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Existence of nondecreasing solutions of some nonlinear integral equations of fractional order

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Abstract

The purpose of this paper is to examine the class of functional integral equations of fractional order in the space of continuous functions on interval $[0,a]$. Using Darbo's fixed point theorem associated with the measure of noncompactness, we present sufficient conditions for existence of nondecreasing solutions of some functional integral equations of fractional order. These existence results include several obtained from previous studies. Finally, we establish some examples to show that our results are applicable.

Keywords: Nonlinear integral equations, measure of noncompactness, nondecreasing solution, Darbo fixed point theorem, Riemann–Liouville fractional integral.
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1. Introduction

Many nonlinear problems arising from areas of the real world, such as natural sciences, can be represented with operator equations. Especially, integral and differential equations of fractional order play a very important role in describing these problems. For example, some problems in physics, mechanics and other fields can be described with the help of integral and differential equations of fractional order. Some of these problems are theory of neutron transport, the theory of radioactive transfer, the kinetic theory of gases [19], the traffic theory and so on.

J. Banaś et al. dealt with the following equations,

$$x(t) = h(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{v(s,x(s))}{(t-s)^{1-\alpha}} ds, \ t \in [0,1], \tag{1.1}$$

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$$x(t)=h(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,\infty)\,,$$
$$x(t)=f(t,x(t))\left(p(t)+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,(Gx)(s))}{(t-s)^{1-\alpha}}ds\right),\ t\in[0,1]\,,$$
$$x(t)=f_1(t)+\frac{f_2(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,\infty)\,,$$
$$x(t)=f_1(t,x(t))+\frac{f_2(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,\infty)\,,$$
$$x(t)=f(t,x(\alpha_1(t)),...,x(\alpha_n(t)))+\int_0^{\beta(t)}u(t,s,x(\gamma_1(s)),...,x(\gamma_m(s)))ds,\ t\in[0,\infty)\,,$$

in [2]-[7] respectively. On the other hand M. Abdalla Darwish et al. considered the following equations,

$$\begin{aligned}
x(t)&=f(t)+\frac{x(t)}{\Gamma(\alpha)}\int_0^t\frac{u(t,x(t))}{(t-s)^{1-\alpha}}ds,\ t\in[0,T]\,,\\
x(t)&=f(t,x(t))+\frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,\infty)\,,\\
x(t)&=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{g(k(t,s))}{(t-s)^{1-\alpha}}\left|x(s)\right|ds,\ t\in[0,1]\,,\\
x(t)&=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,x(s),x(\lambda s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,\infty)\,,\\
x(t)&=a(t)+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{k(t,s)u(t,s,x(s),x(\lambda s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,1]\,,\\
x(t)&=g(t,x(t))+\frac{(Tx)(t)}{\Gamma(\alpha)}\int_0^t\frac{h(u(t,s))}{(t-s)^{1-\alpha}}(Hx)(s)ds,\ t\in[0,1]\,,\\
x(t)&=g(t,x(t))+\frac{f(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,(Hx)(s))}{(t-s)^{1-\alpha}}ds,\ t\in[0,1]\,,
\end{aligned}\tag{1.2}$$

in [12]-[18] respectively.

Also, in 2010 K. Blachandran et al. [1], for $t\in[0,\infty)$, discussed the following equation,

$$x(t)=g(t,x(\alpha(t)))+\frac{f(t,x(\beta(t)))}{\Gamma(\alpha)}\int_0^t\frac{u(t,s,x(\gamma(s)))}{(t-s)^{1-\alpha}}ds.$$

On the other hand, the authors considered the following equation in [10] and [21],

$$x(t)=g(t,x(\beta(t)))+f(t,x(\alpha(t)))\int_0^{\varphi(t)}u(t,s,x(\gamma(s)))ds,\ t\in[0,a]\,.$$

Then İ. Özdemir and Ü. Çakan dealt with the following equations,

$$x(t)=g(t,x(\alpha(t)))+f\left(t,\int_0^{\varphi(t)}u(t,s,x(\gamma(s)))ds,x(\beta(t))\right),\ t\in[0,a]\,,$$

$$\begin{aligned}
x(t)&=g(t,x(\beta_1(t)),...,x(\beta_s(t)))\\
&\quad+f(t,x(\xi_1(t)),...,x(\xi_m(t)))\int_0^{\varphi(t)}u(t,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))d\tau,\ t\in[0,a]\,,
\end{aligned}\tag{1.3}$$

in [22], [11] respectively.

In this paper, we will consider the following equation for $t \in [0, a]$ and $0 < \alpha \leq 1$,

$$x(t) = g(t, x(\beta_1(t)), ..., x(\beta_s(t))) + \frac{f(t, x(\xi_1(t)), ..., x(\xi_m(t)))}{\Gamma(\alpha)} \int_0^{\varphi(t)} \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t) - \tau)^{1-\alpha}} d\tau. \quad (1.4)$$

We present some definitions and preliminary results about the concept of measure of noncompactness and fractional integral equations in next section. In the last section, we give our main results concerning the existence of nondecreasing and continuous solutions of integral equation (1.4) by applying Darbo's fixed point theorem associated with the measures of noncompactness defined by J. Banaś et al. [8] and [9], as well as some examples to show that these results are applicable.

2. Notations, definitions and auxiliary facts

First of all we will remind the concept of Riemann–Liouville fractional integral of order α for the function $x(t)$.

Definition 2.1 ([20])**.** Let $x \in C[a, b]$ and $a < t < b$, then

$$I_{a^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)}{(t - s)^{1-\alpha}} ds, \;\; \alpha \in (-\infty, \infty)$$

is called the Riemann-Liouville fractional integral of order α, where symbol of Γ denotes the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt.$$

Let $(E, \|.\|)$ be an infinite dimensional Banach space with zero element θ. We write $B(x, r)$ to denote the closed ball centered at x with radius r and especially, we write B_r in case of $x = \theta$. We write $\overline{X}$ and Conv X to denote the closure of X and convex hull of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty bounded subsets of E and let $\mathfrak{N}_E$ indicate its subfamily of all relatively compact sets. Finally, the standard algebraic operations on sets are denoted by λX and $X + Y$, respectively.

We use the following definition of the measure of noncompactness given in [8].

Definition 2.2. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv } X)$.
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5. If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $\bigcap_{n=1}^{\infty} X_n$ is nonempty.

