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Coupled coincidence point results for Geraghty-type contraction by using monotone property in partially ordered S-metric spaces

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Abstract

In this paper, we introduce a new concept of generalized compatibility for a pair of mappings defined on a product S-metric and prove certain coupled coincidence point results for mappings satisfying Geraghty-type contraction by using g-monotone instead of the usually mixed monotone property. We also give some sufficient conditions for the uniqueness of a coupled coincidence point. Our results generalize the corresponding results of Zhou and Liu [M. Zhou, X.-L. Liu, J. Funct. Spaces, **2016** (2016), 9 pages], without mixed weakly monotone property and Kadelburg et al. [Z. Kadelburg, P. Kuman, S. Radenović, W. Sintunavarat, Fixed Point Theory Appl., **2015** (2015), 14 pages] from usually metric to S-metric. An illustrative example is presented to support our results.

Keywords: Coupled coincidence point, generalized compatibility, monotone property, partially ordered S-metric space, Geraghty-type contraction.
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1. Introduction

It is well-known that fixed point theory in partially ordered metric spaces are one of the most important

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tools of nonlinear analysis has been widely applied to matrix equations, ordinary differential equations, fuzzy differential equations, integral equations and intermediate value theorems.

In 2006, Bhaskar and Lakshmikantham [4] introduced the notion of a coupled fixed point and proved some interesting coupled fixed point theorems for the mappings satisfying a mixed monotone property, then Lakshmikantham and Ćirić [8] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems that extended the theorems due to Bhaskar and Lakshmikantham. Subsequently, many authors obtained coupled coincidence and coupled fixed point theorems in ordered metric spaces. Recently, in [9–11], the authors established common fixed theorems by using g-monotone property instead of g-mixed monotone property. These kinds of results can be applied in another type of situations, so they give an opportunity to widen the field of applications. In particular the so-called tripled fixed point (and, more generally, n-tuple results) can be more easily handled by using monotone property instead of mixed monotone property (see for example [1, 2, 7]).

On the other hand, several authors have studied fixed point theory in generalized metric spaces. In 2012, Sedghi et al. [13] have introduced the notion of an S-metric space and proved that this notion is a generalization of a metric space. Also, they have proved some properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space. After that, Sedghi and Dung [12] proved a general fixed point theorem in S-metric spaces which is a generalization of [13, Theorem 3.1] and obtained many analogues of fixed point theorems in metric spaces for S-metric spaces. In [5], Gordji et al. have introduced the concept of a mixed weakly monotone pair of maps and proved some coupled common fixed point theorems for a contractive-type maps by using the mixed weakly monotone property in partially ordered metric spaces. These results are of particular interest to state coupled common fixed point theorems for maps with mixed weakly monotone property in partially ordered S-metric spaces. In 2013, Dung [3] used the notion of a mixed weakly monotone pair of maps to state a coupled common fixed point theorem for maps on partially ordered S-metric spaces and generalized the main results of [3–5] into the structure of S-metric spaces. In 2015, Zhou and Liu [14] established some coupled common fixed point theorems under Geraghty-type contraction by using mixed weakly monotone property in partially ordered S-metric spaces.

In this manuscript, we firstly employ a new concept of generalized compatibility of a pair of mappings defined on a product S-metric space, then give some coupled coincidence point results of a pair mappings under Geraghty-type contraction using g-monotone property instead of mixed monotone property. This result generalizes the main results of [14] and [6] into the structure of S-metric spaces. Also, an illustrative example is presented showing the validity of our results.

2. Preliminaries

We now recall some basic definitions and important results for our discussion in the sequel.

Definition 2.1 ([13, Definition 2.1])**.** Let X be a nonempty set. An S-metric on X is a function $S : X^3 \mapsto [0,\infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S(x, y, z) = 0$ if and only if $x = y = z = 0$;
2. $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Immediate examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x, y, z) = \|x + 2y - 3z\| + \|x - z\|$ is an S-metric on X.

(2) Let $\mathbb{R}$ be a real line, then $S(x, y, z) = |x - z| + |y - z|$ is an S-metric on $\mathbb{R}$.

(3) Let X be a nonempty set, d is ordinary metric on X, the $S_d(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.

Lemma 2.2 ([13, Lemma 1.4]). *Let (X, S) be an $S-$metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

and

$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Lemma 2.3 ([13, Lemma 2.5]). *Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$, for all $x, y \in X$.*

Lemma 2.4. *Let (X, d) be a metric space. Then $X \times X$ is a metric space with metric D_d^{max} given by*

$$D_d^{max}((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}$$

for all $x, y, u, v \in X$.

Proof. For all $x, y, u, v \in X$, we have $D_d^{max}((x, y), (u, v)) \in [0, \infty)$ and $D_d^{max}((x, y), (u, v))= 0$, if and only if $d(x, u) = d(y, v) = 0$, if and only if $x = u, y = v$, that is $(x, y) = (u, v)$, and

$$\begin{aligned} &D_d^{max}((x, y), (u, v)) \\ &\quad = \max\{d(x, u), d(y, v)\} \\ &\quad \leq \max\{d(x, a), d(u, a)\} + \max\{d(y, b), d(v, b)\} \\ &\quad = D_d^{max}((x, y), (a, b)) + D_d^{max}((u, v), (a, b)). \end{aligned}$$

Therefore, D_d^{max} is a metric on $X \times X$. □

Lemma 2.5. *Let (X, S) be an S-metric space. Then $X \times X$ is an S-metric space with S-metric D_s^{max} given by*

$$D_s^{max}((x, y), (u, v), (w, s)) = \max\{S(x, u, w), S(y, v, t)\}$$

for all $x, y, u, v, w, t \in X$.

Proof. For all $x, y, u, v, w, t \in X$, we have $D_s^{max}((x, y), (u, v), (w, t)) \in [0, \infty)$ and $D_s^{max}((x, y), (u, v), (w, t))=$ 0, if and only if $S(x, u, w) = S(y, v, t) = 0$, if and only if $x = u = w, y = v = t$, that is, $(x, y) = (u, v) = (w, t)$, and

$$\begin{aligned} &D_s^{max}((x, y), (u, v), (w, t)) \\ &\quad = \max\{S(x, u, w), S(y, v, t)\} \\ &\quad \leq \max\{S(x, x, a), S(u, u, a)\} + \max\{S(w, w, a), S(y, y, b)\} + \max\{S(v, v, b), S(t, t, b)\} \\ &\quad = D_s^{max}((x, y), (x, y), (a, b)) + D_s^{max}((u, v), (u, v), (a, b)) + D_s^{max}((w, t), (w, t), (a, b)). \end{aligned}$$

Thus, D_s^{max} is an S-metric on $X \times X$. □

Remark 2.6. Let (X, d) be a metric space. By using Lemma 2.5 with $S = S_d$, we get

$$\begin{aligned} D_s^{max}((x, y), (x, y), (u, v)) &= \max\{S_d(x, x, u), S_d(y, y, v)\} \\ &= 2 \max\{d(x, u), d(y, v)\} \\ &= 2D_d^{max}((x, y), (u, v)) \end{aligned}$$

for all $x, y, u, v \in X$.

Definition 2.7 ([13, Definition 2.8]). Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\} \subset X$ is said to be convergent to $x \in X$, if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$.

(2) A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence, if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x_m) < \epsilon$.

(3) The S-metric space (X, S) is said to be complete, if every Cauchy sequence is a convergent sequence.

Lemma 2.8. *Let (X, S) be an S-metric space. Then (X, S) is complete, if and only if $(X \times X, D_s^{max})$ is complete.*

Proof. It is obvious to get the conclusion from the definition of completeness on (X, S). □

Definition 2.9. Suppose that $f, g : X \times X \mapsto X$ are two maps. f is said to be g-nondecreasing with a partial order $\preceq$, if for all $x, y, u, v \in X$, with $g(x, y) \preceq g(u, v)$, we have $f(x, y) \preceq f(u, v)$.

