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# Existence and uniqueness for solutions of parabolic quasi-variational inequalities with impulse control and nonlinear source terms

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**Abstract**

In this paper, we present a new proof for the existence and uniqueness of solutions of parabolic quasi-variational inequalities with impulse control. We prove some properties of the presented algorithm (see [S. Boulaaras, M. Haiour, Appl. Math. Comput., **217** (2011), 6443–6450], [S. Boulaaras, M. Haiour, Indaga. Math., **24** (2013), 161–173]) using a semi-implicit scheme with respect to the $t$-variable combined with a finite element spatial approximation. 

*Keywords:* Parabolic quasi-variational inequality, finite elements, semi-implicit scheme, contraction, fixed point, impulse control problem.
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## 1. Introduction

The aim of this paper is to extend the results of M. Boulbrachene and M. Haiour [7] and P. Cortey-Dumont [8], who established the existence, uniqueness and error estimates for the solutions of elliptic variational and quasi-variational inequalities. Here we use a new idea based on the algorithm of Bensoussan and Lions, which has been given for evolutionary free boundary problems, using the concept of $L^\infty$-stability [7], in order to present a new proof for the existence and uniqueness of the solutions of Parabolic Quasi-Variational Inequalities (PQVIs) with respect to the right-hand side as a nonlinear source term and an obstacle defined as an impulse control problem.

Namely, we consider the following PQVIs: find $u \in L^2\left(0, T, H^1\right)$ such that

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$$\begin{cases} \dfrac{\partial u}{\partial t} + Au \leq f(u) \text{ in } \Sigma, \\ u \leq Mu, \\ \left( \dfrac{\partial u}{\partial t} + Au - f(u) \right)(u - Mu) = 0, \\ u(0,x) = u_0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where

- $\Sigma = \Omega \times [0, T]$ is a set in $\mathbb{R} \times \mathbb{R}^n$ such that $T < +\infty$ and $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$ with sufficiently smooth boundary $\Gamma$
- $A$ is an operator defined over $H^1(\Omega)$ by

$$Au = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} + a_0(x) u, \tag{1.2}$$

and $a(\cdot, \cdot)$ is the bilinear form associated with operator $A$, given by

$$a(u,v) = \int_\Omega \left( \sum_{ij=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} v + a_0(x) uv \right) dx, \tag{1.3}$$

assumed to be noncoercive, and whose coefficients $a_{i,j}(x)$, $b_j(x)$, $a_0(x) \in L^\infty(\Omega) \cap C_2(\bar{\Omega})$, $x \in \bar{\Omega}$, $1 \leq i, j \leq n$, are sufficiently smooth and satisfy the following conditions:

$$a_{ij}(x) = a_{ji}(x), \ a_0(x) \geq \beta > 0, \quad \beta \in \mathbb{R} - \text{constant}, \tag{1.4}$$

$$\sum_{i\ j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma |\xi|^2, \ \xi \in \mathbb{R}^2, \ \gamma > 0, \ x \in \bar{\Omega}. \tag{1.5}$$

- $f(\cdot)$ is a Lipschitz increasing nonlinear source term such that

$$f \in L^2(0, T, L^\infty(\Omega)) \cap C^1\left(0, T, H^{-1}(\Omega)\right), \ f \geq 0, \tag{1.6}$$

with rate $c$ satisfying

$$c \leq \beta. \tag{1.7}$$

- $M$ is an operator given by

$$Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x + \xi), \tag{1.8}$$

where $k > 0$ and

$$Mu \in L^2\left(0, T, W^{2,\infty}(\Omega)\right). \tag{1.9}$$

As shown in [14], $M$ is concave, i.e., for $u, v \in C(\Omega)$,

$$M(\delta u + (1-\delta) v) \geq \delta M(u) + (1-\delta) M(v). \tag{1.10}$$

Additionally, the following holds:

$$\forall \eta \in \mathbb{R}, M(u + \eta) = M(u) + \eta. \tag{1.11}$$

We will use the notation $(\cdot,\cdot)_\Omega$ for the inner product in $L^2(\Omega)$.

Stationary free boundary problems are encountered in several applications. For example, in stochastic control, the solution of (1.1) characterizes the infimum of the cost function associated to an optimally controlled stochastic switching process without costs for switching and for the calculus of quasi-stationary states for the simulation of petroleum or gaseous deposits (see [2]). From the mathematical analysis point of view, the elliptic case of the problem (1.1) was studied intensively in the late 1980s ([1, 9, 10, 11, 12]; for the numerical and computational side see [1], [5, 6, 7]). However, as far as finite element approximation is concerned, only a few works are known in the literature ([6, 7, 12]).

