Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 529-539
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some generalized fixed point theorems in the context of ordered metric spaces

Mehmet Kir*, Hukmi Kiziltunc

Department of Mathematics, Faculty of Science, Ataturk University, 25240, Erzurum, Turkey.

Abstract

In this paper, we give three main theorems which are new generalizations of Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem in the context of the ordered metric space.

Keywords: Fixed point, Chatterjea fixed point theorem, Kannan fixed point theorem, contraction mappings
2010 MSC: 41A65, 41A15, 47H09, 47H10, 54H25.

---

1. Introduction and Preliminaries

A mapping $T : X \to X$ where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y).$$

Banach proved that a contraction mapping has a unique fixed point in a complete metric space (X, d) [1] (see also [2, 5, 6, 9]).

In [11], Ran and Reurings established the Banach fixed point theorem in the context of ordered metric spaces.

Theorem 1.1 ([11])**.** *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Furthermore, every pair $x, y \in X$ has a lower bound and an upper bound. If $f : X \to X$ is a continuous, monotone (i.e., either order-preserving or order-reversing) map from X into X such that*

$$\exists\, 0 < c < 1 : d(fx, fy) \leq cd(x, y), x \geq y$$

*Corresponding author
Email addresses: mehmetkir04@gmail.com (Mehmet Kir), hukmu@atauni.edu.tr (Hukmi Kiziltunc)

Received 2015-1-7

and

$$\exists x_0 \in X : x_0 \leq f x_0 \ or \ x_0 \geq f x_0,$$

then f has a unique fixed point $\overline{x}$. Morever, for every $x \in X$ $\lim_{n\to\infty} f^n x = \overline{x}$.

Nieto and López [10] gave an alternative condition for the continuity of the mapping f as following;

"if any nondecreasing sequence $\{x_n\}$ in Xconverges to z then $x_n \preceq z$ for all $n \geq 0$"

Also, to guarantee the uniqueness of the fixed point, Nieto and López [10] gave an alternative condition met the requirement of " every pair $x, y \in X$ has a lower bound and an upper bound" as following;

$$\text{"for every } x, y \in X; \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y \text{ ".} \tag{1.1}$$

Theorem 1.2 ([10])**.** *Let $(X, \preceq)$ be an ordered set endowed with a metric d and the mapping $f : X \to X$ be given. Suppose that the following conditions hold:*
i) (X, d) is complete;
ii) if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$;
iii) f is nondecreasing;
iv) there exists $x_0 \in X$ such that $x_0 \preceq T x_0$;
v) there exists a constant $k \in (0, 1)$ such that for all $x, y \in X$ with $x \succeq y$,

$$d(fx, fy) \leq kd(x, y).$$

Then f has a fixed point. Moreover, if for all $(x, y) \in X \times X$ there exists a $z \in X$ such that $x \preceq z$ and $y \preceq z$, then the fixed point is unique.

Two of the well known fixed point theorems are Kannan fixed point theorem and Chatterjea fixed point theorem.

Theorem 1.3 ([4])**.** *If a mapping $T : X \to X$ where (X, d) is a complete metric space, satisfies the inequality*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

where $a \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point. The mappings satisfying (1.2) *are called Kannan type mappings.*

Theorem 1.4 ([3])**.** *If a mapping $T : X \to X$ where (X, d) is a complete metric space, satisfies the inequality*

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)]$$

such that $b \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point.

Also, in 2011, Moradi and Davood introduced a new extension of Kannan fixed point theorem on complete metric space as following;

Theorem 1.5 ([8])**.** *Let (X, d) be a complete metric space and $T, S : X \to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\mu \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$,*

$$d(TSx, TSy) \leq \mu[d(Tx, TSy) + d(Ty, TSx)],$$

then, S has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to the fixed point.

Definition 1.6 ([8])**.** Let (X,d) be a metric space.

i) A mapping $T:X\to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ is also convergence.

ii) T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

In 2014, Mustafa et. al., [7] introduced fixed point theorems for weakly T-Chatterjea and weakly T-Kannan-contractive mappings in complete $b-$ metric spaces.

In this paper we have three main theorems which are new generalization of Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem in the context of the ordered metric space.

2. Main Results

For the simplicity in writing, we will use the following symbols. We donote by Ψ the set of all functions $F:[0,\infty)\to[0,\infty)$ satisfying:
F1) F is continuous and monotone nondecreasing,
F2) $F(t)=0$ if and only if $t=0$.