Example 2.10. Let $X = (0, \infty)$ be endowed with the natural ordering of real numbers $\leq$. Define mappings $f, g : X \times X \mapsto X$ by $f(x, y) = ex + y$ and $g(x, y) = x + y$, for all $(x, y) \in X \times X$. Then f is g-nondecreasing with respect to $\leq$.

Example 2.11. Let $X = \mathbb{N}$ be endowed with the partial order $\preceq$ defined by $x \preceq y$, if and only if y divides x. Define the mappings $f, g : X \times X \mapsto X$ by $f(x, y) = x^2y^2$ and $g(x, y) = xy$, for all $(x, y) \in X \times X$. Then f is g-nondecreasing with respect to $\preceq$.

Definition 2.12. Let $f, g : X \times X \mapsto X$ be two maps. An element $(x, y) \in X \times X$ is called a

(1) coupled fixed point of a mapping $f : X \times X \mapsto X$, if $x = f(x, y)$ and $y = f(y, x)$.

(2) coupled coincidence point of two mappings $f, g : X \times X \mapsto X$, if $f(x, y) = g(x, y)$ and $f(y, x) = g(y, x)$.

(3) coupled common fixed point of a mapping $f, g : X \times X \mapsto X$, if $x = f(x, y) = g(x, y)$ and $y = f(y, x) = g(y, x)$.

Definition 2.13. Let (X, S) be an S-metric space and let $f, g : X \times X \mapsto X$ be two maps. We say that the pair (f, g) is generalized compatible, if

$$S(f(g(x_n, y_n), g(y_n, x_n)), f(g(x_n, y_n), g(y_n, x_n)), g(f(x_n, y_n), f(y_n, x_n))) \to 0, \quad \text{as} \quad n \to \infty,$$
$$S(f(g(y_n, x_n), g(x_n, y_n)), f(g(y_n, x_n), g(x_n, y_n)), g(f(y_n, x_n), f(x_n, y_n))) \to 0, \quad \text{as} \quad n \to \infty,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n, y_n) = t_1,$$
$$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n, x_n) = t_2.$$

Definition 2.14. Let $f, g : X \times X \mapsto X$ be two maps. We say the pair (f, g) is commuting, if

$$f(g(x, y), g(y, x)) = g(f(x, y), f(y, x))$$

for all $x, y \in X$.

3. Main results

Let Θ denote the set of all functions $\theta : [0, \infty)^2 \mapsto [0, 1)$ which satisfy the following conditions:

(θ_1) $\theta(s, t) = \theta(t, s)$ for all $s, t \in [0, \infty)$;

(θ_2) for two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\theta(s_n, t_n) \to 1 \Rightarrow s_n, t_n \to 0.$$

Some examples of such a function are as follows:

Example 3.1. Let $\theta : [0,\infty)^2 \mapsto [0,1)$ be defined by

$$\theta(x,y) = \begin{cases} \frac{sin(k_1x+k_2y)}{k_1x+k_2y}, & x>0 \quad \text{or} \quad y>0, \quad \text{and} \quad k_1,k_2 \in (0,1); \\ l \in [0,1), & x=y=0. \end{cases}$$

Example 3.2. Let $\theta : [0,\infty)^2 \mapsto [0,1)$ be defined by

$$\theta(x,y) = \begin{cases} \frac{ln(1+\max\{k_1x,k_2y\})}{\max\{k_1x,k_2y\}}, & x>0 \quad \text{or} \quad y>0, \quad \text{and} \quad k_1,k_2 \in (0,1); \\ l \in [0,1), & x=y=0. \end{cases}$$

Example 3.3. Let $\theta : [0,\infty)^2 \mapsto [0,1)$ be defined by

$$\theta(x,y) = \begin{cases} 1-k(x+y), & x+y>0, \quad \text{and} \quad k(x+y) \le 1; \\ l \in [0,1), & k(x+y)>1. \end{cases}$$

Now we prove our main results.

Theorem 3.4. *Let $(X,S,\preceq)$ be a partially ordered S-metric space; $f,g : X\times X \mapsto X$ be two maps such that*

(1) *X is S-complete;*

(2) *f,g are two generalized compatible maps such that f is g-nondecreasing with respect to $\preceq$, and there exist $x_0,y_0 \in X$ such that $g(x_0,y_0) \preceq f(x_0,y_0)$ and $g(y_0,x_0) \preceq f(y_0,x_0)$;*

(3) *there exists $\theta \in \Theta$ such that*

$$\begin{aligned} S(f(x,y),f(x,y),f(u,v)) \le \theta(&S(g(x,y),g(x,y),g(u,v)),S(g(y,x),g(y,x),g(v,u))) \\ &\times \max\{S(g(x,y),g(x,y),g(u,v)),S(g(y,x),g(y,x),g(v,u))\} \end{aligned} \tag{3.1}$$

for all $x,y,u,v \in X$ with $g(x,y) \preceq g(u,v)$ and $g(y,x) \preceq g(v,u)$;

(4) *for any $x,y \in X$, there exist $u,v \in X$ such that*

$$f(x,y) = g(u,v), f(y,x) = g(v,u). \tag{3.2}$$

(5) *(a) f and g are continuous; or*
(b) X has the following property:
if a nondecreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f and g have a coupled coincidence point in X.

Proof. Let $x_0,y_0 \in X$ be such that $g(x_0,y_0) \preceq f(x_0,y_0)$ and $g(y_0,x_0) \preceq f(y_0,x_0)$.

From (3.2), there exists $(x_1,y_1) \in X\times X$ such that $f(x_0,y_0) = g(x_1,y_1)$ and $f(y_0,x_0) = g(y_1,x_1)$.

Again by assumption (4), for (x_1,y_1) there exists $(x_2,y_2) \in X \times X$ such that $f(x_1,y_2) = g(x_2,y_2)$ and $f(y_1,x_1) = g(y_2,x_2)$. By continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$f(x_n,y_n) = g(x_{n+1},y_{n+1}), f(y_n,x_n) = g(y_{n+1},x_{n+1}) \tag{3.3}$$

for all $n \in \mathbb{N}$.

First we show that for all $n \in \mathbb{N}$, we have

$$g(x_n, y_n) \preceq g(x_{n+1}, y_{n+1}), \quad \textit{and} \quad g(y_n, x_n) \preceq g(y_{n+1}, x_{n+1}). \tag{3.4}$$

As $g(x_0, y_0) \preceq f(x_0, y_0)$ and $g(y_0, x_0) \preceq f(y_0, x_0)$ and as $f(x_0, y_0) = g(x_1, y_1)$ and $f(y_0, x_0) = g(y_1, x_1)$, we have $g(x_0, y_0) \preceq g(x_1, y_1)$ and $g(y_0, x_0) \preceq g(y_1, x_1)$. Thus, (3.4) holds for $n = 0$. Suppose now that (3.4) holds for some fixed $n \in \mathbb{N}$.

Since f is g-nondecreasing with respect to $\preceq$, we have

$$g(x_{n+1}, y_{n+1}) = f(x_n, y_n) \preceq f(x_{n+1}, y_{n+1}) = g(x_{n+2}, y_{n+2}),$$

and

$$g(y_{n+1}, x_{n+1}) = f(y_n, x_n) \preceq f(y_{n+1}, x_{n+1}) = g(y_{n+2}, x_{n+2}).$$

Hence (3.4) holds for all $n \in \mathbb{N}$.

If $g(x_{n_0}, y_{n_0}) = g(x_{n_0+1}, y_{n_0+1})$ and $g(y_{n_0}, y_{n_0}) = g(y_{n_0+1}, x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then (x_{n_0}, y_{n_0}) is a coupled coincidence point of f and g.

Therefore, in what follows, we assume that for each $n \in \mathbb{N}$, $g(x_n, y_n) \neq g(x_{n+1}, y_{n+1})$ or $g(y_n, x_n) \neq g(y_{n+1}, x_{n+1})$ holds.