In [7] we applied a new time-space discretization using the semi-implicit time scheme combined with a finite element spatial approximation. We found that (1.1) can be transformed into a full-discrete system of elliptic quasi-variational inequalities, we proposed a new iterative discrete algorithm to show the existence and uniqueness of the discrete solution, and we gave a simple proof for asymptotic behavior in the $L^\infty$ -norm using the theta time scheme combined with a finite element spatial approximation. Also, in [3], we analyzed the stability in the uniform norm for the theta-scheme with respect to the $t$-variable combined with a finite element spatial approximation for the evolutionary variational inequalities and quasi-variational inequalities with an obstacle defined as an impulse control problem.

In this paper we present a new proof for the existence and uniqueness for PQVIs. It consists of four steps, and it is based on some properties of the presented discrete iterative algorithm using the semi-implicit scheme with respect to the $t$-variable combined with a finite element spatial approximation. This paper is structured as follows. In Sections 2 and 3 we provide some definitions, assumptions, notations and standard propositions needed throughout the paper, and we associate with the discrete system of EQVIs a fixed point mapping, which we use to define the discrete algorithm based on the semi-implicit time scheme. We introduce a monotone iterative scheme based on Bensoussan's algorithm, and study some of its properties. These properties together with the subsolutions concepts will play a crucial role in proving the existence and uniqueness of solutions for the problem introduced in this paper, knowing that the proof is based on the $L^\infty$-stability of the solution with respect to the right-hand side and its characterization as the least upper bound of the subsolutions set (see also [6, 7]). It is worth mentioning that this approach is entirely different from the one developed for the evolutionary problem. Also, it is used for the first time in the case of QVIs. In Section 4 we present the main result, with a new proof for the existence and uniqueness of solutions of PQVIs with nonlinear source terms. Finally, we provide some conclusions and perspectives for further studies.

## 2. Parabolic quasi-variational inequalities

After a few simple computations and by using Green's formula, (1.1) can be transformed into the following continuous parabolic quasi-variational inequality: find $u \in \left(L^2\left(0,T,H^1\left(\Omega\right)\right)\right)$ satisfying

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v-u\right) + a\left(u,v-u\right) \geq \left(f\left(u\right), v-u\right), \\ u \leq Mu, \ v \leq Mu, \\ u\left(0,x\right) = u_0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

where $a\left(\cdot,\cdot\right)$ is the bilinear form associated with operator $A$ defined in (1.2).

### *2.1. The time discretization*

We discretize the problem (2.1) with respect to time by using the semi-implicit scheme. Therefore, we search for a sequence of elements $u^k \in H_0^1\left(\Omega\right)$ which approaches $u\left(t_k\right)$, $t_k = k\Delta t$, with initial data $u^0 = u_0$. For $k = 1, \ldots, n$, we have

$$\begin{cases} \dfrac{u^k - u^{k-1}}{\Delta t} + Au^k \le f^k\left(u^k\right) \text{ in } \Sigma, \\ u \le Mu,\ v \le Mu, \\ u^0\left(x\right) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

First we define the mapping

$$T : L^\infty_+\left(\Omega\right) \longrightarrow L^\infty\left(\Omega\right),\ W \longrightarrow TW = \xi^k = \partial\left(F^k\left(w\right), Mu^k\right), \tag{2.3}$$

where $L^\infty_+\left(\Omega\right)$ denotes the positive cone of $L^\infty\left(\Omega\right)$, such that $\xi^k$ is the solution of the following problem:

$$\begin{cases} \dfrac{\xi^k - \xi^{k-1}}{\Delta t} + A\xi^k \le f\left(\xi^k\right) \text{ in } \Sigma, \\ \xi^k \le Mu. \end{cases} \tag{2.4}$$

*2.2. An iterative semi-discrete algorithm*

We choose $u^0 = u_0$ the solution of the semi-discrete equation

$$A^0 u = g^0. \tag{2.5}$$

$g^0$ is an $M$ regular function.

Now we give the semi-discrete algorithm

$$u^k = Tu^{k-1}, k = 1, \ldots, n,\ k = 1, \ldots, n, \tag{2.6}$$

where $u^k$ the solution of the problem (2.2).

*Remark* 2.1. Let

$$\mathbf{Q} = \left\{w \in L^\infty_+ : 0 \le w \le u^0\right\}, \tag{2.7}$$

where $u^0$ is the solution of (2.5). Since $f^k\left(\cdot\right) \ge 0$ and $u^0_h = u_{h0} \ge 0$, combining comparison results in variational inequalities with a simple induction, it follows that $u^k \ge 0$, i.e., $u^k \ge 0$, $\forall k = 1, \ldots, n$ and $Tw \ge 0$. Furthermore, by (2.6) and (2.7) we have

$$u^1 = Tu^0 \le u^0.$$

Similarly as in [6, 7], the mapping $T$ is monotone increasing for the stationary free boundary problem with nonlinear source term. Then it can be easily verified that

$$u^2 = Tu^1 \le Tu^0 = u^1 \le u^0,$$

thus, inductively,

$$u^{k+1} = Tu^k \le u^k \le \ldots \le u^0,\ \forall k = 1, \ldots, n,$$

and also it can be seen that the sequence $\left(u^k\right)_k$ stays in $\mathbf{Q}$.

