, Sx_{n(k)-1}\right) + d_p\left(Sx_{n(k)-1}, Sx_{n(k)}\right) \\ &< \varepsilon + d_p\left(Sx_{n(k)-1}, Sx_{n(k)}\right)\end{aligned}$$

Hence,

$$\varepsilon \leq d_p\left(Sx_{m(k)}, Sx_{n(k)}\right) < \varepsilon + d_p\left(Sx_{n(k)-1}, Sx_{n(k)}\right). \tag{2.9}$$

We know that,

$$\begin{aligned}d_p\left(Sx_{n(k)-1}, Sx_{n(k)}\right) = 2p\left(Sx_{n(k)-1}, Sx_{n(k)}\right) &- p\left(Sx_{n(k)}, Sx_{n(k)}\right) \\ &-p\left(Sx_{n(k)-1}, Sx_{n(k)-1}\right).\end{aligned}$$

Let $k \to \infty$, using (2.4) and (2.5) we get

$$\lim_{k\to\infty} d_p\left(Sx_{n(k)-1}, Sx_{n(k)}\right) = 0. \tag{2.10}$$

Using (2.10) in (2.9) we have

$$\lim_{k\to\infty} d_p\left(Sx_{m(k)}, Sx_{n(k)}\right) = \varepsilon. \tag{2.11}$$

As

$$\begin{aligned}d_p\left(Sx_{m(k)}, Sx_{n(k)}\right) = 2p\left(Sx_{m(k)}, Sx_{n(k)}\right) &- p\left(Sx_{m(k)}, Sx_{m(k)}\right) \\ &-p\left(Sx_{n(k)}, Sx_{n(k)}\right)\end{aligned}$$

Let $k \to \infty$, using (2.5) and (2.11) we get

$$\lim_{k\to\infty} d_p\left(Sx_{m(k)}, Sx_{n(k)}\right) = 2\lim_{k\to\infty} p\left(Sx_{m(k)}, Sx_{n(k)}\right).$$

Therefore

$$\lim_{k\to\infty} p\left(Sx_{m(k)}, Sx_{n(k)}\right) = \frac{\varepsilon}{2}. \tag{2.12}$$

From the triangular inequality

$$\begin{aligned}d_p\left(Sx_{n(k)}, Sx_{m(k)}\right) \leq d_p\left(Sx_{n(k)}, Sx_{n(k)+1}\right) &+ d_p\left(Sx_{n(k)+1}, Sx_{m(k)+1}\right) \\ &+d_p\left(Sx_{m(k)+1}, Sx_{m(k)}\right)\end{aligned}$$

and

$$\begin{aligned}d_p\left(Sx_{n(k)+1}, Sx_{m(k)+1}\right) \leq d_p\left(Sx_{n(k)+1}, Sx_{n(k)}\right) &+ d_p\left(Sx_{n(k)}, Sx_{m(k)}\right) \\ &+ d_p\left(Sx_{m(k)}, Sx_{m(k)+1}\right)\end{aligned}$$

Let $k \to \infty$, and using (2.10) and (2.11) we get

$$\lim_{k\to\infty} d_p\left(Sx_{n(k)}, Sx_{m(k)}\right) \leq \lim_{k\to\infty} d_p\left(Sx_{n(k)+1}, Sx_{m(k)+1}\right)$$

and

$$\lim_{k\to\infty} d_p\left(Sx_{n(k)+1}, Sx_{m(k)+1}\right) \leq \lim_{k\to\infty} d_p\left(Sx_{n(k)}, Sx_{m(k)}\right)$$

Hence,
$$\lim_{k\to\infty} d_p\left(Sx_{n(k)+1}, Sx_{m(k)+1}\right) = \lim_{k\to\infty} d_p\left(Sx_{n(k)}, Sx_{m(k)}\right) = \varepsilon. \tag{2.13}$$

By definition of d_p,
$$\begin{aligned} d_p\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right) = 2\,p\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right) - p\left(Sx_{m(k)+1}, Sx_{m(k)+1}\right)\\ -p\left(Sx_{n(k)+1}, Sx_{n(k)+1}\right)\end{aligned}$$

Let $k \to \infty$, and using (2.5) we get
$$\lim_{k\to\infty} d_p\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right) = 2\lim_{k\to\infty} p\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right) = \varepsilon.$$

Hence, we obtain
$$\lim_{k\to\infty} p\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right) = \frac{\varepsilon}{2}. \tag{2.14}$$

By substituting $x = x_{m(k)}$ and $y = y_{n(k)}$ in (2.1) we get

$$\begin{aligned}\alpha\left(p\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \le \psi\left(p\left(Sx_{m(k)}, Sx_{n(k)}\right), p\left(Sx_{m(k)}, Tx_{m(k)}\right)\right)\\ -\phi\left(p\left(Sx_{m(k)}, Sx_{n(k)}\right), p\left(Sx_{m(k)}, Tx_{m(k)}\right)\right)\end{aligned}$$

$$\begin{aligned}\alpha\left(p\left(Sx_{m(k)+1}, Sx_{n(k)+1}\right)\right) \le \psi\left(p\left(Sx_{m(k)}, Sx_{n(k)}\right), p\left(Sx_{m(k)}, Sx_{m(k)+1}\right)\right)\\ -\phi\left(p\left(Sx_{m(k)}, Sx_{n(k)}\right), p\left(Sx_{m(k)}, Sx_{m(k)+1}\right)\right)\end{aligned}$$