Since $g(x_n, y_n) \preceq g(x_{n+1}, y_{n+1})$ and $g(y_n, x_n) \preceq g(y_{n+1}, x_{n+1})$, by using (3.1) and (3.3), we get for all $n \in \mathbb{N}$,

$$\begin{aligned} S(g(x_{n+1}, & y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})) \\ & = S(f(x_n, y_n), f(x_n, y_n), f(x_{n+1}, y_{n+1})) \\ & \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) \\ & \quad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} S(g(y_{n+1}, & x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2})) \\ & = S(f(y_n, x_n), f(y_n, x_n), f(y_{n+1}, x_{n+1})) \\ & \leq \theta(S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})), S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1}))) \\ & \quad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\}. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get for all $n \in \mathbb{N}$,

$$\begin{aligned} \max\{ & S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\} \\ & \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) \\ & \quad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\} \\ & \leq \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\}. \end{aligned} \tag{3.7}$$

Thus the sequence

$$d_n := \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\}$$

is decreasing. It follows that $d_n \to d$ as $n \to \infty$, for some $d \geq 0$. Next, we claim that $d = 0$.

Suppose to the contrary that $d > 0$, then from (3.7), we obtain that

$$\begin{aligned} & \frac{\max\{S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\}}{\max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\}} \\ & \qquad \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) < 1. \end{aligned}$$

On taking the limit as $n \to \infty$, we get

$$\theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) \to 1.$$

Since $\theta \in \Theta$, we have

$$S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})) \to 0,$$

and

$$S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})) \to 0,$$

as $n \to \infty$.

Hence, $d_n \to 0$ as $n \to \infty$, which contradicts the assumption $d > 0$. Thus $d = 0$, that is,

$$d_n := \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))\} \to 0, \tag{3.8}$$

as $n \to \infty$.

We shall prove that $\{g(x_n, y_n), g(y_n, x_n)\}$ is a Cauchy sequence in $X \times X$ endowed with S-metric D_s^{max} defined in Lemma 2.5.

If $\{g(x_n, y_n), g(y_n, x_n)\}$ is not a Cauchy sequence in $(X \times X, D_s^{max})$, then there exists $\epsilon > 0$ for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for all positive integer k with $n_k > m_k > k$, we have

$$D_s^{max}((g(x_{n_k}, y_{n_k}), g(y_{n_k}, x_{n_k})), (g(x_{n_k}, y_{n_k}), g(y_{n_k}, x_{n_k})), (g(x_{m_k}, y_{m_k}), g(y_{m_k}, x_{m_k}))) \geq \epsilon,$$
$$D_s^{max}((g(x_{n_k-1}, y_{n_k-1}), g(y_{n_k-1}, x_{n_k-1})), (g(x_{n_k-1}, y_{n_k-1}), g(y_{n_k-1}, x_{n_k-1})), (g(x_{m_k}, y_{m_k}), g(y_{m_k}, x_{m_k}))) < \epsilon.$$

By definition of D_s^{max}, we have

$$r_k := \max\{S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))\} \geq \epsilon, \tag{3.9}$$

and

$$\begin{aligned} \max\{&S(g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k-1}, y_{n_k-1}), g(x_{m_k}, y_{m_k})), \\ &S(g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k-1}, x_{n_k-1}), g(y_{m_k}, x_{m_k}))\} < \epsilon. \end{aligned} \tag{3.10}$$

By using (3.9), (3.10) and Lemma 2.2, we have that

$$\begin{aligned} \epsilon \leq r_k &:= \max\{S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))\} \\ &\leq \max\{S(g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k-1}, y_{n_k-1}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k-1}, x_{n_k-1}), g(y_{m_k}, x_{m_k}))\} \\ &\quad + \max\{2S(g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k}, y_{n_k})), 2S(g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k}, x_{n_k}))\} \\ &< 2\max\{S(g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k}, y_{n_k})), S(g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k}, x_{n_k}))\} + \epsilon. \end{aligned}$$

On taking the limit as $k \to \infty$, we have

$$r_k := \max\{S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))\} \to \epsilon. \tag{3.11}$$

By Lemma 2.2, we have

$$\begin{aligned} r_k &= \max\{S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))\} \\ &\leq \max\{2S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{n_k+1}, y_{n_k+1})), 2S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{n_k+1}, x_{n_k+1}))\} \\ &\quad + \max\{S(g(x_{n_k+1}, y_{n_k+1}), g(x_{n_k+1}, y_{n_k+1}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k+1}, x_{n_k+1}), g(y_{n_k+1}, x_{n_k+1}), g(y_{m_k}, x_{m_k}))\} \\ &< 2d_{n_k} + \max\{2S(g(x_{m_k}, y_{m_k}), g(x_{m_k}, y_{m_k}), g(x_{m_k+1}, y_{m_k+1})), 2S(g(y_{m_k}, x_{m_k}), g(y_{m_k}, x_{m_k}), g(y_{m_k+1}, x_{m_k+1}))\} \\ &\quad + \max\{S(g(x_{n_k+1}, y_{n_k+1}), g(x_{n_k+1}, y_{n_k+1}), g(x_{m_k+1}, y_{m_k+1})), \\ &\quad S(g(y_{n_k+1}, x_{n_k+1}), g(y_{n_k+1}, x_{n_k+1}), g(y_{m_k+1}, x_{m_k+1}))\} \end{aligned}$$

$$
\begin{aligned}
&= 2d_{n_k} + 2d_{m_k} + \max\{S(f(x_{n_k}, y_{n_k}), f(x_{n_k}, y_{n_k}), f(x_{m_k}, y_{m_k})), S(f(y_{n_k}, x_{n_k}), f(y_{n_k}, x_{n_k}), f(y_{m_k}, x_{m_k}))\} \\
&< 2d_{n_k} + 2d_{m_k} + \theta(S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))) \\
&\quad \times \max\{S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))\} \\
&\le 2d_{n_k} + 2d_{m_k} + r_k.
\end{aligned}
$$

On taking the limit as $k \to \infty$ and by using (3.8) and (3.11), we get

$$\theta(S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))) \to 1.$$

By using the property of θ, we obtain

$$S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})) \to 0,$$

and

$$S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k})) \to 0,$$

as $k \to \infty$, which implies that

$$\lim_{k\to\infty} r_k = 0,$$

which contradicts with $\epsilon > 0$.

Therefore, $\{g(x_n, y_n), g(y_n, x_n)\}$ is a Cauchy sequence in $(X \times X, D_s^{max})$. Since X is S-complete, by Lemma 2.8, there exists $(u, v) \in X \times X$ such that

$$\lim_{n\to\infty} g(x_n, y_n) = \lim_{n\to\infty} f(x_n, y_n) = u \quad \text{and} \quad \lim_{n\to\infty} g(y_n, x_n) = \lim_{n\to\infty} f(y_n, x_n) = v. \tag{3.12}$$

Since the pair (f, g) satisfies the generalized compatibility, from (3.12), we get that

$$\lim_{n\to\infty} S(f(g(x_n, y_n), g(y_n, x_n)), f(g(x_n, y_n), g(y_n, x_n)), g(f(x_n, y_n), f(y_n, x_n))) = 0, \tag{3.13}$$

and

$$\lim_{n\to\infty} S(f(g(y_n, x_n), g(x_n, y_n)), f(g(y_n, x_n), g(x_n, y_n)), g(f(y_n, x_n), f(x_n, y_n))) = 0.$$

Now, consider the assumption (5) that (a) holds, that is, f and g are continuous. By using Lemma 2.2, we get that

$$
\begin{aligned}
&S(g(u, v), g(u, v), f(g(x_n, y_n), g(y_n, x_n))) \\
&\quad \le 2S(g(u, v), g(u, v), g(f(x_n, y_n), f(y_n, x_n))) \\
&\quad\quad + S(f(g(x_n, y_n), g(y_n, x_n)), f(g(x_n, y_n), g(y_n, x_n)), g(f(x_n, y_n), f(y_n, x_n))).
\end{aligned}
$$

By passing to the limit as $n \to \infty$ and using (3.12), (3.13) and the continuity of f, we get that

$$S(g(u, v), g(u, v), f(u, v)) = 0.$$

Hence, $g(u, v) = f(u, v)$. In similar way, $g(v, u) = f(v, u)$ is obtained.