Theorem 2.3 ([8])**.** *Let C be a nonempty, closed, bounded and convex subset of the Banach space E and let $F : C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(FX) \leq k\mu(X) \quad (2.1)$$

for any nonempty subset X of C, where μ is a measure of noncompactness in E. Then F has a fixed point in the set C.

It is known that the family of all real valued and continuous functions defined on interval $[a, b]$ is a Banach space with the standard norm

$$\|x\| = \max \{|x(t)| : t \in [a, b]\}.$$

Let X be a fixed subset of $\mathfrak{M}_{C[a,b]}$. For $\varepsilon > 0$ and $x \in X$, we denote by $\omega(x, \varepsilon)$ the modulus of continuity of function x defined by

$$\omega(x, \varepsilon) = \sup \{|x(t_1) - x(t_2)| : t_1, t_2 \in [a, b] \text{ and } |t_1 - t_2| \le \varepsilon\}.$$

Furthermore, let $\omega(X, \varepsilon)$ and $\omega_0(X)$ are defined by

$$\omega(X, \varepsilon) = \sup \{\omega(x, \varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon). \tag{2.2}$$

Then, function ω_0 is a measure of noncompactness in space $C[a, b]$, [8]. For $x \in X$ let us consider the following quantities

$$d(x) = \sup \{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in [a, b] \text{ and } t \le s\},$$

$$i(x) = \sup \{|x(s) - x(t)| - [x(t) - x(s)] : t, s \in [a, b] \text{ and } t \le s\}.$$

The quantity $d(x)$ represents the degree of decrease of the function x while $i(x)$ represents the degree of increase. Moreover, $d(x) = 0$ if and only if x is nondecreasing on $[a, b]$ and similarly $i(x) = 0$ if and only if x is nonincreasing on $[a, b]$. Further, let us put

$$d(X) = \sup \{d(x) : x \in X\},$$

$$i(X) = \sup \{i(x) : x \in X\}.$$

Finally, let us denote

$$\begin{aligned} \mu_d(X) &= \omega_0(X) + d(X), \\ \mu_i(X) &= \omega_0(X) + i(X). \end{aligned} \tag{2.3}$$

The authors have shown in [9] that above functions μ_d and μ_i are measures of noncompactness in the space $C[a, b]$.

3. The main results

Hereafter we write I to denote the interval $[0, a]$. We study functional integral equation (1.4) with the following conditions:

($\mathbf{a_1}$) Functions

$$\begin{aligned} \beta_j &: I \to I, \text{ for } 1 \le j \le s, \\ \xi_i &: I \to I, \text{ for } 1 \le i \le m, \\ \gamma_\eta &: [0, C] \to I, \text{ for } 1 \le \eta \le n \end{aligned}$$

and

$$\varphi : I \to \mathbb{R}_+$$

are continuous.

(a₂) $g : I \times \mathbb{R}^s \to \mathbb{R}$ is continuous and there exist nonnegative constants k_i for $1 \le i \le s$ such that

$$|g(t, x_1, x_2, ..., x_s) - g(t, y_1, y_2, ..., y_s)| \le \sum_{i=1}^{s} k_i |x_i - y_i|,$$

for all $t \in I$ and $x_i, y_i \in \mathbb{R}$.

(a₃) $f : I \times \mathbb{R}^m \to \mathbb{R}$ is continuous and there exist nonnegative constants λ_i for $1 \le i \le m$ such that

$$|f(t, x_1, x_2, ..., x_m) - f(t, y_1, y_2, ..., y_m)| \le \sum_{i=1}^{m} \lambda_i |x_i - y_i|,$$

for all $t \in I$ and $x_i, y_i \in \mathbb{R}$.

(a₄) Function $u : I \times I \times \mathbb{R}^n \to \mathbb{R}$ is continuous and there exist functions $h_j : \mathbb{R}_+ \to \mathbb{R}_+$ for $1 \le j \le n$ which are nondecreasing on $\mathbb{R}_+$. Moreover the inequality

$$|u(t, \tau, x_1, x_2, ..., x_n)| \le \sum_{j=1}^{n} h_j(|x_j|) \tag{3.1}$$

holds for all $t, \tau \in I$ and $x_j \in \mathbb{R}$ with $1 \le j \le n$.

(a₅) There exists a positive solution r_0 of the inequality

$$\sum_{i=1}^{s} k_i r + \frac{C^\alpha}{\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \lambda_i r + N \right) \sum_{j=1}^{n} h_j(r) + M \le r, \tag{3.2}$$

where C, M and N are the positive constants such that

$$\varphi(t) \le C, \ \ |g(t, 0, 0, ..., 0)| \le M \text{ and } |f(t, 0, 0, ..., 0)| \le N$$

for all $t \in I$.

Theorem 3.1. *Under assumptions* $(\mathbf{a_1})-(\mathbf{a_5})$*, Eq.* (1.4) *has at least one solution* $x = x(t)$ *which belongs to* $B_{r_0} \subset C(I)$.

Proof. Note that we will use Theorem 2.3 as our main tool. We define operator T as

$$(Tx)(t) = g(t, x(\beta_1(t)), ..., x(\beta_s(t))) + \frac{f(t, x(\xi_1(t)), ..., x(\xi_m(t)))}{\Gamma(\alpha)} \int_0^{\varphi(t)} \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t) - \tau)^{1-\alpha}} d\tau,$$

for any $x \in C(I)$. Firstly we show that Tx is continuous on I. To do this we define operators G, F and U as

$$(Gx)(t) = g(t, x(\beta_1(t)), ..., x(\beta_s(t))),$$
$$(Fx)(t) = \frac{f(t, x(\xi_1(t)), ..., x(\xi_m(t)))}{\Gamma(\alpha)}$$

and

$$(Ux)(t) = \int_0^{\varphi(t)} \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t) - \tau)^{1-\alpha}} d\tau$$

for any $x \in C(I)$. Obviously the conditions of Theorem 3.1 guarantee that functions Gx and Fx are continuous on I. Let us take an arbitrary $\varepsilon > 0$ and $t_1, t_2 \in I$ such that $|t_1 - t_2| \le \varepsilon$. Without loss of generality we can assume that $\varphi(t_2) \le \varphi(t_1)$. Then, taking into account our assumptions we get