Taking limit as $k \to \infty$, using (2.4), (2.12), (2.14) and the continuity of ϕ and ψ we obtain
$$\alpha\left(\frac{\varepsilon}{2}\right) \le \psi\left(\frac{\varepsilon}{2}, 0\right) - \phi\left(\frac{\varepsilon}{2}, 0\right) < \psi\left(\frac{\varepsilon}{2}, 0\right) < \psi\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) = \alpha\left(\frac{\varepsilon}{2}\right),$$

since ψ is monotone increasing in both variables and $\phi\left(\frac{\varepsilon}{2}, 0\right) > 0$, so the above inequality gives a contradiction that is $\varepsilon = 0$. Hence $\{Sx_n\}$ is a Cauchy sequence in (X, d_p). As (X, p) is complete so is (X, d_p) and there exists some $z \in X$ such that
$$\lim_{n\to\infty} d_p\left(Sx_n, z\right) = 0. \tag{2.15}$$

Also by using Lemma 1.4(2), we have
$$\begin{aligned} p(z,z) &= \lim_{n\to\infty} p\left(Sx_n, z\right) = \lim_{n\to\infty} p\left(Sx_n, Sx_m\right)\\ &= \frac{1}{2}\lim_{m,n\to\infty} d_p\left(Sx_n, Sx_m\right) = 0. \end{aligned}\tag{2.16}$$

Which shows that $Sx_n \to z$ as $n \to \infty$ in the partial metric space (X, p). Since all $S(A_i)$'s are closed in X, so $z \in S(A_i)$ for all i. Thus $z \in \bigcap_{i=1}^{m} S(A_i)$ and there exists $z_i \in A_i$ such that $Sz_i = z$. Also S is given as a one-one mapping so we have $Sz_1 = Sz_2 = ... = Sz_m = z$, which implies $z_1 = z_2 = ... = z_m = z'$, therefore $Sz' = z$ for $z' \in \bigcap_{i=1}^{m} A_i$ and $\lim_{n\to\infty} Sx_n = z = Sz'$. By construction, the sequence $\{Sx_n\}$ has infinite number of terms in each A_i. Now fix $i \in \{1, 2, ..., m\}$ such that $z \in A_i$ and $Tz \in A_{i+1}$. We may take a subsequence $\{Sx_{n(k)}\}$ of $\{Sx_n\}$ with $Sx_{n(k)} \in SA_{i-1}$ where $x_{n(k)} \in A_{i-1}$ and also converges to z. Using (2.16) we have
$$p(z,z) = \lim_{n\to\infty} p\left(Sx_n, z\right) = \lim_{n\to\infty} p\left(Sx_{n(k)}, z\right) = 0. \tag{2.17}$$

put $y = x_{n(k)}$ and $x = z'$ in (2.1) we have
$$\begin{aligned}\alpha\left(p\left(Tz', Tx_{n(k)}\right)\right) = \alpha\left(p\left(Tz', Sx_{n(k)+1}\right)\right)\\ \le \psi\left(p\left(Sz', Sx_{n(k)}\right), p\left(Sz', Tz'\right)\right) - \phi\left(p\left(Sz', Sx_{n(k)}\right), p\left(Sz', Tz'\right)\right)\end{aligned}$$

Apply limit as $n \to \infty$, using (2.4) and the property of ϕ and ψ we have

$$\begin{aligned} \psi\left(p\left(Tz', Sz'\right), p\left(Tz', Sz'\right)\right) &= \alpha\left(p\left(Tz', Sz'\right)\right) \\ \leq \psi\left(p\left(Sz', Sz'\right), p\left(Sz', Tz'\right)\right) &- \phi\left(p\left(Sz', Sz'\right), p\left(Sz', Tz'\right)\right) \\ &\leq \psi\left(p\left(Sz', Sz'\right), p\left(Sz', Tz'\right)\right). \end{aligned} \tag{2.18}$$

Since ψ is monotone increasing so we get $p\left(Tz', Sz'\right) \leq p\left(Sz', Sz'\right)$, and by definition of small self-distance $p\left(Sz', Sz'\right) \leq p\left(Tz', Sz'\right)$, hence

$$p\left(Sz', Sz'\right) = \left(Tz', Sz'\right).$$

If $p\left(Sz', Sz'\right) \neq 0$, then $p\left(Sz', Sz'\right) > 0$ and, from(2.18) we get

$$\begin{aligned} \psi\left(p\left(Sz', Sz'\right), p\left(Sz', Sz'\right)\right) = \alpha\left(p\left(Sz', Sz'\right)\right) &\leq \psi\left(p\left(Sz', Sz'\right), p\left(Sz', Sz'\right)\right) \\ &-\phi\left(p\left(Sz', Sz'\right), p\left(Sz', Tz'\right)\right) \\ &\leq \psi\left(p\left(Sz', Sz'\right), p\left(Sz', Sz'\right)\right). \end{aligned} \tag{2.19}$$