According the assumption (1.6), $f\left(\cdot\right)$ is increasing and, by the previous remark, for $k = 1, \ldots, n$ we have

$$f\left(u^k\right) \le f\left(u^{k-1}\right).$$

Then we can rewrite (2.2) as follows:

$$\begin{cases} \dfrac{u^k - u^{k-1}}{\Delta t} + Au^k \leq f\left(u^{k-1}\right) \text{ in } \Sigma, \\ \xi^k \leq Mu, \\ \xi^0(x) = \xi_0 \text{ in } \Omega,\ \xi = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.8}$$

Also, (2.8) can be transformed into the following system of semi-discrete PQVIs:

$$\begin{cases} \left(\dfrac{u^k - u^{k-1}}{\Delta t}, v - u^k\right) + a\left(u^k, v - u^k\right) \geq \left(f\left(u^{k-1}\right), v - u^k\right), \\ u \leq Mu, \\ u(0, x) = u_0 \text{ in } \Omega. \end{cases} \tag{2.9}$$

*2.3. The spatial discretization*

Let $\Omega$ be decomposed into triangles and $\tau_h$ denote the set of all elements with mesh size $h > 0$. We assume that the family $\tau_h$ is regular and quasi-uniform. We consider the usual basis of affine functions $\varphi_l$, $l = \{1, \ldots, m(h)\}$ defined by $\varphi_l(M_s) = \delta_{ls}$ where $M_s$ is a vertex of the considered triangulation. We introduce the following discrete spaces $V^h$ of finite element:

$$V^h = \left\{v \in L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right) :\ v\mid_K \in P_1,\ K \in \tau_h, \text{ and } u(\cdot, 0) = u_0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\right\}, \tag{2.10}$$

where $r_h$ is the usual interpolation operator defined by

$$v \in L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right),\ r_h v = \sum_{i=1}^{m(h)} v(M_i)\varphi_i(x) \tag{2.11}$$

and $P_1$ denotes the space of polynomials with degree at most 1.

In this paper, we shall make use of the discrete maximum principle assumption (dmp). In other words, we shall assume that the matrices $(A)_{ps} = a(\varphi_p, \varphi_s)$ are $M$-matrices ([8]).

We discretize in space the problem (2.9), i.e. we approach the space $H_0^1$ by a space discretization of finite dimension $V^h \subset H_0^1$, and we get the following discrete PQVIs.

$$\begin{cases} \left(\dfrac{u_h^k - u_h^{k-1}}{\Delta t}, v_h - u_h^k\right) + a\left(u_h^k, v_h - u_h^k\right) \geq \left(f\left(u_h^{k-1}\right), v_h - u_h^k\right), \\ u_h^k \leq r_h M u_h^k, \\ u^0(x) = u_0 \text{ in } \Omega, \end{cases} \tag{2.12}$$

which implies

$$\begin{cases} \left(\dfrac{u_h^k}{\Delta t}, v_h - u_h^k\right) + a\left(u_h^k, v_h - u_h^k\right) \geq \left(f\left(u_h^{k-1}\right) + \dfrac{u_h^{k-1}}{\Delta t}, v_h - u_h^k\right), \\ u_h^k \leq r_h M u_h^k, \\ u_h^k(0) = u_{0h}^k \text{ in } \Omega. \end{cases} \tag{2.13}$$

Then, the problem (2.13) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities (EQVIs):

$$\begin{cases} b\left(u_h^k, v_h-u_h^k\right) \geq \left(f\left(u_h^{k-1}\right)+\lambda u_h^{k-1}, v_h-u_h^k\right), \quad u_h^k \in V^h, \\ u_h^k \leq r_h M u_h^k, \\ u_h^k(0)=u_{0h}^k \text{ in } \Omega, \end{cases} \tag{2.14}$$

such that

$$\begin{cases} b\left(u_h^k, v_h-u_h^k\right)=\lambda\left(u_h^k, v_h-u_h^k\right)+a\left(u_h^k, v_h-u_h^k\right), \quad u_h^k \in V^h, \\ \lambda=\dfrac{1}{\Delta t}=\dfrac{1}{k}=\dfrac{T}{n},\ k=1,\ldots,n. \end{cases} \tag{2.15}$$

*2.4. An iterative discrete algorithm*

As we have chosen before in the iterative semi-discrete algorithm, $u_h^0=u_{h0}$ is the solution of the following full-discrete equation

$$b\left(u_h^0, v_h\right)=\left(g^0, v_h\right),\ v_h \in V^h, \tag{2.16}$$

where $g^0$ is a linear and a regular function.

Now we give the full discrete algorithm

$$u_h^k=T_h u^{k-1}, k=1,\ldots,n, \tag{2.17}$$

where $u_h^k$ is the solution of the problem (2.14).

Let $F^{k-1}(w)=f(w)+\lambda w,\ \tilde{F}^{k-1}(\tilde{w})=f(\tilde{w})+\lambda\tilde{w} \in L^{\infty}(\Omega)$ be the corresponding right-hand sides to the EQVIs.