Also, we denote by $SSC(X)$ the set of all mappings $T:X\to X$ such that T is one to one, continuous, subsequentially convergent and preserve the order, by $SC(X)$ the set of all mappings $T:X\to X$ such that T is one to one, continuous, sequentially convergent and preserve the order.

Theorem 2.1. *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $f:X\to X$ be a monotone nondecreasing mapping and $T\in SSC(X)$, $F\in\Psi$. For all $x,y\in X$ with $x\preceq y$, $\alpha\in[0,1)$,*

$$F(d(Tfx,Tfy))\leq \alpha F(d(Tx,Ty)).$$

Also, suppose that either

C1) f is continuous or C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n\preceq z$ for all $n\geq 0$.

If there exists $x_0\in X$ with $x_0\preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x,y\in X$ there exists $z\in X$ which is comparable to x and y, then the fixed point is unique.

Proof. Let $x_0\in X$ be an arbitrary point such that $x_n=fx_{n-1}=f^nx_0$, $n=1,2,3,\cdots$. As f is nondecreasing and $x_0\preceq fx_0$, we have

$$Tx_0\preceq Tfx_0\preceq Tf^2x_0\preceq Tf^3x_0\preceq\cdots\preceq Tf^nx_0\preceq\cdots \tag{2.1}$$

Since $Tx_n\preceq Tx_{n+1}$,

$$\begin{aligned} F(d(Tx_n,Tx_{n+1})) &= F(d(Tfx_{n-1},Tfx_n))\\ &\leq \alpha F(d(Tx_{n-1},Tx_n))\\ &\vdots\\ &\leq \alpha^n F(d(Tx_0,Tx_1)). \end{aligned} \tag{2.2}$$

Letting $n\to\infty$ in (2.2), then we have

$$F(d(Tx_n,Tx_{n+1}))\to 0^+, \text{ as} n\to\infty.$$

$$d(Tx_n,Tx_{n+1})\to 0, \text{ as } n\to\infty.$$

Also, for $m,n\in\mathbb{N}, m>n$

$$F(d(Tx_n,Tx_m))\leq \alpha^n F(d(Tx_0,Tx_{m-n})) \tag{2.3}$$

letting $m, n \to \infty$ (2.3), we get

$$d\left(Tx_n, Tx_m\right) \to 0.$$

Thus, we obtain that $\{Tx_n\}$ is Cauchy sequence. As (X, d) is complete, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v. \tag{2.4}$$

Note that $T \in SSC\left(X\right)$, then $\{x_n\}$ has a convergent subsequence, so there is $u \in X$such that

$$\lim_{k\to\infty} x_{n(k)} = u. \tag{2.5}$$

Also, T is continuous and $x_{n(k)} \to u$, therefore

$$\lim_{k\to\infty} Tx_{n(k)} = Tu. \tag{2.6}$$

and

$$\lim_{k\to\infty} d\left(Tx_{n(k)}, Tu\right) = 0.$$

Now, we will show that $u \in X$ is a fixed point of f. In here we have two cases.

Case 1: Let C1) holds. From the continuity of f, we have

$$Tu = \lim_{k\to\infty} Tx_{n(k)} = \lim_{k\to\infty} Tfx_{n(k)-1} = Tfu.$$

since, T is one to one we get $fu = u$, namely $u \in X$ is a fixed point of f.

Case 2: Let C2) holds. Since $\left\{Tx_{n(k)}\right\}$ converges to $Tu \in X$, for all $\frac{\epsilon}{2} > 0$ there is $N_1 \in \mathbb{N}$ such that for all $n\left(k\right) > N_1$, we have

$$d\left(Tx_{n(k)}, Tu\right) < \frac{\epsilon}{2}.$$

Also, as $\left\{Tx_{n(k)}\right\}$ converges to Tu. From C2), we get $Tx_{n(k)} \preceq Tu$ and we have

$$F\left(d\left(Tf^{n(k)+1}x, Tfu\right)\right) \leq \alpha F\left(d\left(Tf^{n(k)}x, Tu\right)\right). \tag{2.7}$$

Letting $k \to \infty$ in (2.7), we have

$$F\left(d\left(Tu, Tfu\right)\right) \leq 0 \tag{2.8}$$

The inequality (2.8) implies that $Tu = Tfu$. As, $T$is one to one we get $u \in X$ is a fixed point of f.