$$
\begin{aligned}
&|(Ux)(t_1)-(Ux)(t_2)| \\
&= \left| \int_0^{\varphi(t_1)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau - \int_0^{\varphi(t_2)} \frac{u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}}d\tau \right| \\
&\le \int_0^{\varphi(t_2)} \left| \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}} - \frac{u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}} \right| d\tau \\
&\quad + \int_{\varphi(t_2)}^{\varphi(t_1)} \left| \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}} \right| d\tau \\
&\le \int_0^{\varphi(t_2)} \left| \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}} - \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}} \right| d\tau \\
&\quad + \int_0^{\varphi(t_2)} \left| \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}} - \frac{u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}} \right| d\tau \\
&\quad + \int_{\varphi(t_2)}^{\varphi(t_1)} \left| \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}} \right| d\tau \\
&\le \int_0^{\varphi(t_2)} |u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))| \left[\frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}} - \frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}} \right] d\tau \\
&\quad + \int_0^{\varphi(t_2)} \frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}} |u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau))) - u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))| d\tau \\
&\quad + \int_{\varphi(t_2)}^{\varphi(t_1)} \frac{\sum_{j=1}^n h_j(|x(\gamma_j(\tau))|)}{(\varphi(t_1)-\tau)^{1-\alpha}} d\tau \\
&\le \sum_{j=1}^n h_j(\|x\|) \int_0^{\varphi(t_2)} \left[\frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}} - \frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}} \right] d\tau + \omega_{u_1}(I,\varepsilon) \int_0^{\varphi(t_2)} \frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}} d\tau \\
&\quad + \sum_{j=1}^n h_j(\|x\|) \int_{\varphi(t_2)}^{\varphi(t_1)} \frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}} d\tau \\
&\le \sum_{j=1}^n h_j(\|x\|) \frac{(\varphi(t_1)-\varphi(t_2))^\alpha - ([\varphi(t_1)]^\alpha - [\varphi(t_2)]^\alpha)}{\alpha} + \omega_{u_1}(I,\varepsilon) \frac{[\varphi(t_2)]^\alpha}{\alpha} \\
&\quad + \sum_{j=1}^n h_j(\|x\|) \frac{(\varphi(t_1)-\varphi(t_2))^\alpha}{\alpha} \\
&\le \sum_{j=1}^n h_j(\|x\|) \frac{(\varphi(t_1)-\varphi(t_2))^\alpha}{\alpha} + \omega_{u_1}(I,\varepsilon) \frac{[\varphi(t_2)]^\alpha}{\alpha} + \sum_{j=1}^n h_j(\|x\|) \frac{(\varphi(t_1)-\varphi(t_2))^\alpha}{\alpha} \\
&\le 2\sum_{j=1}^n h_j(\|x\|) \frac{[\omega(\varphi,\varepsilon)]^\alpha}{\alpha} + \omega_{u_1}(I,\varepsilon) \frac{C^\alpha}{\alpha},
\end{aligned}
$$

where

$$\omega(\varphi,\varepsilon) = \sup\{|\varphi(t_1)-\varphi(t_2)| : t_1,t_2 \in I \text{ and } |t_1-t_2| \le \varepsilon\}$$

and

$$
\begin{aligned}
\omega_{u_1}(I,\varepsilon) = \sup\{&|u(t_1,\tau,x_1,...,x_n) - u(t_2,\tau,x_1,...,x_n)| : t_1,t_2 \in I,\ \tau \in [0,C], x_i \in R \text{ for } 1 \le i \le n \\
&\text{and } |t_1-t_2| \le \varepsilon\}
\end{aligned}
$$

such that $R = [-\|x\|, \|x\|]$. Hence, taking into account the uniform continuity of function $u(t,\tau,x_1,...,x_n)$ on set $I \times I \times [-\|x\|, \|x\|]^n$ we infer that function Ux is continuous on I. Thus Tx is continuous on I. Using assumptions of Theorem 3.1, for any $x \in B_{r_0}$, we get,

$$\begin{aligned}
|(Tx)(t)| &= \left| g(t, x(\beta_1(t)), ..., x(\beta_s(t))) + \frac{f(t, x(\xi_1(t)), ..., x(\xi_m(t)))}{\Gamma(\alpha)} \int_0^{\varphi(t)} \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t)-\tau)^{1-\alpha}} d\tau \right| \\
&\le |g(t, x(\beta_1(t)), ..., x(\beta_s(t))) - g(t, 0, ..., 0)| + |g(t, 0, ..., 0)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \{ |f(t, x(\xi_1(t)), ..., x(\xi_m(t))) - f(t, 0, ..., 0)| + |f(t, 0, ..., 0)| \} \\
&\quad \times \int_0^{\varphi(t)} \left| \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t)-\tau)^{1-\alpha}} \right| d\tau \\
&\le \sum_{i=1}^{s} k_i |x(\beta_i(t))| + M + \frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{m} \lambda_i |x(\xi_i(t))| + N \right) \int_0^{\varphi(t)} \frac{\sum_{j=1}^{n} h_j(|x(\gamma_j(\tau))|)}{(\varphi(t)-\tau)^{1-\alpha}} d\tau \\
&\le \sum_{i=1}^{s} k_i \|x\| + M + \frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{m} \lambda_i \|x\| + N \right) \sum_{j=1}^{n} h_j(\|x\|) \int_0^{\varphi(t)} \frac{1}{(\varphi(t)-\tau)^{1-\alpha}} d\tau \\
&\le \sum_{i=1}^{s} k_i r_0 + M + \frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{m} \lambda_i r_0 + N \right) \frac{C^\alpha}{\alpha} \sum_{j=1}^{n} h_j(r_0) \\
&= \sum_{i=1}^{s} k_i r_0 + M + \frac{C^\alpha}{\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \lambda_i r_0 + N \right) \sum_{j=1}^{n} h_j(r_0) \\
&\le r_0.
\end{aligned}$$

This result shows that $Tx \in B_{r_0}$. Now, we prove that operator $T : B_{r_0} \to B_{r_0}$ is continuous. To do this, consider $\varepsilon > 0$ and any $x, y \in B_{r_0}$ such that $\|x - y\| \le \varepsilon$. Then we obtain the following inequalities by using the conditions of Theorem 3.1.