**Lemma 2.2** ([4, 6])**.** *Under the previous assumption and the dmp, if*

$$F^{k-1}(w) \geqq F^{k-1}(\tilde{w}),$$

*then*

$$u_h^k=\partial\left(F^{k-1}(w)\right) \geqq \tilde{u}_h^k=\partial\left(F^{k-1}(\tilde{w})\right).$$

We recall some results regarding coercive quasi-variational inequalities that are necessary to prove some useful qualitative properties.

**Definition 2.3.** $\zeta_h^k$ is said to be a subsolution for the system of EQVIs (2.14) if

$$\begin{cases} b\left(\zeta_h^k, \varphi_s\right) \leq \left(f+\lambda\zeta_h^{k-1}, \varphi_s\right),\ \forall\varphi_s, \quad s=1,\ldots,m(h), \\ \zeta_h^k \leq r_h M\zeta_h^k. \end{cases}$$

**Theorem 2.4** ([3])**.** *Under the discrete maximum principle, there exists a constant $\alpha>0$ such that*

$$b\left(u_h^k, u_h^k\right)=a\left(u_h^k, u_h^k\right)+\lambda\left(u_h^k, u_h^k\right) \geq \alpha\left\|u_h^k\right\|_{H^1(\Omega)}, \tag{2.18}$$

*where*

$$\lambda=\left(\frac{\|b_j\|_{\infty}^2}{2\gamma}+\frac{\gamma}{2}+\|a_0\|_{\infty}\right),\ \alpha=\frac{\gamma}{2}.$$

Let $X_h$ be the set of discrete subsolutions. Then, we have the following theorem.

**Theorem 2.5.** *Under the discrete maximum principle, the solution of the system of EQVIs* (2.14) *is the maximum element of* $X_h$.

*Proof.* We denote $\varphi^+ = \max(\varphi, 0)$, $\varphi^- = \max(-\varphi, 0)$.

Let $w_h \in V_h^i$ be a solution of the following of the full discrete system of parabolic quasi-variational inequalities using the theta time scheme combined with a finite element spatial approximation ([3, 4]):

$$\begin{cases} b\left(w_h, \breve{v}_h - w\right) \geq \left(f\left(w_h\right) + \lambda w_h, \tilde{v}_h - w_h^k\right), \ \forall \tilde{v}_h \in V^h, \\ \\ w_h \leq r_h M u_h^k, \ \tilde{v} \leq r_h M u_h^k, \end{cases} \tag{2.19}$$

where $\breve{v}_h = \sum\limits_{s=1}^{m(h)} \tilde{v}_s \varphi_s$. Since $\tilde{v}$ is a trial function, we choose $\tilde{v}_h = w_h - v_h$ and $v_h > 0$. Thus

$$b\left(w_h, \varphi_s\right) \leq \left(f\left(w_h\right) + \lambda w_h, \varphi_s\right), \tag{2.20}$$

that is to say $w_h \in X_h$. On the other hand, let $z_h$ be a subsolution such that

$$w_h \leq z_h. \tag{2.21}$$

Then we have

$$\begin{cases} b\left(z_h, \varphi_s\right) \leq \left(f\left(w_h\right) + \lambda w_h, \varphi_s\right) \\ \\ z_h \leq r_h M u_h^k. \end{cases}$$

Setting $v_h = (z_h - w_h)^+ \geq 0$ as a trial function, we obtain

$$\begin{cases} b\left(z_h, (z_h - w_h)^+\right) \leq \left(f\left(w_h\right) + \lambda w_h, (z_h - w_h)^+\right) \\ \\ z_h \leq r_h M u_h^k \end{cases}$$

and since $w_h$ is a subsolution too, we have

$$\begin{cases} b\left(w_h, (z_h - w_h)^+\right) \leq \left(f\left(w_h\right) + \lambda w_h, (z_h - w_h)^+\right) \\ \\ z_h \leq r_h M u_h^k. \end{cases}$$

Thus, we deduce that

$$-b\left((z_h - w_h)^+, (z_h - w_h)^+\right) \geq 0.$$

Under the coerciveness of the bilinear form, by using Theorem 2.4 we get

$$(z_h - w_h)^+ = 0,$$

therefore

$$z_h \leq w_h. \tag{2.22}$$

Thus, from (2.21) and (2.22) we obtain

$$z_h = w_h.$$

□

In this situation, the existence of a unique continuous solution to the stationary system can be handled in the spirit of [13], or by adapting the algorithmic approach developed for the coercive and noncoercive problems using Bensoussan's algorithm [7]. We provide only a brief description of this approach and skip over the proofs.

## 3. Existence and uniqueness for discrete PQVIs

Now, we shall give proofs for the existence and uniqueness for the solution of the system (2.14), using the algorithm based on a semi-implicit time scheme combined with a finite element approximation which was already used in previous research regarding evolutionary free boundary problems (see [4]).