Adding the condition (1.1), we show the uniqueness of the fixed point. We will do this by showing that

$$\lim_{k\to\infty} f^{n(k)}x = u$$

for every $x \in X$. Here we have two cases.

Firstly, let x and x_0 be comparable, then $x \preceq x_0$ or $x_0 \preceq x$. In both cases, we have $f^{n(k)}x \preceq f^{n(k)}x_0$ or $f^{n(k)}x_0 \preceq f^{n(k)}x$. Hence, we have

$$F\left(d\left(Tf^{n(k)}x, Tf^{n(k)}x_0\right)\right) \leq \alpha^{n(k)} F\left(d\left(Tx, Tx_0\right)\right)$$

letting $k \to \infty$ in the last inequality, we have

$$\lim_{k\to\infty} f^{n(k)}x = \lim_{k\to\infty} f^{n(k)}x_0 = u.$$

Secondly, let x and x_0 are not comparable and let x_1, resp. x_2, be an upper bound, resp. a lower bound, of x and x_0.That is $x_1 \succeq x \succeq x_2$ and $x_1 \succeq x_0 \succeq x_2$. Thus we have

$$\lim_{k\to\infty} f^{n(k)}x = \lim_{k\to\infty} f^{n(k)}x_0 = u.$$

Hence, the proof is completed. □

Remark 2.2. The case of $T \in SC(X)$, the proof of the Theorem 2.1 takes place analogously by replacing $\{n\}$ with $\{n(k)\}$.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$. For all $x, y \in X$ with $x \preceq y$, $\alpha \in [0, 1)$,*

$$d(Tfx, Tfy) \leq \alpha d(Tx, Ty).$$

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $F \in \Psi$. For all $x, y \in X$ with $x \preceq y$, $\alpha \in [0, 1)$,*

$$F(d(fx, fy)) \leq \alpha F(d(x, y)).$$

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be mappings such that f is monotone nondecreasing. For all $x, y \in X$ with $x \preceq y$, $\alpha \in [0, 1)$,*

$$d(fx, fy) \leq \alpha d(x, y).$$

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$. For all $x, y \in X$ with $x \preceq y$, $\alpha \in [0, 1)$,*

$$\int_0^{d(Tfx, Tfy)} \varphi(t)\, dt \leq \alpha \int_0^{d(Tx, Ty)} \varphi(t)\, dt, \tag{2.9}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)\, dt > 0$.

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.7. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be mappings such that f is monotone nondecreasing. For all $x, y \in X$ with $x \preceq y$, $\alpha \in [0, 1)$,*

$$\int_0^{d(fx, fy)} \varphi(t)\, dt \leq \alpha \int_0^{d(x, y)} \varphi(t)\, dt, \tag{2.10}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)\, dt > 0$.

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Theorem 2.8. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$, $F \in \Psi$. For all $x, y \in X$ with $x \preceq y$, $\beta \in \left[0, \frac{1}{2}\right)$,*

$$F(d(Tfx, Tfy)) \leq \beta [F(d(Tx, Tfx)) + F(d(Ty, Tfy))].$$

Also, suppose that

C1) f is continuous or C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Proof. Let $x_0 \in X$ be an arbitrary point such that $x_n = fx_{n-1} = f^n x_0$, $n = 1, 2, 3, \ldots$. As f is nondecreasing and $x_0 \preceq fx_0$, we have

$$Tx_0 \preceq Tfx_0 \preceq Tf^2 x_0 \preceq Tf^3 x_0 \preceq \cdots \preceq Tf^n x_0 \preceq \cdots \tag{2.11}$$

Since $Tx_n \preceq Tx_{n+1}$

$$\begin{aligned} F(d(Tx_n, Tx_{n+1})) &= F(d(Tfx_{n-1}, Tfx_n)) \\ &\leq \beta [F(d(Tx_{n-1}, Tx_n)) + F(d(Tx_n, Tx_{n+1}))]. \end{aligned} \tag{2.12}$$

From, (2.12), we get

$$F(d(Tx_n, Tx_{n+1})) \leq \frac{\beta}{1-\beta} F(d(Tx_{n-1}, Tx_n)). \tag{2.13}$$

Also, by continuing the process (2.13), we obtain that

$$F(d(Tx_n, Tx_{n+1})) \leq \left(\frac{\beta}{1-\beta}\right)^n F(d(Tx_0, Tx_1)). \tag{2.14}$$