$$\begin{aligned}
&|(Tx)(t) - (Ty)(t)| \\
&\le |g(t, x(\beta_1(t)), ..., x(\beta_s(t))) - g(t, y(\beta_1(t)), ..., y(\beta_s(t)))| \\
&\quad + \left| \frac{f(t, x(\xi_1(t)), ..., x(\xi_m(t)))}{\Gamma(\alpha)} - \frac{f(t, y(\xi_1(t)), ..., y(\xi_m(t)))}{\Gamma(\alpha)} \right| \int_0^{\varphi(t)} \left| \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t)-\tau)^{1-\alpha}} \right| d\tau \\
&\quad + \left(\left| \frac{f(t, y(\xi_1(t)), ..., y(\xi_m(t)))}{\Gamma(\alpha)} - \frac{f(t, 0, ..., 0)}{\Gamma(\alpha)} \right| + \left| \frac{f(t, 0, ..., 0)}{\Gamma(\alpha)} \right| \right) \times \\
&\quad \times \int_0^{\varphi(t)} \left| \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t)-\tau)^{1-\alpha}} - \frac{u(t, \tau, y(\gamma_1(\tau)), ..., y(\gamma_n(\tau)))}{(\varphi(t)-\tau)^{1-\alpha}} \right| d\tau \\
&\le \sum_{i=1}^{s} k_i |x(\beta_i(t)) - y(\beta_i(t))| + \sum_{i=1}^{m} \frac{\lambda_i |x(\xi_i(t)) - y(\xi_i(t))|}{\Gamma(\alpha)} \int_0^{\varphi(t)} \frac{\sum_{j=1}^{n} h_j(|x(\gamma_j(\tau))|)}{(\varphi(t)-\tau)^{1-\alpha}} d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{m} \lambda_i |y(\xi_i(t))| + N \right) \int_0^{\varphi(t)} \frac{|u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau))) - u(t, \tau, y(\gamma_1(\tau)), ..., y(\gamma_n(\tau)))|}{(\varphi(t)-\tau)^{1-\alpha}} d\tau \\
&\le \sum_{i=1}^{s} k_i \|x - y\| + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i \|x - y\| \sum_{j=1}^{n} h_j(\|x\|) \frac{C^\alpha}{\alpha} + \frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{m} \lambda_i \|y\| + N \right) \omega_{u_n}(I, \varepsilon) \frac{C^\alpha}{\alpha} \\
&\le \varepsilon \sum_{i=1}^{s} k_i + \frac{\varepsilon C^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} h_j(r_0) + \frac{C^\alpha}{\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \lambda_i r_0 + N \right) \omega_{u_n}(I, \varepsilon), \qquad (3.3)
\end{aligned}$$

where

$$\omega_{u_n}(I,\varepsilon) = \sup\{|u(t,\tau,x_1,...,x_n)-u(t,\tau,y_1,...,y_n)| : t\in I,\ \tau\in[0,C],\ x_i,y_i\in R_1, 1\le i\le n \text{ and } |x_i-y_i|\le\varepsilon\}$$

with $R_1=[-r_0,r_0]$. On the other hand, since the function $u=u(t,\tau,x_1,...,x_n)$ is uniformly continuous on $I\times I\times[-r_0,r_0]^n$, we infer that $\omega_{u_n}(I,\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, the above estimate (3.3) proves that operator T is continuous on B_{r_0}. Now, we show that operator T satisfies (2.1) with respect to measure of noncompactness ω_0 on B_{r_0}. To do this, fix arbitrary $\varepsilon>0$. Let X be any nonempty subset of B_{r_0}, $x\in X$ and $t_1,t_2\in I$ with $\varphi(t_2)\le\varphi(t_1)$ and $|t_1-t_2|\le\varepsilon$, then we get

$$\begin{aligned}
&|(Tx)(t_1)-(Tx)(t_2)|\\
&\le |g(t_1,x(\beta_1(t_1)),...,x(\beta_s(t_1)))-g(t_1,x(\beta_1(t_2)),...,x(\beta_s(t_2)))|\\
&\quad+|g(t_1,x(\beta_1(t_2)),...,x(\beta_s(t_2)))-g(t_2,x(\beta_1(t_2)),...,x(\beta_s(t_2)))|\\
&\quad+\left\{\left|\frac{f(t_1,x(\xi_1(t_1)),...,x(\xi_m(t_1)))}{\Gamma(\alpha)}-\frac{f(t_1,x(\xi_1(t_2)),...,x(\xi_m(t_2)))}{\Gamma(\alpha)}\right|\right.\\
&\quad+\left.\left|\frac{f(t_1,x(\xi_1(t_2)),...,x(\xi_m(t_2)))}{\Gamma(\alpha)}-\frac{f(t_2,x(\xi_1(t_2)),...,x(\xi_m(t_2)))}{\Gamma(\alpha)}\right|\right\}\\
&\quad\times\int_0^{\varphi(t_1)}\left|\frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}}\right|d\tau+\left|\frac{f(t_2,x(\xi_1(t_2)),...,x(\xi_m(t_2)))}{\Gamma(\alpha)}\right|\\
&\quad\times\int_0^{\varphi(t_1)}\frac{|u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))-u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))|}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau\\
&\quad+\left|\frac{f(t_2,x(\xi_1(t_2)),...,x(\xi_m(t_2)))}{\Gamma(\alpha)}\right|\\
&\quad\times\int_0^{\varphi(t_2)}|u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))|\left[\frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}}-\frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}}\right]d\tau\\
&\quad+\left|\frac{f(t_2,x(\xi_1(t_2)),...,x(\xi_m(t_2)))}{\Gamma(\alpha)}\right|\int_{\varphi(t_2)}^{\varphi(t_1)}\left|\frac{u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}}\right|d\tau\\
&\le\sum_{i=1}^{s}k_i|x(\beta_i(t_1))-x(\beta_i(t_2))|+\omega_g(I,\varepsilon)\\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\sum_{i=1}^{m}\lambda_i|x(\xi_i(t_1))-x(\xi_i(t_2))|+\omega_f(I,\varepsilon)\right)\int_0^{\varphi(t_1)}\frac{\sum_{j=1}^{n}h_j(|x(\gamma_j(\tau))|)}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\{|f(t_2,x(\xi_1(t_2)),...,x(\xi_m(t_2)))-f(t_2,0,...,0)|+|f(t_2,0,...,0)|\}\\
&\quad\times\left(\omega_{u_1}(I,\varepsilon)\int_0^{\varphi(t_1)}\frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau\right.\\
&\quad+\left.\int_0^{\varphi(t_2)}\sum_{j=1}^{n}h_j(|x(\gamma_j(\tau))|)\left[\frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}}-\frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}}\right]d\tau+\int_{\varphi(t_2)}^{\varphi(t_1)}\frac{\sum_{j=1}^{n}h_j(|x(\gamma_j(\tau))|)}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau\right)\\
&\le\sum_{i=1}^{s}k_i\omega(x,\omega(\beta_i,\varepsilon))+\omega_g(I,\varepsilon)+\frac{1}{\Gamma(\alpha)}\left(\sum_{i=1}^{m}\lambda_i\omega(x,\omega(\xi_i,\varepsilon))+\omega_f(I,\varepsilon)\right)\sum_{j=1}^{n}h_j(r_0)\frac{C^\alpha}{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\left(\sum_{i=1}^{m}\lambda_i\left|x(\xi_i(t_2))\right|+N\right)\times \\
&\times\left(\omega_{u_1}(I,\varepsilon)\frac{C^\alpha}{\alpha}+\sum_{j=1}^{n}h_j(r_0)\frac{(\varphi(t_1)-\varphi(t_2))^\alpha-([\varphi(t_1)]^\alpha-[\varphi(t_2)]^\alpha)}{\alpha}+\sum_{j=1}^{n}h_j(r_0)\frac{(\varphi(t_1)-\varphi(t_2))^\alpha}{\alpha}\right) \\
&\le\sum_{i=1}^{s}k_i\omega(x,\omega(\beta_i,\varepsilon))+\omega_g(I,\varepsilon)+\frac{C^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\lambda_i\omega(x,\omega(\xi_i,\varepsilon))+\omega_f(I,\varepsilon)\right)\sum_{j=1}^{n}h_j(r_0) \\
&+\frac{1}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\lambda_i r_0+N\right)\left(C^\alpha\omega_{u_1}(I,\varepsilon)+2\left[\omega(\varphi,\varepsilon)\right]^\alpha\sum_{j=1}^{n}h_j(r_0)\right), \qquad (3.4)
\end{aligned}
$$