*3.1. A fixed point mapping associated with the system of EQVIs*

We define the mapping

$$T_h : L^{\infty}_{+}(\Omega) \longrightarrow V^h,\ u \longrightarrow T_h u = \xi_h^k = \partial_h \left( F^k(u), r_h M u^k \right), \tag{3.1}$$

such that $\xi_h^k$ is the solution of the full discrete problem

$$\begin{cases} b\left(\xi_h^k, v_h - \zeta_h^k\right) \geq \left(F^{k-1}, v_h - \xi_h^k\right),\ v_h \in V^h, \\ \xi^k \leq M u^k,\ k = 1, \dots, n, \\ \xi^0(x) = \xi_0 \text{ in } \Omega,\ \xi = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $\xi_h^k = \partial_h\left(F^{k-1}(v), r_h M u^k\right)$, $\tilde{\xi}_h^k = \partial_h\left(G^{k-1}(w), r_h M w^k\right)$ be the corresponding solutions to the discrete EQVIs defined in (2.14).

**Proposition 3.1.** *Under the above assumptions, the solution $\partial_h(\cdot,\cdot)$ of (2.14) is increasing according the obstacle $r_h M w^k$ and the right hand side $F^{k-1} = f + \lambda w^{k-1}$, i.e., if we have*

$$F^{k-1} \leq G^{k-1} \text{ and } Mv^k \leq Mw^k,$$

*then*

$$\partial_h\left(F^{k-1}, r_h M v^k\right) \leq \partial_h\left(G^{k-1}, r_h M w^k\right).$$

*Proof.* Suppose that $F^{k-1} \leq G^{k-1}$ and $Mv^k \leq Mw^k$. Setting $u_1 = \partial_h\left(F^{k-1}, r_h M v^k\right)$ and $w_1 = \partial_h\left(G^{k-1}, r_h M w^k\right)$, we have from the proof of Theorem 2.5 that

$$\begin{cases} b\left(u_h^k, \varphi_s\right) \leq \left(F^{k-1}, \varphi_s\right), \\ u_h^k \leq r_h M u_h^k, \end{cases}$$

hence

$$\begin{cases} b\left(u_h^k, \varphi_s\right) \leq \left(F^{k-1}, \varphi_s\right) \leq \left(G^{k-1}, \varphi_s\right), \\ u_h^k \leq r_h M u^k \leq r_h M w_h^k, \end{cases}$$

and thus,

$$\begin{cases} b\left(u_h^k, \varphi_s\right) \leq \left(G^{k-1}, \varphi_s\right), \\ u_h^k \leq r_h M w_h^k. \end{cases}$$

It follows that $u_h^k$ is a subsolution for the solution $w_h^k$, that is to say that $u_h^k \leq w_h^k$. Therefore

$$\partial_h\left(F^{k-1}, r_h M u^k\right) \leq \partial_h\left(G^{k-1}, r_h M w^k\right).$$

□

**Lemma 3.2** (see [7])**.** *Let $\delta$ be a positive constant. Then*

$$\partial_h\left(F^{k-1}, r_h M u^k + \delta\right) = \partial_h\left(F^{k-1}, r_h M u^k\right) + \delta.$$

*Proof.* The proof is similar to that in [7] for the noncoercive case with a simple obstacle. □

**Proposition 3.3.** *Under the previous assumptions,*

$$\partial_h\left(F^{k-1}+G^{k-1},r_hMu^k+r_hMw^k\right)\geq\partial_h\left(F^{k-1},r_hMu^k\right)+\partial_h\left(G^{k-1},r_hMu^k\right),$$

*where* $\partial_h\left(F^{k-1},r_hMu^k\right)$ *is a solution of the problem* (2.14) *with the obstacle* $Mu^k$ *and the right hand side* $F^{k-1}$*, and* $\partial_h\left(G^{k-1},r_hMu^k\right)$ *is a solution of the problem* (2.14) *with the obstacle* $Mw^k$ *and the right hand side* $G^{k-1}$*.*

*Proof.* We set

$$u_h^k=\partial_h\left(F^k,r_hMu^k\right) \tag{3.3}$$

and

$$w_h^k=\partial_h\left(G^{k-1},r_hMw^k\right).$$

It is clear that (3.3) verify the system of EQVIs

$$\begin{cases} b\left(u_h^k,v_h-u_h^k\right)\geq\left(F^{k-1},v_h-u_h^k\right),\ v_h\in V^h,\\ u_h^k\leq r_hMu_h^k. \end{cases} \tag{3.4}$$

It follows that

$$\begin{cases} b\left(u_h^k+w_h,(v_h+w_h)-\left(u_h^k+w_h\right)\right)\geq\left(F^{k-1}+G^{k-1},(v_h+w_h)-\left(u_h^k+w_h\right)\right),\\ v_h+w_h^k\leq r_hMu_h^k+r_hMw_h^k,\\ u_h^k+w_h^k\leq r_hMu_h^k+r_hMw_h^k. \end{cases}$$