Letting $n \to \infty$ in (2.14), we obtain that

$$F(d(Tx_n, Tx_{n+1})) \to 0^+ \text{ as } n \to \infty.$$

Again using (2.14), for all $m, n \in \mathbb{N}$, taking $m > n$, we have

$$\begin{aligned} F(d(Tx_n, Tx_m)) &= F(d(Tf^n x_0, Tf^m x_0)) \\ &\leq \left(\frac{\beta}{1-\beta}\right)^n F\left(d\left(Tx_0, Tf^{m-n} x_0\right)\right). \end{aligned} \tag{2.15}$$

Letting $m, n \to \infty$ in (2.15) ,we have

$$F(d(Tx_n, Tx_m)) \to 0^+ \text{as } m, n \to \infty.$$

So, we have $d(Tx_n, Tx_m) \to 0$ as, $m, n \to \infty$.

In the next stage, by using similar methods in Theorem 2.1, we obtain that $\{Tx_n\}$ is Cauchy sequence in complete metric space (X, d) and there exist $u \in X$ such that $\{Tx_n\}$ converges to $Tu \in X$. Now, we will show that $u \in X$ is a fixed point of f. In here we have two cases.

Case 1: Let C1) holds. From the continuity of f, we have

$$Tu = \lim_{k \to \infty} Tx_{n(k)} = \lim_{k \to \infty} Tfx_{n(k)-1} = Tfu.$$

Since, T is one to one we get that $u \in X$ is a fixed point of f.

Case 2: Let C2) holds. Since $\{Tx_n\}$ converges to $Tu \in X$, for all $\frac{\epsilon}{2} > 0$ there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, we have

$$d(Tx_n, Tu) < \frac{\epsilon}{2}.$$

Also, as $\{Tx_n\}$ converges to Tu, From C2) we get $Tx_n \preceq Tu$ and

$$\begin{aligned} F(d(Tu, Tfu)) &\leq F(d(Tu, Tx_n) + d(Tx_n, Tfu)) \\ &= F(d(Tu, Tx_n) + d(Tf^n x_0, Tfu)) \\ &\leq F(d(Tu, Tx_n) + d(Tf^n x_0, Tf^n x_1) + d(Tf^n x_1, Tfu)) \\ &= F(d(Tu, Tx_n) + d(Tfx_n, Tfx_{n+1}) + d(Tfx_n, Tfu)). \end{aligned} \tag{2.16}$$

Letting $n \to \infty$ in (2.7), we have

$$F(d(Tu, Tfu)) \leq 0 \tag{2.17}$$

The inequality (2.17) implies that $Tu = Tfu$. As, T is one to one we get $u \in X$ is a fixed point of f. Adding the condition (1.1), we show the uniqueness of the fixed point.

Let $u' \in X$ be another fixed point of f. From the (1.1), there exists an element z in X such that z comparable to u and u'.The monotonicity of f implies that $f^n z$ comparable to $f^n u = u$ and $f^n u' = u'$ for all $n \in \mathbb{N}$. As, $T \in SSC(X)$, then $Tf^n z$ is comparable to Tu and Tu'. Also, $F \in \Psi$ we have

$$\begin{aligned} F\left(d\left(Tu, Tu'\right)\right) &\leq F\left(d(Tu, Tf^n z) + d\left(Tf^n z, Tu'\right)\right) \\ &= F(d(Tf^n u, Tf^n z)) + F\left(d\left(Tf^n z, Tf^n u'\right)\right) \\ &\leq \beta^n F(d(u, z)) + \beta^n F\left(d\left(z, u'\right)\right). \end{aligned} \tag{2.18}$$

Letting $n \to \infty$ in the inequality (2.18),

$$F\left(d\left(Tu, Tu'\right)\right) \leq 0.$$

The last inequality implies $Tu = Tu'$. As T is one to one we get $u = u'$. Hence, the proof is completed. □

Remark 2.9. The case of $T \in SC(X)$, the proof of the Theorem 2.8 takes place analogously by replacing $\{n\}$ with $\{n(k)\}$.