Now, consider the assumption 5 that (b) holds. By (3.4) and (3.12), we have $\{g(x_n, y_n)\}$ and $\{g(y_n, x_n)\}$ are nondecreasing sequences, $g(x_n, y_n) \to u$ and $g(y_n, x_n) \to v$ as $n \to \infty$. Thus for all $n \in \mathbb{N}$, we have

$$g(x_n, y_n) \preceq u, g(y_n, x_n) \preceq v.$$

By using (3.1) and Lemma 2.2, we get

$$
\begin{aligned}
S(f(u, v), f(u, v), g(u, v)) &\le 2S(f(u, v), f(u, v), g(x_{n+1}, y_{n+1})) + S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(u, v)) \\
&= 2S(f(u, v), f(u, v), f(x_n, y_n)) + S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(u, v)) \\
&\to 0,
\end{aligned}
$$

as $n \to \infty$. Hence, $g(u, v) = f(u, v)$. In similar way, we get $g(v, u) = f(v, u)$.

Note that in the case (b), continuity and generalized compatibility assumptions are not necessary in the proof. □

Remark 3.5. In Theorem 3.4, the condition that f has g-nondecreasing property is a substitution for the mixed weakly monotone property of the pair of (f, g) that was used in [5]-[14].

In Theorem 3.4, the following contractive condition was studied:

$$\begin{aligned} S(f(x,y), f(x,y), f(u,v)) \leq \theta(&S(g(x,y), g(x,y), g(u,v)), S(g(y,x), g(y,x), g(v,u))) \\ &\times \max\{S(g(x,y), g(x,y), g(u,v)), S(g(y,x), g(y,x), g(v,u))\}. \end{aligned}$$

The above condition is, in some sense, an extension of the contractive condition:

$$d(F(x,y), F(u,v)) \leq \theta(d(gx, gu), d(gy, gv) \times \max\{d(gx, gu), d(gy, gv)\},$$

of Kadelburg et al. [6] from metric to S-metric spaces.

Now, we give a more general contractive condition to extend Theorem 3.4.

Theorem 3.6. *Let $(X, S, \preceq)$ be a partially ordered S-metric space; $f, g : X \times X \mapsto X$ be two maps such that*

1. *X is S-complete;*
2. *f, g are two generalized compatible maps such that f is g-nondecreasing with respect to $\preceq$ and there exist $x_0, y_0 \in X$ such that $g(x_0, y_0) \preceq f(x_0, y_0)$ and $g(y_0, x_0) \preceq f(y_0, x_0)$;*
3. *there exists $\theta \in \Theta$ such that*

$$\begin{aligned} S(f(x,y), f(x,y), f(u,v)) \leq \theta(&S(g(x,y), g(x,y), g(u,v)), S(g(y,x), g(y,x), g(v,u))) \\ &\times \max\{S(g(x,y), g(x,y), g(u,v)), S(g(y,x), g(y,x), g(v,u)), \\ &S(g(x,y), g(x,y), f(x,y)), S(g(y,x), g(y,x), f(y,x)), \\ &S(g(u,v), g(u,v), f(u,v)), S(g(v,u), g(v,u), f(v,u))\} \end{aligned} \tag{3.14}$$

for all $x, y, u, v \in X$ with $g(x,y) \preceq g(u,v)$ and $g(y,x) \preceq g(v,u)$;

4. *for any $x, y \in X$, there exist $u, v \in X$ such that*

$$f(x,y) = g(u,v), f(y,x) = g(v,u); \tag{3.15}$$

5. (a) *f and g are continuous; or*
   (b) *X has the following property:*
   if a nondecreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f and g have a coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0, y_0) \preceq f(x_0, y_0)$ and $g(y_0, x_0) \preceq f(y_0, x_0)$.

From (3.15), there exists $(x_1, y_1) \in X \times X$ such that $f(x_0, y_0) = g(x_1, y_1)$ and $f(y_0, x_0) = g(y_1, x_1)$.

Again by assumption 4, for (x_1, y_1) there exists $(x_2, y_2) \in X \times X$ such that $f(x_1, y_2) = g(x_2, y_2)$ and $f(y_1, x_1) = g(y_2, x_2)$. By continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$f(x_n, y_n) = g(x_{n+1}, y_{n+1}), f(y_n, x_n) = g(y_{n+1}, x_{n+1}) \tag{3.16}$$

for all $n \in \mathbb{N}$.

First we show that for all $n \in \mathbb{N}$, we have

$$g(x_n, y_n) \preceq g(x_{n+1}, y_{n+1}) \quad \text{and} \quad g(y_n, x_n) \preceq g(y_{n+1}, x_{n+1}). \tag{3.17}$$

As $g(x_0, y_0) \preceq f(x_0, y_0)$ and $g(y_0, x_0) \preceq f(y_0, x_0)$ and as $f(x_0, y_0) = g(x_1, y_1)$ and $f(y_0, x_0) = g(y_1, x_1)$, we have $g(x_0, y_0) \preceq g(x_1, y_1)$ and $g(y_0, x_0) \preceq g(y_1, x_1)$. Thus (3.17) holds for $n = 0$. Suppose now that (3.17) holds for some fixed $n \in \mathbb{N}$.

Since f is g-nondecreasing with respect to $\preceq$, we have

$$g(x_{n+1}, y_{n+1}) = f(x_n, y_n) \preceq f(x_{n+1}, y_{n+1}) = g(x_{n+2}, y_{n+2}),$$

and

$$g(y_{n+1}, x_{n+1}) = f(y_n, x_n) \preceq f(y_{n+1}, x_{n+1}) = g(y_{n+2}, x_{n+2}).$$

Hence (3.17) holds for all $n \in \mathbb{N}$.

If $g(x_{n_0}, y_{n_0}) = g(x_{n_0+1}, y_{n_0+1})$ and $g(y_{n_0}, y_{n_0}) = g(y_{n_0+1}, x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then (x_{n_0}, y_{n_0}) is a coupled coincidence point of f and g.

Therefore, in what follows, we assume that for each $n \in \mathbb{N}$, $g(x_n, y_n) \neq g(x_{n+1}, y_{n+1})$ or $g(y_n, x_n) \neq g(y_{n+1}, x_{n+1})$ holds.

Since $g(x_n, y_n) \preceq g(x_{n+1}, y_{n+1})$ and $g(y_n, x_n) \preceq g(y_{n+1}, x_{n+1})$, by using (3.14) and (3.16) with $x = x_n$, $y = y_n$, $u = x_{n+1}$, $v = y_{n+1}$ we get for all $n \in \mathbb{N}$,

$$\begin{aligned}
&S(g(x_{n+1},y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2}))\\
&\quad = S(f(x_n, y_n), f(x_n, y_n), f(x_{n+1}, y_{n+1}))\\
&\quad \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})))\\
&\qquad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})),\\
&\qquad S(g(x_n, y_n), g(x_n, y_n), f(x_n, y_n)), S(g(y_n, x_n), g(y_n, x_n), f(y_n, x_n)),\\
&\qquad S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), f(x_{n+1}, y_{n+1})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), f(y_{n+1}, x_{n+1}))\}\\
&\quad = \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})))\\
&\qquad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})),\\
&\qquad S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})),\\
&\qquad S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&S(g(x_{n+1},y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2}))\\
&\quad \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})))\\
&\qquad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})), \qquad (3.18)\\
&\qquad S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\}.
\end{aligned}$$

Similarly, by using (3.14) and (3.16), we get

$$\begin{aligned}
&S(g(y_{n+1},x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\\
&\quad = S(f(y_n, x_n), f(y_n, x_n), f(y_{n+1}, x_{n+1}))\\
&\quad \leq \theta(S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})), S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1}))) \qquad (3.19)\\
&\qquad \times \max\{S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})), S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})),\\
&\qquad S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2})), S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2}))\}.
\end{aligned}$$