where

$$\omega_g(I,\varepsilon)=\sup\left\{|g(t_1,x_1,...,x_s)-g(t_2,x_1,...,x_s)|:t_1,t_2\in I,x_i\in R_1 \text{ for } 1\le i\le s \text{ and } |t_1-t_2|\le\varepsilon\right\},$$

$$
\begin{aligned}
\omega_f(I,\varepsilon) \;=\; &\sup\left\{|f(t_1,x_1,...,x_m)-f(t_2,x_1,...,x_m)|:t_1,t_2\in I,x_j\in R_1 \text{ for } 1\le j\le m\right. \\
&\left.\text{and } |t_1-t_2|\le\varepsilon\right\},
\end{aligned}
$$

also,

$$
\begin{aligned}
\omega(\varphi,\varepsilon)&=\sup\left\{|\varphi(t_1)-\varphi(t_2)|:t_1,t_2\in I \text{ and } |t_1-t_2|\le\varepsilon\right\}, \\
\omega(\xi_j,\varepsilon)&=\sup\left\{|\xi_j(t_1)-\xi_j(t_2)|:t_1,t_2\in I \text{ and } |t_1-t_2|\le\varepsilon\right\}, \\
\omega(\beta_i,\varepsilon)&=\sup\left\{|\beta_i(t_1)-\beta_i(t_2)|:t_1,t_2\in I \text{ and } |t_1-t_2|\le\varepsilon\right\},
\end{aligned}
$$

for $1\le j\le m$ and $1\le i\le s$. Thus, by using the above estimate (3.4), we get

$$
\begin{aligned}
\omega(TX,\varepsilon) \;\le\; &\sum_{i=1}^{s}k_i\omega(X,\omega(\beta_i,\varepsilon))+\omega_g(I,\varepsilon)+\frac{C^\alpha}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\lambda_i\omega(X,\omega(\xi_i,\varepsilon))+\omega_f(I,\varepsilon)\right)\sum_{j=1}^{n}h_j(r_0) \\
&+\frac{1}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\lambda_i r_0+N\right)\left(C^\alpha\omega_{u_1}(I,\varepsilon)+2\left[\omega(\varphi,\varepsilon)\right]^\alpha\sum_{j=1}^{n}h_j(r_0)\right). \qquad (3.5)
\end{aligned}
$$

We obtain that $\omega(\varphi,\varepsilon)\to 0$, $\omega(\beta_j,\varepsilon)\to 0$ and $\omega(\xi_i,\varepsilon)\to 0$ for $1\le j\le s$, $1\le i\le m$ as $\varepsilon\to 0$ since functions φ, β_j and ξ_i are uniformly continuous on set I. Similarly, we get $\omega_g(I,\varepsilon)$, $\omega_f(I,\varepsilon)\to 0$ and $\omega_{u_1}(I,\varepsilon)\to 0$ as $\varepsilon\to 0$ because of the fact that functions g, f and u are uniformly continuous on sets $I\times[-r_0,r_0]^s$, $I\times[-r_0,r_0]^m$ and $I\times I\times[-r_0,r_0]^n$, respectively. Hence, we conclude from (3.5) that

$$\omega_0(TX)\le Q\omega_0(X), \qquad (3.6)$$

where

$$Q=\sum_{i=1}^{s}k_i+\frac{C^\alpha}{\Gamma(\alpha+1)}\sum_{i=1}^{m}\lambda_i\sum_{j=1}^{n}h_j(r_0).$$

Since $Q<1$, from inequality (3.2) we obtain that operator T is a contraction on ball B_{r_0} with respect to measure of noncompactness ω_0 in (2.2). Hence, from Theorem 2.3 we get that T has at least one fixed point in B_{r_0}. Consequently, the nonlinear functional integral equation (1.4) has at least one continuous solution in $B_{r_0}\subset C(I)$. This completes the proof. □

Theorem 3.2. *Let φ, β_j, ξ_i, γ_η $(1\le j\le s,\ 1\le i\le m,\ 1\le\eta\le n)$, restriction of f to $I\times\mathbb{R}_+^m$, restriction of g to $I\times\mathbb{R}_+^s$ and restriction of u to $I\times I\times\mathbb{R}_+^n$ are nonnegative and nondecreasing functions for each variable separately, in addition to asumptions $(\mathbf{a_1})-(\mathbf{a_5})$. Then Eq.(1.4) has at least one nondecreasing positive solution $x=x(t)$ which belongs to $B_{r_0}\subset C(I)$.*

Before starting the proof of the theorem, we present the following lemma.