Corollary 2.10. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$. For all $x, y \in X$ with $x \preceq y$, $\beta \in \left[0, \frac{1}{2}\right)$,*

$$d(Tfx, Tfy) \leq \beta\left[d(Tx, Tfx) + d(Ty, Tfy)\right].$$

Also, suppose that

C1) f is continuous or C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.11. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $F \in \Psi$. For all $x, y \in X$ with $x \preceq y$, $\beta \in \left[0, \frac{1}{2}\right)$,*

$$F(d(fx, fy)) \leq \beta\left[F(d(x, fx)) + F(d(y, fy))\right].$$

Also, suppose that

C1) f is continuous or

C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.12. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping. For all $x, y \in X$ with $x \preceq y$, $\beta \in \left[0, \frac{1}{2}\right)$,*

$$d(fx, fy) \leq \beta \left[d(x, fx) + d(y, fy)\right].$$

Also, suppose that

C1) f is continuous or

C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Remark 2.13. The Corollary 2.12 is Kannan fixed point theorem in the context of ordered metric space.

Corollary 2.14. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$. For all $x, y \in X$ with $x \preceq y$, $\beta \in \left[0, \frac{1}{2}\right)$,*

$$\int_0^{d(Tfx, Tfy)} \varphi(t)\, dt \leq \beta \int_0^{d(Tx, Tfx) + d(Ty, Tfy)} \varphi(t)\, dt, \tag{2.19}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\, dt > 0$.

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.15. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping. For all $x, y \in X$ with $x \preceq y$, $\beta \in \left[0, \frac{1}{2}\right)$,*

$$\int_0^{d(fx, fy)} \varphi(t)\, dt \leq \beta \int_0^{d(x, fx) + d(y, fy)} \varphi(t)\, dt, \tag{2.20}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\, dt > 0$.

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Remark 2.16. Also, by taking $\varphi(t) = 1$ in Corollary 2.15, we obtain Kannan fixed point theorem in the context of the ordered metric space.

Theorem 2.17. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X), F \in \Psi$. For all $x, y \in X$ with $x \preceq y$, $\mu \in \left[0, \frac{1}{2}\right)$,*

$$F(d(Tfx, Tfy)) \leq \mu \left[F(d(Tx, Tfy)) + F(d(Ty, Tfx))\right].$$

Also, suppose that

C1) f is continuous or C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Proof. Let $x_0 \in X$ be an arbitrary point such that $x_n = fx_{n-1} = f^n x_0$, $n = 1, 2, 3, ...$ As f is nondecreasing and $x_0 \preceq fx_0$, we have

$$Tx_0 \preceq Tfx_0 \preceq Tf^2x_0 \preceq Tf^3x_0 \preceq \cdots \preceq Tf^nx_0 \preceq \cdots \tag{2.21}$$

Considering $F \in \Psi$ and $Tx_n \preceq Tx_{n+1}$

$$\begin{aligned} F\left(d\left(Tx_n, Tx_{n+1}\right)\right) &= F\left(d\left(Tfx_{n-1}, Tfx_n\right)\right) \\ &\leq \mu\left[F\left(d\left(Tx_{n-1}, Tx_{n+1}\right)\right) + F\left(d\left(Tx_n, Tx_n\right)\right)\right] \\ &\leq \mu\left[F\left(d\left(Tx_{n-1}, Tx_n\right)\right) + F\left(d\left(Tx_n, Tx_{n+1}\right)\right)\right]. \end{aligned} \tag{2.22}$$

From inequality (2.22), we have

$$F\left(d\left(Tx_n, Tx_{n+1}\right)\right) \leq \frac{\mu}{1-\mu} F\left(d\left(Tx_{n-1}, Tx_n\right)\right),$$

and we get

$$F\left(d\left(Tx_n, Tx_{n+1}\right)\right) \leq \left(\frac{\mu}{1-\mu}\right)^n F\left(d\left(Tx_0, Tx_1\right)\right). \tag{2.23}$$

Letting $n \to \infty$ in (2.23), we obtain that

$$F\left(d\left(Tx_n, Tx_{n+1}\right)\right) \to 0^+, \ (\text{ as } n \to \infty \ ).$$

Also, using (2.23), for all $m, n \in \mathbb{N}$, taking $m > n$, we have

$$\begin{aligned} F\left(d\left(Tx_n, Tx_m\right)\right) &= F\left(d\left(Tf^nx_0, Tf^mx_0\right)\right) \\ &\leq \left(\frac{\mu}{1-\mu}\right)^n F\left(d\left(Tx_0, Tf^{m-n}x_0\right)\right) \end{aligned} \tag{2.24}$$