From (3.18) and (3.19), as $\theta(s, t) = \theta(t, s)$, we get for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\max\{S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\}\\
&\quad \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) \qquad (3.20)\\
&\qquad \times \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})),\\
&\qquad S(g(x_{n+1}, y_{n+1}), g(x_{n+1}, y_{n+1}), g(x_{n+2}, y_{n+2})), S(g(y_{n+1}, x_{n+1}), g(y_{n+1}, x_{n+1}), g(y_{n+2}, x_{n+2}))\}.
\end{aligned}$$

Put

$$d_n = \max\{S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})\}. \tag{3.21}$$

Then from (3.20) we have, as $\theta(s, t) = \theta(t, s) < 1$ for all $s, t > 0$.

$$\begin{aligned} d_{n+1} &\leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) \max\{d_n, d_{n+1}\} \\ &< \max\{d_n, d_{n+1}\}. \end{aligned}$$

Since $d_{n+1} < d_{n+1}$ is impossible, hence we have

$$d_{n+1} \leq \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})))d_n. \tag{3.22}$$

Hence we get $d_{n+1} < d_n$ for all $n \in \mathbb{N}$. Therefore, there is some $d \geq 0$ such that

$$\lim_{n\to\infty} d_n = d.$$

Then from (3.22) we have

$$d \leq \lim_{n\to\infty} \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})))d,$$

and hence, if suppose that $d > 0$, then

$$\lim_{n\to\infty} \theta(S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})), S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1}))) = 1.$$

Since $\theta \in \Theta$, we have

$$S(g(x_n, y_n), g(x_n, y_n), g(x_{n+1}, y_{n+1})) \to 0 \quad \text{and} \quad S(g(y_n, x_n), g(y_n, x_n), g(y_{n+1}, x_{n+1})) \to 0.$$

Hence, by (3.21), $d_n \to 0$. Therefore, our supposition $d > 0$ was wrong, that is,

$$\lim_{n\to\infty} d_n = d = 0. \tag{3.23}$$

Now, we prove that $\{g(x_n, y_n), g(y_n, x_n)\}$ is a Cauchy sequence in $X \times X$ endowed with S-metric D_s^{max} defined in Lemma 2.5.

If $\{g(x_n, y_n), g(y_n, x_n)\}$ is not a Cauchy sequence in $(X \times X, D_s^{max})$, then there exists $\epsilon > 0$ for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for all positive integer k with $n_k > m_k > k$, we have

$$D_s^{max}((g(x_{n_k}, y_{n_k}), g(y_{n_k}, x_{n_k})), (g(x_{n_k}, y_{n_k}), g(y_{n_k}, x_{n_k})), (g(x_{m_k}, y_{m_k}), g(y_{m_k}, x_{m_k}))) \geq \epsilon,$$
$$D_s^{max}((g(x_{n_k-1}, y_{n_k-1}), g(y_{n_k-1}, x_{n_k-1})), (g(x_{n_k-1}, y_{n_k-1}), g(y_{n_k-1}, x_{n_k-1})), (g(x_{m_k}, y_{m_k}), g(y_{m_k}, x_{m_k}))) < \epsilon.$$

By definition of D_s^{max}, we have

$$\begin{aligned} r_k := \max\{&S(g(x_{n_k}, y_{n_k}), g(x_{n_k}, y_{n_k}), g(x_{m_k}, y_{m_k})), \\ &S(g(y_{n_k}, x_{n_k}), g(y_{n_k}, x_{n_k}), g(y_{m_k}, x_{m_k}))\} \\ \geq& \epsilon, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \max\{&S(g(x_{n_k-1}, y_{n_k-1}), g(x_{n_k-1}, y_{n_k-1}), g(x_{m_k}, y_{m_k})), \\ &S(g(y_{n_k-1}, x_{n_k-1}), g(y_{n_k-1}, x_{n_k-1}), g(y_{m_k}, x_{m_k}))\} < \epsilon. \end{aligned} \tag{3.25}$$

By using (3.24), (3.25) and Lemma 2.2, we have that

$$\begin{aligned}\epsilon \le r_k &:= \max\{S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k})),S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k}))\}\\ &\le \max\{S(g(x_{n_k-1},y_{n_k-1}),g(x_{n_k-1},y_{n_k-1}),g(x_{m_k},y_{m_k})),S(g(y_{n_k-1},x_{n_k-1}),g(y_{n_k-1},x_{n_k-1}),g(y_{m_k},x_{m_k}))\}\\ &\quad+\max\{2S(g(x_{n_k-1},y_{n_k-1}),g(x_{n_k-1},y_{n_k-1}),g(x_{n_k},y_{n_k})),2S(g(y_{n_k-1},x_{n_k-1}),g(y_{n_k-1},x_{n_k-1}),g(y_{n_k},x_{n_k}))\}\\ &< 2\max\{S(g(x_{n_k-1},y_{n_k-1}),g(x_{n_k-1},y_{n_k-1}),g(x_{n_k},y_{n_k})),S(g(y_{n_k-1},x_{n_k-1}),g(y_{n_k-1},x_{n_k-1}),g(y_{n_k},x_{n_k}))\}+\epsilon.\end{aligned}$$

On taking the limit as $k\to\infty$, we have

$$r_k := \max\{S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k})),S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k}))\}\to\epsilon. \tag{3.26}$$

By Lemma 2.2, we have

$$\begin{aligned}r_k &= \max\{S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k})),S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k}))\}\\ &\le \max\{2S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{n_k+1},y_{n_k+1})),2S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{n_k+1},x_{n_k+1}))\}\\ &\quad+\max\{S(g(x_{n_k+1},y_{n_k+1}),g(x_{n_k+1},y_{n_k+1}),g(x_{m_k},y_{m_k})),S(g(y_{n_k+1},x_{n_k+1}),g(y_{n_k+1},x_{n_k+1}),g(y_{m_k},x_{m_k}))\}\\ &< 2d_{n_k}+\max\{2S(g(x_{m_k},y_{m_k}),g(x_{m_k},y_{m_k}),g(x_{m_k+1},y_{m_k+1})),2S(g(y_{m_k},x_{m_k}),g(y_{m_k},x_{m_k}),g(y_{m_k+1},x_{m_k+1}))\}\\ &\quad+\max\{S(g(x_{n_k+1},y_{n_k+1}),g(x_{n_k+1},y_{n_k+1}),g(x_{m_k+1},y_{m_k+1})),\\ &\quad S(g(y_{n_k+1},x_{n_k+1}),g(y_{n_k+1},x_{n_k+1}),g(y_{m_k+1},x_{m_k+1}))\}\\ &= 2d_{n_k}+2d_{m_k}+\max\{S(f(x_{n_k},y_{n_k}),f(x_{n_k},y_{n_k}),f(x_{m_k},y_{m_k})),S(f(y_{n_k},x_{n_k}),f(y_{n_k},x_{n_k}),f(y_{m_k},x_{m_k}))\}\\ &< 2d_{n_k}+2d_{m_k}+\theta(S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k})),S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k})))\\ &\quad\times\max\{S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k})),S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k}))\}\\ &\le 2d_{n_k}+2d_{m_k}+r_k.\end{aligned}$$

On taking the limit as $k\to\infty$ and using (3.23) and (3.26), we get

$$\theta(S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k})),S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k})))\to 1.$$

By using the property of θ, we obtain

$$S(g(x_{n_k},y_{n_k}),g(x_{n_k},y_{n_k}),g(x_{m_k},y_{m_k}))\to 0,$$

and

$$S(g(y_{n_k},x_{n_k}),g(y_{n_k},x_{n_k}),g(y_{m_k},x_{m_k}))\to 0,$$

as $k\to\infty$, which implies that

$$\lim_{k\to\infty} r_k = 0,$$

which contradicts with $\epsilon > 0$.