Considering the trial function $v_h=u_h^k-\zeta_h$ with $\zeta_h\geq 0$, we find

$$\begin{cases} b\left(u_h^k+w_h^k,\zeta_h\right)\leq\left(F^{k-1}+G^{k-1},\zeta_h\right),\zeta_h\geq 0\\ u_h^k+w_h^k\leq r_hMu_h^k+r_hMw_h^k. \end{cases}$$

Therefore

$$u_h^k+w_h^k=\partial_h\left(F^{k-1},r_hMu^k\right)+\partial_h\left(G^{k-1},r_hMw^k\right)$$

is a subsolution for the obstacle $r_hMu_h^k+r_hMw_h^k$ and the right hand side $F^{k-1}+G^{k-1}$. However, we know by Theorem 2.5 that the solution

$$\partial_h\left(F^{k-1}+G^{k-1},r_hMu^k+r_hMw_h^k\right)$$

is the greatest element in the subsolutions set. Then

$$\partial_h\left(F^{k-1}+G^{k-1},r_hMu^k+r_hMw\right)\geq\partial_h\left(F^{k-1},r_hMu^k\right)+\partial_h\left(G^{k-1},r_hMw^k\right).$$

□

**Proposition 3.4.** *Under the previous assumptions, the result from* Lemma 3.2 *can be extended as*

$$\partial_h\left(F^{k-1}+\delta a_0+\lambda,r_hMu^k+\delta\right)=\partial_h\left(F^{k-1},r_hMu^k\right)+\delta,$$

*where* $\delta$ *is a positive constant and* $\lambda$ *is defined in* (2.15)*.*

*Proof.* We can deduce the inequality

$$\partial_h \left(F^{k-1} + \delta a_0 + \mu, r_h M u^k + \delta\right) \geq \partial_h \left(F^{k-1}, r_h M u^k\right) + \delta$$

from Proposition 3.3. It remains only to prove that

$$\partial_h \left(F^{k-1} + \delta a_0 + \mu, r_h M u^k + \delta\right) \leq \partial_h \left(F^{k-1}, r_h M u^k\right) + \delta.$$

We consider the following system of inequalities:

$$\begin{cases} b\left(\rho_h^k, v_h - \rho_h^k\right) \geq \left(F^{k-1} + G^{k-1}, v_h - \rho_h^k\right), \\ v_h, \rho_h^k \leq r_h M u_h^k + \delta. \end{cases} \tag{3.5}$$

It can be verified that

$$\left(F^k + \delta a_0 + \mu, v_h - \rho_h^k\right) = \left(F^k, v_h - \rho_h^k\right) + \left(\delta a_0 + \mu, v_h - \rho^k\right) = \left(F^{,k}, v_h - \rho^k\right) + \left(\left(\delta a_0 + \mu\right), \left(v_h - \rho^k\right)\right).$$

Using (1.3) we can show that

$$a\left(\delta, v_h - \rho_h^k\right) = \delta a_0, \left(v_h - \rho_h^k\right), \ \delta \geq 0,$$

thus

$$b\left(\delta, v_h - \rho_h^k\right) = \left(\delta a_0 + \mu, v_h - \rho_h^k\right).$$

Consequently,

$$\left(F^{k-1} + \delta a_0 + \mu, v_h - \rho_h^k\right) = \left(F^{k-1}, v_h - \rho_h^k\right) + b\left(\delta, v_h - \rho_h^k\right). \tag{3.6}$$

From (3.5) and (3.6), we have that

$$\begin{cases} b\left(\rho_h^k, v_h - \rho_h^k\right) \geq \left(F^{k-1}, v_h - \rho_h^k\right) + b\left(\delta, v_h - \rho_h^k\right), \\ v_h, \rho_h^k \leq r_h M u_h^k + \delta, \ v_h \in V^h. \end{cases}$$

Then

$$\begin{cases} b\left(\rho_h^k - \delta, v_h - \rho_h^k\right) \geq \left(F^{k-1}, v_h - \rho_h^k\right), \ v_h \in V^h, \\ v_h, \rho_h^k - \delta \leq r_h M u_h^k. \end{cases} \tag{3.7}$$

Taking $v_h = \rho_h^k - \tilde{v}_h$ with $\tilde{v}_h \geq 0$ in (3.7), we get

$$\begin{cases} b\left(\rho_h^k - \delta, \varphi_l\right) \leq \left(F^{k-1}, \varphi_l\right), \ \varphi_l = 1, ..., m\left(h\right), \\ \rho_h^k - \delta \leq r_h M u_h^k. \end{cases}$$