Lemma 3.3. *Assume that the hypothesis of Theorem 3.2 is satisfied. Then*

$$d(Gx) \le \sum_{i=1}^{s} k_i d(x)$$

and

$$d(Fx) \le \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i d(x).$$

Proof. If $t_1 = t_2$, then $d(Gx) = d(Fx) = 0$.
Let $t_1 \neq t_2$. Without loss of the generality we can assume that $t_2 < t_1$. Now let us define sets A and B by

$$A = \{(t_2, t_1) \in I \times I : t_2 < t_1 \text{ and } x(\beta_i(t_1)) = x(\beta_i(t_2)), \text{ for some } 1 \le i \le s\}$$

and

$$B = \{(t_2, t_1) \in I \times I : t_2 < t_1 \text{ and } x(\xi_i(t_1)) = x(\xi_i(t_2)), \text{ for some } 1 \le i \le m\}.$$

If $(t_2, t_1) \notin A$. Then we get

$$\begin{aligned}
&|(Gx)(t_1) - (Gx)(t_2)| - [(Gx)(t_1) - (Gx)(t_2)] \\
&= |g(t_1, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))| \\
&\quad - [g(t_1, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))] \\
&\le |g(t_1, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1)))| \\
&\quad + |g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))| \\
&\quad - [g(t_1, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1)))] \\
&\quad - [g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))] \\
&= |g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))| \\
&\quad - [g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))] \\
&\le |g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))| \\
&\quad + |g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))| \\
&\quad + \cdots + |g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))| \\
&\quad - [g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))] \\
&\quad - [g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))] \\
&\quad - \cdots - [g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))] \\
&= \frac{|g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_1(t_1)) - x(\beta_1(t_2))|} |x(\beta_1(t_1)) - x(\beta_1(t_2))| \\
&\quad + \frac{|g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_2(t_1)) - x(\beta_2(t_2))|} \times \\
&\quad \times |x(\beta_2(t_1)) - x(\beta_2(t_2))| \\
&\quad + \cdots + \frac{|g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))|}{|x(\beta_s(t_1)) - x(\beta_s(t_2))|} \times \\
&\quad \times |x(\beta_s(t_1)) - x(\beta_s(t_2))| \\
&\quad - \frac{[g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))]}{[x(\beta_1(t_1)) - x(\beta_1(t_2))]} [x(\beta_1(t_1)) - x(\beta_1(t_2))] \\
&\quad - \frac{[g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))]}{[x(\beta_2(t_1)) - x(\beta_2(t_2))]} \times
\end{aligned}$$

$$
\begin{aligned}
& \times [x(\beta_2(t_1)) - x(\beta_2(t_2))] \\
& - \cdots - \frac{[g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))]}{[x(\beta_s(t_1)) - x(\beta_s(t_2))]} \times \\
& \times [x(\beta_s(t_1)) - x(\beta_s(t_2))] \\
= \; & \frac{|g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_1(t_1)) - x(\beta_1(t_2))|} |x(\beta_1(t_1)) - x(\beta_1(t_2))| \\
& + \frac{|g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_2(t_1)) - x(\beta_2(t_2))|} \times \\
& \times |x(\beta_2(t_1)) - x(\beta_2(t_2))| \\
& + \cdots + \frac{|g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))|}{|x(\beta_s(t_1)) - x(\beta_s(t_2))|} \times \\
& \times |x(\beta_s(t_1)) - x(\beta_s(t_2))| \\
& - \frac{|g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_1(t_1)) - x(\beta_1(t_2))|} [x(\beta_1(t_1)) - x(\beta_1(t_2))] \\
& - \frac{|g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_2(t_1)) - x(\beta_2(t_2))|} \times \\
& \times [x(\beta_2(t_1)) - x(\beta_2(t_2))] \\
& - \cdots - \frac{|g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))|}{|x(\beta_s(t_1)) - x(\beta_s(t_2))|} \times \\
& \times [x(\beta_s(t_1)) - x(\beta_s(t_2))] \\
= \; & \frac{|g(t_2, x(\beta_1(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_1(t_1)) - x(\beta_1(t_2))|} \times \\
& \times \{|x(\beta_1(t_1)) - x(\beta_1(t_2))| - [x(\beta_1(t_1)) - x(\beta_1(t_2))]\} \\
& + \frac{|g(t_2, x(\beta_1(t_2)), x(\beta_2(t_1)), ..., x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), x(\beta_2(t_2)), x(\beta_3(t_1)), ..., x(\beta_s(t_1)))|}{|x(\beta_2(t_1)) - x(\beta_2(t_2))|} \times \\
& \times \{|x(\beta_2(t_1)) - x(\beta_2(t_2))| - [x(\beta_2(t_1)) - x(\beta_2(t_2))]\} \\
& + \cdots + \frac{|g(t_2, x(\beta_1(t_2)), ..., x(\beta_{s-1}(t_2)), x(\beta_s(t_1))) - g(t_2, x(\beta_1(t_2)), ..., x(\beta_s(t_2)))|}{|x(\beta_s(t_1)) - x(\beta_s(t_2))|} \times \\
& \times \{|x(\beta_s(t_1)) - x(\beta_s(t_2))| - [x(\beta_s(t_1)) - x(\beta_s(t_2))]\} \\
\leq \; & \frac{k_1 |x(\beta_1(t_1)) - x(\beta_1(t_2))|}{|x(\beta_1(t_1)) - x(\beta_1(t_2))|} \{|x(\beta_1(t_1)) - x(\beta_1(t_2))| - [x(\beta_1(t_1)) - x(\beta_1(t_2))]\} \\
& + \frac{k_2 |x(\beta_2(t_1)) - x(\beta_2(t_2))|}{|x(\beta_2(t_1)) - x(\beta_2(t_2))|} \{|x(\beta_2(t_1)) - x(\beta_2(t_2))| - [x(\beta_2(t_1)) - x(\beta_2(t_2))]\} \\
& + \cdots + \frac{k_s |x(\beta_s(t_1)) - x(\beta_s(t_2))|}{|x(\beta_s(t_1)) - x(\beta_s(t_2))|} \{|x(\beta_s(t_1)) - x(\beta_s(t_2))| - [x(\beta_s(t_1)) - x(\beta_s(t_2))]\} \\
\leq \; & k_1 d(x) + k_2 d(x) + \cdots + k_s d(x) \\
= \; & \sum_{i=1}^{s} k_i d(x). \qquad (3.7)
\end{aligned}
$$

Then, we conclude that $d(Gx) \leq \sum_{i=1}^{s} k_i d(x)$ from (3.7).