Therefore, $\{g(x_n,y_n),g(y_n,x_n)\}$ is a Cauchy sequence in $(X\times X, D_s^{max})$. Since X is S-complete, by Lemma 2.8, there exists $(u,v)\in X\times X$ such that

$$\lim_{n\to\infty} g(x_n,y_n)=\lim_{n\to\infty} f(x_n,y_n)=u \quad\text{and}\quad \lim_{n\to\infty} g(y_n,x_n)=\lim_{n\to\infty} f(y_n,x_n)=v. \tag{3.27}$$

Since the pair (f,g) satisfies the generalized compatibility, from (3.27), we get that

$$\lim_{n\to\infty} S(f(g(x_n,y_n),g(y_n,x_n)),f(g(x_n,y_n),g(y_n,x_n)),g(f(x_n,y_n),f(y_n,x_n)))=0. \tag{3.28}$$

and

$$\lim_{n\to\infty} S(f(g(y_n,x_n),g(x_n,y_n)),f(g(y_n,x_n),g(x_n,y_n)),g(f(y_n,x_n),f(x_n,y_n)))=0.$$

Now, consider the assumption 5 that (a) holds, that is, f and g are continuous. By using Lemma 2.2, we get that

$$\begin{aligned} S(g(u,v),g(u,v),f(g(x_n,y_n),g(y_n,x_n))) \le\ & 2S(g(u,v),g(u,v),g(f(x_n,y_n),f(y_n,x_n))) \\ & + S(f(g(x_n,y_n),g(y_n,x_n)),f(g(x_n,y_n), \\ & g(y_n,x_n)),g(f(x_n,y_n),f(y_n,x_n))). \end{aligned}$$

By passing to the limit as $n \to \infty$ and using (3.27), (3.28) and the continuity of f, we get that

$$S(g(u,v),g(u,v),f(u,v)) = 0.$$

Hence, $g(u,v) = f(u,v)$. In similar way $g(v,u) = f(v,u)$ is obtained.

Now, consider the assumption 5 that (b) holds. By (3.17) and (3.27), we have $\{g(x_n,y_n)\}$ and $\{g(y_n,x_n)\}$ are nondecreasing sequences, $g(x_n,y_n) \to u$ and $g(y_n,x_n) \to v$, as $n \to \infty$. Thus, for all $n \in \mathbb{N}$, we have

$$g(x_n,y_n) \preceq u, g(y_n,x_n) \preceq v.$$

By using (3.14) and Lemma 2.2, we get

$$\begin{aligned} S(f(u,v),f(u,v),g(u,v)) &\le 2S(f(u,v),f(u,v),g(x_{n+1},y_{n+1})) + S(g(x_{n+1},y_{n+1}),g(x_{n+1},y_{n+1}),g(u,v)) \\ &= 2S(f(u,v),f(u,v),f(x_n,y_n)) + S(g(x_{n+1},y_{n+1}),g(x_{n+1},y_{n+1}),g(u,v)) \\ &\to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, $g(u,v) = f(u,v)$. In similar way, we get $g(v,u) = f(v,u)$. □

The commuting maps (f,g) are obviously generalized compatible, then we obtain the following corollary.

Corollary 3.7. *Let $(X,S,\preceq)$ be a partially ordered S-metric space, $f,g : X \times X \mapsto X$ be two maps such that*

1. *X is S-complete;*
2. *f,g are two commuting maps such that f is g-nondecreasing with respect to $\preceq$ and there exist $x_0,y_0 \in X$ such that $g(x_0,y_0) \preceq f(x_0,y_0)$ and $g(y_0,x_0) \preceq f(y_0,x_0)$;*
3. *there exists $\theta \in \Theta$ such that*

$$\begin{aligned} S(f(x,y),f(x,y),f(u,v)) \le\ & \theta(S(g(x,y),g(x,y),g(u,v)),S(g(y,x),g(y,x),g(v,u))) \\ & \times \max\{S(g(x,y),g(x,y),g(u,v)),S(g(y,x),g(y,x),g(v,u))\} \end{aligned}$$

   for all $x,y,u,v \in X$ with $g(x,y) \preceq g(u,v)$ and $g(y,x) \preceq g(v,u)$;

4. *for any $x,y \in X$, there exist $u,v \in X$ such that*

$$f(x,y) = g(u,v), f(y,x) = g(v,u);$$

5. (a) *f and g are continuous; or*
   (b) *X has the following property:*
   if a nondecreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f and g have a coupled coincidence point in X.

Definition 3.8 ([11, Definition 1.1])**.** Let $(X,\preceq)$ be a partially ordered set and let $f : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping f is said to have the g-monotone property, if f is monotone

g-nondecreasing in both of its arguments, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow f(x_1, y) \preceq f(x_2, y),$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow f(x, y_1) \preceq f(x, y_2).$$

If we take $g = I_X$ (an identity mapping on X), then f is a monotone mapping on X.

Definition 3.9. Let (X, S) be an S-metric space and let $f : X \times X \mapsto X$ and $g : X \mapsto X$ be two maps. We say the pair (f, g) is compatible, if

$$S(f(g(x_n), g(y_n)), f(g(x_n), g(y_n)), g(f(x_n, y_n))) \to 0, \quad \text{as} \;\; n \to \infty,$$
$$S(f(g(y_n), g(x_n)), f(g(y_n), g(x_n)), g(f(y_n, x_n))) \to 0, \quad \text{as} \;\; n \to \infty,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = t_1,$$
$$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = t_2.$$

Definition 3.10. Let $f : X \times X \mapsto X$ and $g : X \mapsto X$ be two maps. We say the pair (f, g) is commuting if, $f(gx, gy) = g(f(x, y), f(y, x))$ for all $x, y \in X$.

Now, we deduce some analogous results to Kadelburg et al. [6] in partially ordered S-metric spaces.

Corollary 3.11. *Let $(X, S, \preceq)$ be a partially ordered complete S-metric space, $f : X \times X \mapsto X$ and $g : X \mapsto X$ be such that f has g-monotone property. Suppose that the following hold:*

1. *g is continuous and $g(X)$ is closed;*
2. *$f(X \times X) \subset g(X)$ and f and g are compatible;*
3. *there exist $x_0, y_0 \in X$ such that $gx_0 \preceq f(x_0, y_0)$ and $gy_0 \preceq f(y_0, x_0)$;*
4. *there exists $\theta \in \Theta$ such that for all $x, y, u, v \in X$*

$$S(f(x, y), f(x, y), f(u, v)) \le \theta(S(gx, gx, gu), S(gy, gy, gv)) \times \max\{S(gx, gx, gu), S(gy, gy, gv)\},$$

   with $gx \preceq gu$ and $gy \preceq gv$;

5. (a) *f is continuous; or*
   (b) *X has the following property:*
   if a nondecreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f and g have a coupled coincidence point in X.

Proof. By starting from x_0, y_0 in assumption 3 and using $f(X \times X) \subset g(X)$ in assumption 2, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_n = f(x_{n-1}, y_{n-1}), gy_n = f(y_{n-1}, x_{n-1}), \quad \text{for} \;\; n = 1, 2, \cdots .$$

Then, proceeding the proof of Theorem 3.4 by using the two sequences mentioned above, we can draw the conclusion. □

Remark 3.12. Since commuting maps (f, g) are necessarily compatible, then the conclusion of Corollary 3.11 holds true by using (f, g) is commuting instead of compatibility of f and g.

By putting $g = I_x$, where I_x is an identity mapping on X in Corollary 3.11, we obtain the following corollary.

Corollary 3.13. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and let $f : X \times X \mapsto X$ have the monotone property. Suppose that the following hold:*

1. *there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \preceq f(y_0, x_0)$;*
2. *there exists $\theta \in \Theta$ such that for all $x, y, u, v \in X$,*
$$S(f(x,y), f(x,y), f(u,v)) \leq \theta(S(x,x,u), S(y,y,v)) \times \max\{S(x,x,u), S(y,y,v)\},$$
with $x \preceq u$ and $y \preceq v$;
3. (a) *f is continuous; or*
   (b) *X has the following property:*
   if a nondecreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f has a fixed point in X.

By taking $\theta(t_1, t_2) = k$ with $k \in [0, 1)$ for all $t_1, t_2 \in [0, \infty)$ in Corollary 3.11 and Corollary 3.13, we obtain the following corollaries.