Therefore, $\rho_h^k - \delta$ is the subsolution for the obstacle $r_h M u_h^k$ and the right hand side $F^k$. As we know that $\partial_h \left(F^{k-1}, r_h M u_h^{,k}\right)$ is the greatest element in the subsolutions set, it follows that

$$\rho_h^k - \delta \leq \partial_h \left(F^{k-1}, r_h M u_h^k\right),$$

i.e.,

$$\rho_h^k \leq \partial_h \left(F^{k-1}, r_h M u_h^k\right) + \delta.$$

Thus

$$\partial \left(F^{k-1}, +a_0 \delta + \mu, r_h M u_h^{,k} + \delta\right) \leq \partial_h \left(F^{k-1}, r_h M u_h^k\right) + \delta. \tag{3.8}$$

From the first inequality which was deduced by Proposition 3.3 and (3.8), we infer that

$$\partial \left(F^{k-1}, +a_0 \delta + \mu, r_h M u_h^k + \delta\right) = \partial_h \left(F^{k-1}, r_h M u_h^k\right) + \delta. \tag{3.9}$$

□

*3.2. Some properties of the mapping $T_h$*

Let $\bar{u}_h^0$ be the finite element approximation of the discrete equation (2.5).

**Proposition 3.5.** *Under the above assumptions, the mapping $T_h$ satisfies the following relations for all $v,\ w \in\ L_+^\infty(\Omega)$:*

(i) $T_h v \leq T_h w$ *whenever* $V \leq W$*,*
(ii) $T_h w \geq 0$*,*
(iii) $T_h w \leq \bar{u}_h^0$.

*Proof.* (i) Let $v,\ w\ \in\ L_+^\infty(\Omega)$ such that $v \leq w$. Then, since $\partial_h$ is increasing in two cases (the coercive and noncoercive cases [6, 7]), it follows that

$$\partial_h\left(f^k(v) + \lambda v, r_h M v^k\right) \leq \partial_h\left(f(w) + \lambda w, r_h M w^k\right),$$

that is to say,

$$T_h v \leq T_h w.$$

(ii) This follows directly from the fact that $f \geq 0$ and $Mw^k \geq 0$. Thus, we have $T_h w \geq 0$.
(iii) The fact that both the solutions $\xi_h^k$ of (2.14) and $\bar{u}_h^0$ of (2.5) belong to $V^h$ readily implies that

$$\xi_h^k - \left(\xi_h^k + \bar{u}^0\right)^+ \in L^\infty(\Omega).$$

Moreover, as $\left(\xi + \bar{u}^0\right)^+ \geq 0$, it follows that

$$\xi_h^k - \left(\xi_h^k + \bar{u}^0\right)^+ \leq \xi_h^k \leq M w^k.$$

Therefore, we can take $v_h = \xi_h^k - \left(\xi_h^k + \bar{u}^0\right)^+$ as a trial function in (2.14). This gives

$$b\left(\xi_h^k, -\left(\xi_h^k + \bar{u}^0\right)^+\right) \geq \left(f(w) + \lambda w, -\left(\xi_h^k + \bar{u}^0\right)^+\right).$$

Also, for $v_h = \left(\xi_h^k + \bar{u}^0\right)^+$ as trial function in (2.5), we obtain

$$b\left(u^0, \left(\xi_h^k + \bar{u}^0\right)^+\right) = \left(f^k, \left(\xi_h^k + \bar{u}^0\right)^+\right),\ \forall v_h \in V^h, \tag{3.10}$$

so, by addition, we find that

$$-b\left(\left(\xi_h^k + \bar{u}^0\right)^+, \left(\xi_h^k + \bar{u}^0\right)^+\right) \geq 0.$$

By Theorem 2.4 it follows that

$$\left(\xi_h^k + \bar{u}^0\right)^+ = 0,$$

and thus

$$\xi_h^k \leq \bar{u}^0.$$

□

**Proposition 3.6.** *The mapping $T_h$ is concave on $L_+^\infty(\Omega)$, i.e.,*

$$T_h(\eta v + (1-\eta) w) \geq \eta T_h(v) + (1-\eta) T_h w,\ \forall v, w \in L_+^\infty(\Omega).$$

*Proof.* Let $v,\ w \in L_+^\infty(\Omega)$, and let $F^k = f^k + \mu v^k,\ G^k = f^k + \mu w^{k-1}$ be the right hand sides of the systems of inequalities (2.14). We have

$$T_h\left(\eta v+(1-\eta)w\right)=\partial_h\left(\eta F^{k-1}+(1-\eta)G^{k-1},r_h\eta Mv_h^k+r_h(1-\eta)Mw_h^k\right).$$