On the other hand, in the case of $(t_2, t_1) \in A$, if we define

$$A'_{(t_2,t_1)} = \{i : 1 \leq i \leq s \text{ and } x(\beta_i(t_2)) \neq x(\beta_i(t_1))\}$$

for every fixed (t_2, t_1) then there are the following two cases.

- Case of $A'_{(t_2,t_1)} = \emptyset$: In this case, obviously $d(Gx) = 0$, so the claim is clear.

- Case of $A'_{(t_2,t_1)} \neq \emptyset$: In this case, if the above process (multiplication and division with $|x(\beta_i(t_1)) - x(\beta_i(t_2))|$ and $[x(\beta_i(t_1)) - x(\beta_i(t_2))]$) is applied only for $i \in A'_{(t_2,t_1)}$, then

$$d(Gx) \leq \sum_{i \in A'_{(t_2,t_1)}} k_i d(x).$$

Similarly, if $(t_2, t_1) \notin B$ we have

$$d(Fx) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i d(x),$$

if $(t_2, t_1) \in B$, we get $d(Fx) = 0$ or

$$d(Fx) \leq \frac{1}{\Gamma(\alpha)} \sum_{i \in B'_{(t_2,t_1)}} \lambda_i d(x),$$

where $B'_{(t_2,t_1)}$ is defined as

$$B'_{(t_2,t_1)} = \{i : 1 \leq i \leq m \text{ and } x(\xi_i(t_2)) \neq x(\xi_i(t_1))\}$$

for every fixed $(t_2, t_1) \in B$. □

Proof of Theorem 3.2. We will use measure of noncompactness μ_d defined by (2.3) as an application of Theorem 2.3. We know from the proof of Theorem 3.1 that operator T defined as

$$(Tx)(t) = g(t, x(\beta_1(t)), ..., x(\beta_s(t))) + \frac{f(t, x(\xi_1(t)), ..., x(\xi_m(t)))}{\Gamma(\alpha)} \int_0^{\varphi(t)} \frac{u(t, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t) - \tau)^{1-\alpha}} d\tau$$

transforms B_{r_0} into itself and is continuous on B_{r_0}. Assume that X is any nonempty subset of B_{r_0} and t_1, t_2 are arbitrarily fixed in I with $t_2 \leq t_1$. Then, for any $x \in X$, we get

$$\begin{aligned}
&|(Tx)(t_1) - (Tx)(t_2)| - [(Tx)(t_1) - (Tx)(t_2)] \\
&\quad \leq |(Gx)(t_1) - (Gx)(t_2)| - [(Gx)(t_1) - (Gx)(t_2)] \\
&\qquad + |(Fx)(t_1)(Ux)(t_1) - (Fx)(t_2)(Ux)(t_1)| + |(Fx)(t_2)(Ux)(t_1) - (Fx)(t_2)(Ux)(t_2)| \\
&\qquad - [(Fx)(t_1)(Ux)(t_1) - (Fx)(t_2)(Ux)(t_1)] - [(Fx)(t_2)(Ux)(t_1) - (Fx)(t_2)(Ux)(t_2)] \\
&\quad = |(Gx)(t_1) - (Gx)(t_2)| - [(Gx)(t_1) - (Gx)(t_2)] \\
&\qquad + \{|(Fx)(t_1) - (Fx)(t_2)| - [(Fx)(t_1) - (Fx)(t_2)]\} (Ux)(t_1) \\
&\qquad + \{|(Ux)(t_1) - (Ux)(t_2)| - [(Ux)(t_1) - (Ux)(t_2)]\} (Fx)(t_2) \\
&\quad \leq d(Gx) + d(Fx)(Ux)(t_1) + d(Ux)(Fx)(t_2) \\
&\quad \leq d(Gx) + \frac{C^\alpha}{\alpha} \sum_{i=1}^{n} h_i(r_0) d(Fx) + \left(\frac{\sum_{i=1}^{m} \lambda_i r_0 + N}{\Gamma(\alpha)} \right) d(Ux).
\end{aligned} \tag{3.8}$$

On the other hand, for any $x \in X$,

$$\begin{aligned}
&(Ux)(t_1) - (Ux)(t_2) \\
&\quad = \int_0^{\varphi(t_1)} \frac{u(t_1, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t_1) - \tau)^{1-\alpha}} d\tau - \int_0^{\varphi(t_2)} \frac{u(t_2, \tau, x(\gamma_1(\tau)), ..., x(\gamma_n(\tau)))}{(\varphi(t_2) - \tau)^{1-\alpha}} d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\varphi(t_2)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau + \int_{\varphi(t_2)}^{\varphi(t_1)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_1)-\tau)^{1-\alpha}}d\tau \\
&\quad - \int_0^{\varphi(t_2)} \frac{u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}}d\tau + \int_0^{\varphi(t_2)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}}d\tau \\
&\quad - \int_0^{\varphi(t_2)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}}d\tau \\
&\geq p\left\{\int_0^{\varphi(t_2)} \left(\frac{1}{(\varphi(t_1)-\tau)^{1-\alpha}} - \frac{1}{(\varphi(t_2)-\tau)^{1-\alpha}}\right)d\tau + \int_{\varphi(t_2)}^{\varphi(t_1)} \frac{d\tau}{(\varphi(t_1)-\tau)^{1-\alpha}}\right\} \\
&\quad + \int_0^{\varphi(t_2)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau))) - u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}}d\tau \\
&\geq p\frac{[\varphi(t_1)]^\alpha - [\varphi(t_2)]^\alpha}{\alpha} + \int_0^{\varphi(t_2)} \frac{u(t_1,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau))) - u(t_2,\tau,x(\gamma_1(\tau)),...,x(\gamma_n(\tau)))}{(\varphi(t_2)-\tau)^{1-\alpha}}d\tau,
\end{aligned}
\tag{3.9}
$$

where

$$p = \min\{u(t,\tau,x_1,...,x_n) : t \in I,\ \tau \in [0,C],\ x_i \in [-r_0,r_0] \text{ and } 1 \leq i \leq n\}.$$