Corollary 3.14. *Let $(X, S, \preceq)$ be a partially ordered complete S-metric space, $f : X \times X \mapsto X$ and $g : X \mapsto X$ be such that f has g-monotone property. Suppose that the following hold:*

1. *g is continuous and $g(X)$ is closed;*
2. *$f(X \times X) \subset g(X)$ and f and g are compatible;*
3. *there exist $x_0, y_0 \in X$ such that $gx_0 \preceq f(x_0, y_0)$ and $gy_0 \preceq f(y_0, x_0)$;*
4. *there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$,*
$$S(f(x,y), f(x,y), f(u,v)) \leq k \times \max\{S(gx, gx, gu), S(gy, gy, gv)\},$$
with $gx \preceq gu$ and $gy \preceq gv$;
5. (a) *f is continuous; or*
   (b) *X has the following property:*
   if a nondecreasing sequence $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f and g have a coupled coincidence point in X.

Corollary 3.15. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and let $f : X \times X \mapsto X$ have the monotone property. Suppose that the following hold:*

1. *there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \preceq f(y_0, x_0)$;*
2. *there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$,*
$$S(f(x,y), f(x,y), f(u,v)) \leq k \times \max\{S(x,x,u), S(y,y,v)\},$$
with $x \preceq u$ and $y \preceq v$;
3. (a) *f is continuous; or*

(b) *X has the following property:*
if a nondecreasing sequence $\{x_n\} \to x$ *as* $n \to \infty$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

Then f has a fixed point in X.

Remark 3.16. Since, for $k, l \geq 0$, $k + l \leq 1$,

$$kS(gx, gx, gu) + lS(gy, gy, gv) \leq \max\{S(gx, gx, gu), S(gy, gy, gv)\},$$

Corollary 3.14 (resp. Corollary 3.15) remains valid, if the right-hand side of assumption 5 of Corollary 3.14 (resp. assumption 2 of Corollary 3.15) is replaced by

$$kS(gx, gx, gu) + lS(gy, gy, gv) \quad (\text{resp. } kS(x, x, u) + lS(y, y, v))$$

for some $k, l \geq 0$, $k + l < 1$.

Now, we prove the uniqueness of the coupled coincidence point. Note that if $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x, y), (u, v) \in X \times X$

$$(x, y) \preceq (u, v) \Leftrightarrow g(x, y) \preceq g(u, v) \wedge g(y, x) \preceq g(v, u),$$

where $g : X \times X \mapsto X \times X$ is one-one.

Theorem 3.17. *In addition to the hypotheses of Theorem* 3.4*, suppose the following condition holds:*

(∗) *for every* $(x, y), (z, t) \in X \times X$, *there exists another* $(u, v) \in X \times X$ *which is comparable to* (x, y) *and* (z, t).

Then f and g have a unique coupled coincidence point.

Proof. From Theorem 3.4, the set of coupled coincidence points of f and g is nonempty.

Suppose (x, y) and (z, t) are coupled coincidence points of f and g, that is,

$$f(x, y) = g(x, y), \quad f(y, x) = g(y, x);$$
$$f(z, t) = g(z, t), \quad f(t, z) = g(t, z).$$

Now we prove that $g(x, y) = g(z, t)$ and $g(y, x) = g(t, z)$. By assumption, there exists $(u, v) \in X \times X$ that is comparable to (x, y) and (z, t). We define sequences $\{g(u_n, v_n)\}$ and $\{g(v_n, u_n)\}$ as follows, with $u_0 = u, v_0 = v$:

$$f(u_n, v_n) = g(u_{n+1}, v_{n+1}), \quad f(v_n, u_n) = g(v_{n+1}, u_{n+1}), \quad \text{for all } n \in \mathbb{N}.$$

Since (u, v) is comparable to (x, y), we assume that $(x, y) \preceq (u, v) = (u_0, v_0)$, which implies that $g(x, y) \preceq g(u_0, v_0)$ and $g(y, x) \preceq g(v_0, u_0)$. We suppose that $(x, y) \preceq (u_n, v_n)$ for some n.

We will prove that

$$(x, y) \leq (u_{n+1}, v_{n+1}).$$

Since f is g-monotone, we have that $g(x, y) \preceq g(u_n, v_n)$ implies $f(x, y) \preceq f(u_n, v_n)$ and $g(y, x) \preceq g(v_n, u_n)$ implies $f(y, x) \preceq f(v_n, u_n)$. Now

$$g(x, y) = f(x, y) \preceq f(u_n, v_n) = g(u_{n+1}, v_{n+1}),$$
$$g(y, x) = f(y, x) \preceq f(v_n, u_n) = g(v_{n+1}, u_{n+1}).$$

Thus we have

$$(x, y) \preceq (u_n, v_n), \quad \text{for all } n \in \mathbb{N}. \tag{3.29}$$

By using (3.1) and (3.29), we have

$$\begin{aligned}S(g(x,y),g(x,y),g(u_{n+1},v_{n+1})) &= S(f(x,y),f(x,y),f(u_n,v_n)\\ &\le \theta(S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n)))\\ &\quad\times \max\{S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))\}.\end{aligned}$$

Similarly

$$\begin{aligned}S(g(y,x),g(y,x),g(v_{n+1},u_{n+1})) &= S(f(y,x),f(y,x),f(v_n,u_n)\\ &\le \theta(S(g(y,x),g(y,x),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n)))\\ &\quad\times \max\{S(g(y,x),g(y,x),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))\}.\end{aligned}$$

This implies that

$$\begin{aligned}\max\{S(g(x,y),g(x,y),&g(u_{n+1},v_{n+1})),S(g(y,x),g(y,x),g(v_{n+1},u_{n+1}))\}\\ &\le \theta(S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n)))\\ &\quad\times \max\{S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))\}\\ &< \max\{S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))\}.\end{aligned}$$

Therefore, we get that $d_n := \max\{S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))\}$ is decreasing and hence $d_n \to d$ as $n \to \infty$, for some $d \ge 0$. Now, we claim that $d = 0$. Assume to the contrary that $d > 0$, from above inequality, we get

$$\begin{aligned}&\frac{\max\{S(g(x,y),g(x,y),g(u_{n+1},v_{n+1})),S(g(y,x),g(y,x),g(v_{n+1},u_{n+1}))\}}{\max\{S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))\}}\\ &\qquad\le \theta(S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n)))\\ &\qquad< 1.\end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality, we have

$$\theta(S(g(x,y),g(x,y),g(u_n,v_n)),S(g(y,x),g(y,x),g(v_n,u_n))) \to 1, \quad \text{as} \quad n \to \infty.$$

By the property (θ_2) of $\theta \in \Theta$, we have

$$S(g(x,y),g(x,y),g(u_n,v_n)) \to 0, \quad S(g(y,x),g(y,x),g(v_n,u_n)) \to 0, \quad \text{as} \quad n \to \infty.$$

Now we have $d_n \to 0$ as $n \to \infty$, which contradicts with $d > 0$. Therefore, we conclude that $d_n \to 0$, as $n \to \infty$ and then

$$\lim_{n\to\infty} S(g(x,y),g(x,y),g(u_n,v_n)) = 0, \quad \lim_{n\to\infty} S(g(y,x),g(y,x),g(v_n,u_n)) = 0. \tag{3.30}$$

Similarly, we have

$$\lim_{n\to\infty} S(g(z,t),g(z,t),g(u_n,v_n)) = 0, \quad \lim_{n\to\infty} S(g(t,z),g(t,z),g(v_n,u_n)) = 0. \tag{3.31}$$

By using (3.30) and (3.31), we have $g(x,y) = g(z,t)$ and $g(y,x) = g(t,z)$. □

Corollary 3.18. *In addition to the hypotheses of Corollary* 3.14*, assume that for any* $(x,y),(z,t) \in X \times X$*, there exists another* $(u,v) \in X \times X$ *which is comparable to* (x,y) *and* (z,t)*, then* f *and* g *have a unique coupled fixed point.*