Letting $m, n \to \infty$ in (2.24), we have

$$F\left(d\left(Tx_n, Tx_m\right)\right) \to 0^+ \text{ as } m, n \to \infty.$$

The last inequality implies $d\left(Tx_n, Tx_m\right) \to 0$ as, $m, n \to \infty$. we obtain that $\{Tx_n\}$ is Cauchy sequence in complete metric space (X, d) and there exist $u \in X$ such that $\{Tx_n\}$ converges to $Tu \in X$. In the next stage, by using similar methods in Theorem 2.1 or Theorem 2.8, the proof can be completed. □

Corollary 2.18. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$. For all $x, y \in X$ with $x \preceq y$, $\mu \in \left[0, \frac{1}{2}\right)$,*

$$d\left(Tfx, Tfy\right) \leq \mu\left[d\left(Tx, Tfy\right) + d\left(Ty, Tfx\right)\right].$$

Also, suppose that

C1) f is continuous or C2) assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.19. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $F \in \Psi$. For all $x, y \in X$ with $x \preceq y$, $\mu \in \left[0, \frac{1}{2}\right)$,*

$$F\left(d\left(fx, fy\right)\right) \leq \mu\left[F\left(d\left(x, fy\right)\right) + F\left(d\left(y, fx\right)\right)\right].$$

Also, suppose that

C1) f is continuous or C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for all $(x,y) \in X \times X$ there exists a $z \in X$ such that $x \preceq z$ and $y \preceq z$, then the fixed point is unique.

Corollary 2.20. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping. For all $x, y \in X$ with $x \preceq y$, $\mu \in \left[0, \frac{1}{2}\right)$,*

$$d(fx, fy) \leq \mu \left[d(x, fy) + d(y, fx)\right].$$

Also, suppose that

C1) f is continuous or C2) Assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X.Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Remark 2.21. The Corollary 2.20 is Chatterjea fixed point theorem in the context of ordered metric space.

Corollary 2.22. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in SSC(X)$. For all $x, y \in X$ with $x \preceq y$, $\mu \in \left[0, \frac{1}{2}\right)$,*

$$\int_0^{d(Tfx,Tfy)} \varphi(t)\, dt \leq \mu \int_0^{d(Tx,Tfy)+d(Ty,Tfx)} \varphi(t)\, dt, \tag{2.25}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0,\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)\, dt > 0$.

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Corollary 2.23. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping. For all $x, y \in X$ with $x \preceq y$, $\mu \in \left[0, \frac{1}{2}\right)$,*

$$\int_0^{d(fx,fy)} \varphi(t)\, dt \leq \mu \int_0^{d(x,fy)+d(y,fx)} \varphi(t)\, dt, \tag{2.26}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0,\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)\, dt > 0$.

Also, suppose that the condition C1) or C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Remark 2.24. Also, taking $\varphi(t) = 1$ in Corollary 2.23, we get Chatterjea fixed point theorem in the context of ordered metric space.

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integerales*, Fund. Math., **3** (1922), 133–181. 1

[2] A. Branciari, *A fixed point theorem for mapping satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29** (2002), 531–536. 1

[3] S. K. Chatterjea, *Fixed point theorems*, C.R. Acad. Bulgare Sci., **25** (1972), 727–730. 1, 4

[4] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., **60** (1968), 71–76. 1.3

[5] M. Kır, H. Kızıltuc, *T_F-contractive conditions for Kannan and Chatterjea fixed point theorems*, Adv. Fixed Point Theory, **4** (2014), 140–148. 1

[6] N. Malhotra, B. Bansal, *Some common coupled fixed point theorems for generalized contraction in $b-$metric spaces,* J. Nonlinear Sci. Appl., **8** (2015), 8–16. 1

[7] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in $b-$metric spaces*, J. Inequal. Appl., **2014** (2014), 14 pages. 1

[8] S. Moradi, A. Davood, *New extension of Kannan fixed point theorem on complete metric and generalized metric spaces*, Int. J. Math. Anal., **5** (2011), 2313–2320. 1.5, 1.6

[9] S. Moradi, A. Beiranvand, *Fixed point of T_F-contractive single-valued mappings*, Iran. J. Math. Sci. Inform., **5** (2010), 25–32. 1

[10] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239. 1, 1.2

[11] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1, 1.1