Since function $t \to u(t,\tau,x_1,...,x_n)$ is nondecreasing on I, (3.9) implies that

$$(Ux)(t_1) - (Ux)(t_2) \geq 0$$

and so

$$d(Ux) = 0. \tag{3.10}$$

Hence, taking into account Lemma 3.3, (3.8) and (3.10), we can write

$$d(Tx) \leq \left(\sum_{i=1}^{s} k_i + \frac{C^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n} h_j(r_0)\right) d(x) = Qd(x)$$

for any $x \in X$. Thus we get

$$d(TX) \leq Qd(X). \tag{3.11}$$

So, we conclude from (3.6) and (3.11) that

$$\mu_d(TX) \leq Q\mu_d(X). \tag{3.12}$$

Since $Q < 1$, from inequality (3.2) we obtain that operator T is a contraction on ball B_{r_0} with respect to measure of noncompactness μ_d. Then, from Theorem 2.3 we get that T has at least one fixed point in B_{r_0}. Consequently, nonlinear functional integral equation (1.4) has at least one positive and continuous solution in $B_{r_0} \subset C(I)$. Finally, we prove that these solutions are nondecreasing on I. Assume that $D = \{x \in B_{r_0} : Tx = x\}$. Since D is nonempty subset of B_{r_0}, taking into account (3.12), then we can write $\mu_d(D) = 0$. This means that $d(x) = 0$, for every $x \in D$. So x is nondecreasing on I. □

4. Examples

Example 4.1. Consider the following nonlinear functional integral equation in $C[0,1]$:

$$x(t) = \frac{x(t^2)}{5} + \frac{x(t)}{13} + \frac{1+t}{8} + \frac{t^3 + x(\sin t)}{18\Gamma(2)} \int_0^{t^2} \left(\frac{2t^2+\tau^3}{9} + \sqrt[3]{x(\tau)} + \ln\left(1 + |x(\tau^2)|\right) + \frac{x^2(\tau)}{6}\right) d\tau. \tag{4.1}$$

Put

$$\beta_1(t) = t,\ \beta_2(t) = \varphi(t) = t^2,\ \xi_1(t) = \sin t,\ \gamma_1(\tau) = \gamma_3(\tau) = \tau,\ \gamma_2(\tau) = \tau^2,$$

$$g(t, x_1, x_2) = \frac{5x_1 + 13x_2}{65} + \frac{1+t}{8},\ f(t, x_1) = \frac{x_1 + t^3}{18},$$

$$u(t, \tau, x_1, x_2, x_3) = \frac{2t^2 + \tau^3}{9} + \sqrt[3]{x_1} + \ln(1 + |x_2|) + \frac{x_3^2}{6}$$

and

$$h_1(r) = \frac{1}{3} + \sqrt[3]{r},\ h_2(r) = \ln(1 + |r|),\ h_3(r) = \frac{r^2}{6}.$$

Moreover it is easy to see that

$$\alpha = a = m = C = 1,\ \lambda_1 = N = \frac{1}{18},\ M = \frac{1}{4},\ k_1 = \frac{1}{13},\ k_2 = \frac{1}{5},\ s = 2,\ n = 3.$$

In order to verify assumption ($\mathbf{a_5}$) observe that the inequality appearing in this assumption has the form

$$\frac{18r}{65} + \frac{r+1}{108}\left(2 + 6\sqrt[3]{r} + 6\ln(1 + |r|) + r^2\right) + \frac{1}{4} \leq r.$$

It is easy to verify that the number $r_0 \in [0.538, 6.216]$ satisfies the above inequality. On the other hand, it is easy to verify that the other assumptions of Theorem 3.1 hold. Therefore, Theorem 3.1 guarantees that Eq. (4.1) has at least one solution $x = x(t) \in B_{r_0} \subset C\,[0, 1]$ for any $r_0 \in [0.538, 6.216]$.

Example 4.2. Consider the following nonlinear functional integral equation in $C\left[0, \frac{\pi}{2}\right]$:

$$x(t) = \frac{tx(\sqrt{t}) + x(t)}{7 + \cos t} + \frac{\exp t + x(t)}{12\Gamma(\frac{1}{4})} \int_0^t \frac{tx(\sqrt{\tau}) \sin t + \ln\left(1 + |x(\tau)|\right)}{3\sqrt[4]{(t-\tau)^3}} d\tau. \tag{4.2}$$

Put

$$\varphi(t) = \xi_1(t) = \beta_2(t) = t,\ \beta_1(t) = \sqrt{t},\ \gamma_1(\tau) = \sqrt{\tau},\ \gamma_2(\tau) = \tau,$$

$$g(t, x_1, x_2) = \frac{tx_1 + x_2}{7 + \cos t},\ f(t, x_1) = \frac{\exp t + x_1}{12},$$

$$u(t, \tau, x_1, x_2) = \frac{tx_1 \sin t + \tau \ln\left(1 + |x_2|\right)}{3}$$

and

$$h_1(r) = \frac{\pi}{6} r,\ h_2(r) = \frac{\pi}{6} \ln\left(1 + |r|\right),\ n = s = 2,\ m = 1,\ \alpha = \frac{1}{4},\ a = C = \frac{\pi}{2},$$

$$k_1 = \frac{\pi}{14},\ k_2 = \frac{1}{7},\ \lambda_1 = \frac{1}{12},\ M = 0,\ N = \frac{\exp\left(\frac{\pi}{2}\right)}{12}.$$

In order to verify assumption ($\mathbf{a_5}$) observe that the inequality appearing in this assumption has the form

$$\frac{r\,(\pi + 2)}{14} + \frac{\pi\left(\frac{\pi}{2}\right)^{\frac{1}{4}}}{6\Gamma(\frac{5}{4})}\left(\frac{r + \exp\left(\frac{\pi}{2}\right)}{12}\right)\left(r + \ln\left(1 + |r|\right)\right) \leq r.$$

It is easy to verify that number $r_0 \in (0, 3.34906]$ satisfies the above inequality. On the other hand, it is easy to show that the other assumptions of Theorem 3.2 hold. Therefore, Theorem 3.2 guarantees that Eq. (4.2) has at least one nondecreasing positive solution $x = x(t) \in B_{r_0} \subset C\left[0, \frac{\pi}{2}\right]$ for $r_0 \in (0, 3.34906]$.

5. Conclusion

It should be noted that Eq. (1.4) is the more general than some equations considered in previous studies. For example, if $n = m = 1$, $g(t, x_2, ..., x_s) = h(t)$, $\varphi(t) = t$, $\gamma_1(\tau) = \tau$, $\xi_1(t) = t$, and $u(t, \tau, x_1) = v(t, x_1)$ then Eq. (1.1) is obtained from Eq. (1.4) . If $n = 2$, $m = 1$, $g(t, x_2, ..., x_s) = a(t)$, $\gamma_1(\tau) = \tau$ and $\gamma_2(\tau) = \lambda\tau$ then Eq. (1.4) can be reduced to Eq. (1.2) .

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