Theorem 3.19. *In addition to the hypotheses of Corollary* 3.13*, let the condition* $(*)$ *of Theorem* 3.17 *be satisfied. Then the coupled fixed point of* f *is unique. Moreover, if for the terms of sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by* $x_n = f(x_{n-1}, y_{n-1})$ *and* $y_n = f(y_{n-1}, x_{n-1})$*,* $x_n \preceq y_n$ *holds for* n *sufficiently large, then the coupled fixed point of* f *has the form* (x, x)*.*

Proof. We only prove the last assertion. Suppose that for n sufficiently large, $x_n \preceq y_n$. Then by (3.1) of Theorem 3.4 (with $g = I_x$), we get

$$\begin{aligned} S(x_{n+1}, x_{n+1}, y_{n+1}) &= S(f(x_n, y_n), f(x_n, y_n), f(y_n, x_n)) \\ &\leq \theta(S(x_n, x_n, y_n), S(y_n, y_n, x_n)) \times \max\{S(x_n, x_n, y_n), S(y_n, y_n, x_n)\} \\ &= \theta(S(x_n, x_n, y_n), S(y_n, y_n, x_n)) \times S(x_n, x_n, y_n) \\ &< S(x_n, x_n, y_n). \end{aligned}$$

Therefore, we get that $d_n := S(x_n, x_n, y_n)$ is decreasing. Hence, $d_n \to d$ as $n \to \infty$ for some $d \geq 0$.
Next, we prove that $d = 0$. Assume to the contrary that $d > 0$, then from above inequality, we have

$$\frac{S(x_{n+1}, x_{n+1}, y_{n+1})}{S(x_n, x_n, y_n)} \leq \theta(S(x_n, x_n, y_n), S(y_n, y_n, x_n)) < 1.$$

By letting $n \to \infty$, we get $\theta(S(x_n, x_n, y_n), S(y_n, y_n, x_n)) \to 1$. Since $\theta \in \Theta$, we have $S(x_n, x_n, y_n) \to 0$, which contradicts with $d > 0$. Therefore, we have $d_n \to 0$ as $n \to \infty$.

By Lemma 2.2,

$$\begin{aligned} S(x, x, y) &\leq 2S(x, x, x_{n+1}) + S(x_{n+1}, x_{n+1}, y) \\ &\leq 2S(x, x, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, y_{n+1}) + S(y_{n+1}, y_{n+1}, y) \\ &\leq 2S(x, x, x_{n+1}) + 2S(x_n, x_n, y_n) + S(y_{n+1}, y_{n+1}, y). \end{aligned}$$

By passing to the limit as $n \to \infty$, since $x_n \to x$, $y_n \to y$ and $S(x_n, x_n, y_n) \to 0$, we get that $S(x, x, y) \leq 0$ and thus $x = y$. □

Corollary 3.20. *In addition to the hypotheses of Corollary* 3.15*, let the condition* $(*)$ *of Theorem* 3.17 *be satisfied. Then the coupled fixed point of* f *is unique. Moreover, if for the terms of sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by* $x_n = f(x_{n-1}, y_{n-1})$ *and* $y_n = f(y_{n-1}, x_{n-1})$*,* $x_n \preceq y_n$ *holds for* n *sufficiently large, then the coupled fixed point of* f *has the form* (x, x)*.*

At last, we present an example to show that our result can be used when many results in this field cannot.

Example 3.21. Let $X = [0, 1]$ endowed with the natural ordering of real numbers and S-metric defined by $S(x, y, z) = \frac{1}{16}(|x - z| + |y - z|)$, for any $x, y, z \in X$. Then (X, S) is a complete S-metric space.
Define the mappings $f, g : X \times X \mapsto X$ as follows:

$$f(x, y) = \begin{cases} x^2 - y^2, & x \geq y, \\ 0, & x < y. \end{cases}$$

and

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{4}, & x \geq y, \\ 0, & x < y. \end{cases}$$

First of all, we prove that f is g-monotone with respect to the natural ordering of real numbers.

Let $(x, y), (u, v) \in X \times X$ with $g(x, y) \leq g(u, v)$. We consider the following cases.
Case1: If $x < y$, then $f(x, y) = 0 \leq f(u, v)$.

Case2: If $x \geq y$ and if $u \geq v$, then

$$g(x,y) \leq g(u,v) \Rightarrow \frac{x^2-y^2}{4} \leq \frac{u^2-v^2}{4} \Rightarrow x^2-y^2 \leq u^2-v^2 \Rightarrow f(x,y) \leq f(u,v).$$

But if $u < v$, then

$$g(x,y) \leq g(u,v) \Rightarrow 0 \leq \frac{x^2-y^2}{4} \leq 0 \Rightarrow x^2 = y^2 \Rightarrow f(x,y) = 0 \leq f(u,v).$$

Thus we see that f is g-monotone.

Now we prove that for any $x, y \in X$ there exist $u, v \in X$ such that

$$f(x,y) = g(u,v),$$
$$f(y,x) = g(v,u).$$

Let $(x,y) \in X \times X$ be fixed. We consider the following cases.

Case1: If $x = y$, then we have $f(x,y) = 0 = g(x,y)$ and $f(y,x) = 0 = g(y,x)$.

Case2: If $x > y$, then we have $f(x,y) = x^2 - y^2 = g(2x,2y)$ and $f(y,x) = 0 = g(2y,2x)$.

Case3: If $x < y$, then we have $f(x,y) = 0 = g(2x,2y)$ and $f(y,x) = y^2 - x^2 = g(2y,2x)$.

Clearly, f and g are continuous.

Now, we prove that the pair (f,g) satisfies the generalized compatibility hypothesis.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty} g(x_n,y_n) = t_1,$$
$$\lim_{n\to\infty} f(y_n,x_n) = \lim_{n\to\infty} g(y_n,x_n) = t_2.$$

Then we must have that $t_1 = t_2 = 0$ and one can easily check that

$$\lim_{n\to\infty} S(f(g(x_n,y_n),g(y_n,x_n)), f(g(x_n,y_n),g(y_n,x_n)), g(f(x_n,y_n),f(y_n,x_n))) = 0, \quad \text{as} \quad n\to\infty,$$
$$\lim_{n\to\infty} S(f(g(y_n,x_n),g(x_n,y_n)), f(g(y_n,x_n),g(x_n,y_n)), g(f(y_n,x_n),f(x_n,y_n))) = 0, \quad \text{as} \quad n\to\infty.$$

Next, we prove that there exist two $x_0, y_0 \in X$ with $g(x_0,y_0) \leq f(x_0,y_0)$ and $g(y_0,x_0) \leq f(y_0,x_0)$.

Since we have $g(0,\frac{1}{2}) = 0 = f(0,\frac{1}{2})$ and $g(\frac{1}{2},0) = \frac{1}{16} < f(\frac{1}{2},0)) = \frac{1}{4}$.

Finally, we verify the contraction condition (3) of Theorem 3.4, for all $x, y, u, v \in X$ with $g(x,y) \leq g(u,v)$ and $g(y,x) \leq g(v,u)$.

Let $\theta \in \Theta$ be defined by $\theta(s,t) = \frac{1}{2}$, for $s, t \in [0,\infty)$.

$$\begin{aligned} S(f(x,y),f(x,y),f(u,v)) &= \frac{1}{16}(|f(x,y)-f(u,v)| + |f(x,y)-f(u,v)|) \\ &= \frac{1}{8}|f(x,y)-f(u,v)| \\ &= \frac{1}{2}|g(x,y)-g(u,v)| \\ &\leq \theta(S(g(x,y),g(x,y),g(u,v),S(g(y,x),g(y,x),g(v,u))) \\ &\quad \times \max\{S(g(x,y),g(x,y),g(u,v),S(g(y,x),g(y,x),g(v,u))\}. \end{aligned}$$

Hence, the condition (3) of Theorem 3.4 is satisfied.

All requirements of Theorem 3.4 are satisfied and $(0,0)$ is a coupled coincidence point of f and g. Moreover, $(0,0)$ is a unique coupled common fixed point of f and g.

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