A contradiction, since $\psi \in \Omega$. Thus $p\left(Sz', Sz'\right) = p\left(Tz', Sz'\right) = p\left(Tz', Tz'\right) = 0$ and $Tz' = Sz' = z$. Since S and T are weakly compatible so $TTz' = TSz' = STz' = SSz'$, that is $Tz = Sz$. Now, we prove that $Tz = z$. Since $Tz' \in X$ hence $Tz' \in A_i$ for some $i \in \{1, 2, ..., m\}$. By $z' \in \bigcap_{i=1}^m A_i$, we have $z' \in A_{i-1}$. Put $x = z'$ and $y = Tz'$ in (2.1),

$$\begin{aligned} \alpha\left(p\left(Tz', TTz'\right)\right) &\leq \psi\left(p\left(Sz', STz'\right), p\left(Sz', Tz'\right)\right) \\ &-\phi\left(p\left(Sz', STz'\right), p\left(Sz', Tz'\right)\right) \\ &\leq \psi\left(p\left(Sz', STz'\right), p\left(Sz', Tz'\right)\right). \end{aligned}$$

Since $Sz' = Tz'$ so from the last inequality we have

$$\begin{aligned} \alpha\left(p\left(Tz', TTz'\right)\right) &\leq \psi\left(p\left(Tz', TTz'\right), p\left(Tz', Tz'\right)\right) \\ &-\phi\left(p\left(Tz', TTz'\right), p\left(Tz', Tz'\right)\right) \\ &\leq \psi\left(p\left(Tz', TTz'\right), p\left(Tz', Tz'\right)\right) \\ &\leq \psi\left(p\left(Tz', TTz'\right), p\left(Tz', TTz'\right)\right). \end{aligned}$$

Since $\psi \in \Omega$ and $p\left(Tz', Tz'\right) \leq p\left(Tz', TTz'\right)$. Thus we have $p\left(Tz', TTz'\right) = 0$ and, consequently $Tz = Tz' = TTz' = Tz = Sz$. Let $z^* \in X$ be another common fixed point of S and T such that $z \neq z^*$ then $p\left(z, z^*\right) \neq 0$. Since both z and z^* are common fixed point of S and T so by given conditions $z, z' \in \bigcap_{i=1}^m A_i$. Using (2.1) we have

$$\begin{aligned} \alpha\left(p\left(z, z^*\right)\right) = \alpha\left(p\left(Tz, Tz^*\right)\right) &\leq \psi\left(p\left(Sz, Sz^*\right), p\left(Sz, Tz\right)\right) \\ &-\phi\left(p\left(Sz, Sz^*\right), p\left(Sz, Tz\right)\right) \\ &< \psi\left(p\left(Sz, Sz^*\right), p\left(Sz, Tz\right)\right) = \psi\left(p\left(z, z^*\right), p\left(z, z\right)\right). \end{aligned}$$

Since $\psi \in \Omega$ and $p\left(z, z\right) \leq p\left(z, z^*\right)$. Which forces $p\left(z, z^*\right) = 0$, that is $z = z^*$. □

Corollary 2.2. *Let (X, p) be a complete partial metric space, m be a positive integer, $A_1, A_2, ..., A_m$ be non-empty closed subsets of X, $X = \bigcup_{i=1}^m A_i$ be the cyclic representation of X with respect to the self map T defined on X. Suppose that there exists functions $\psi, \phi \in \Omega$, such that*

$$\alpha\left(p\left(Tx, Ty\right)\right) \leq \psi\left(p\left(x, y\right), p\left(x, Tx\right)\right) - \phi\left(p\left(x, y\right), p\left(x, Tx\right)\right)$$

is satisfied for any $x \in A_i, y \in A_{i+1}$ $i \in \{1, 2, ..., m\}$, where $A_{m+1} = A_1$, and for $x \in [0, \infty)$, $\alpha\left(x\right) = \psi\left(x, x\right)$. Then T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

Proof. Take $Sx = x$ in Theorem 2.1. □

Example 2.3. Let $X = [0,1]$ and define the function $p : X \times X \to R^+$ by $p(x,y) = \max\{x,y\}$. Then (X,p) is a complete partial metric space. Let $S, T : X \to X$ are such that $Tx = \frac{x^2}{16}$ and $Sx = \frac{x}{4}$ for all $x \in X$. Suppose that $\phi, \psi : [0,\infty)^2 \to [0,\infty)$ are defined by $\psi(x,y) = x + y$ and $\phi(x,y) = \max\{x,y\}$ for all $x, y \in [0,\infty)$. Let $A_i = [0,1]$ for $i = 1,2,...,m$. All the conditions of Theorem 2.1 are satisfied and we obtain $0 \in \bigcap_{i=1}^m A_i$ as the common fixed point of S and T.

3. Conclusion

A common fixed point theorem for a complete partial metric space has been proved by utilizing the idea of generalized cyclic representation of a non-empty set and the generalized distance altering function of two variables. An illustrative example is also given.

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