Then, by using Proposition 3.4, we get

$$T_h\left(\eta v+(1-\eta)w\right)\geq\eta.\partial_h\left(F^{k-1},r_hMv_h^k\right)+(1-\eta).\partial_h\left(G^{k-1},r_hMv_h^k\right),$$

and thus

$$T_h\left(\eta v+(1-\eta)w\right)\geq\eta T_h(v)+(1-\eta)T_hw,$$

which shows that $T_h$ is concave. □

**Proposition 3.7.** *Under the results of* Propositions 3.4 *and* 3.5 *and using the properties of the operator* $Mu$ *(cf.* [14]*) the mapping* $T_h$ *is Lipschitz on* $L_+^\infty(\Omega)$ *i.e.,*

$$\|T_hv-T_hw\|_\infty\leq\|v-w\|_\infty,\ \forall v,w\in L_+^\infty(\Omega).$$

*Proof.* We clearly have

$$\|T_hv-T_hw\|_{L^\infty(\Omega)}=\left\|\partial_h\left(F^{k-1},r_hMv_h^k\right)-\partial_h\left(G^{k-1},r_hMw_h\right)\right\|_\infty.$$

Setting

$$\phi=\max\left(\|r_hMv_h-r_hMw_h\|_\infty,\frac{1}{\beta+\lambda}\left\|F^{k-1}-G^{k-1}\right\|_\infty\right),$$

we find that

$$r_hMv_h\leq r_hMw_h+\|r_hMv_h-r_hMw_h\|_\infty\leq r_hMw_h^k+\phi^k.$$

On the other hand, we have

$$\begin{aligned}\left\|\partial_h\left(F^{k-1}(v),r_hMv_h^k\right)-\partial_h\left(G^{k-1}(w),r_hMw_h\right)\right\|_\infty&\leq\frac{1}{\beta+\lambda}\left\|F^{k-1}(v)-G^{k-1}(w)\right\|_\infty\\&\leq\frac{\lambda+c}{\lambda+\beta}\|v-w\|_\infty\\&\leq\frac{1+(\Delta t)c}{1+(\Delta t)\beta}\|v-w\|_\infty.\end{aligned}$$

This finally yields

$$\|T_hv-T_hw\|_\infty\leq\left(\frac{1+(\Delta t)c}{1+(\Delta t)\beta}\right)\|v-w\|_\infty.$$

By Proposition 3.4, it follows that

$$\partial_h\left(F^{k-1},r_hMv_h^k\right)\leq\partial_h\left(G^{k-1}+a_0\phi+\lambda,r_hMw_h+\phi\right)\leq\partial_h\left(G^{k-1},r_hMw_h\right)+\phi,$$

whence

$$T_hv\leq T_hw+\phi.$$

Similarly, interchanging the roles of $v_h$ and $w_h$, we also get

$$T_hw\leq T_hv+\phi.$$

Knowing that $M$ is Lipschitz ([14]), we can easily deduce that

$$\begin{aligned}\|T_hv-T_hw\|_\infty&\leq\max\left(\|r_hMv_h-r_hMw_h\|_\infty,\frac{1}{\beta+\lambda}\left\|F^{k-1}-G^{k-1}\right\|_\infty\right)\\&\leq\max\left(1,\frac{1+(\Delta t)c}{1+(\Delta t)\beta}\right)\|v_h-w_h\|_\infty\leq\|v_h-w_h\|_\infty.\end{aligned}$$

□

## 4. The main result

**Lemma 4.1.** *For* $0 \leq \mu \leq \inf\left(\frac{k}{\|\hat{u}^0\|_\infty}, 1\right)$, *where* $k$ *is defined in* (1.8), *we have*

$$T_h(0) \geq \lambda \left\|\hat{u}^0\right\|_\infty . \tag{4.1}$$

*Proof.* From (2.19), $T_h(0) = u^1$, where $\check{u}^1$ is a solution of the following system of quasi-variational inequalities:

$$\begin{cases} b\left(\check{u}_h^1, v_h - \check{u}_h^1\right) \geq \left(f + \mu\check{u}_h^0, v_h - \check{u}_h^1\right), \ v_h \in V^h \\ \check{u}_h^{i1} \leq r_h M \ \hat{u}_h^{i,0}. \end{cases} \tag{4.2}$$

We can take the trial functions

$$v_h = \left(\check{u}_h^1 - \lambda\hat{u}_h^0\right)^- + \check{u}_h^1$$

in the EQVIs (4.2), and

$$-\left(\check{u}_h^1 - \lambda\hat{u}_h^0\right)^-$$

in the problem (2.5). Using the fact that $F^0 \geq 0$, by adding (2.8) and (3.1) we get

$$b^i\left(\check{u}_h^1 - \mu\hat{u}_h^0, \left(\check{u}_h^1 - \mu\hat{u}_h^0\right)^-\right) \geq \left(F^0 - \mu F^0, \left(\check{u}_h^1 - \mu\hat{u}_h^0\right)^-\right) \geq (1-\mu)\left(F^0, \left(\check{u}_h^1 - \mu\hat{u}_h^0\right)^-\right) \geq 0,$$

where $F^0 = f + \lambda\check{u}_h^0$. Thus, by using Theorem 2.4, it follows that

$$\left(\check{u}_h^1 - \lambda\hat{u}_h^0\right)^- = 0,$$

i.e.,

$$\check{u}_h^1 \geq \lambda\hat{u}_h^0, \ i = 1, \dots, M.$$

Then

$$T_h(0) \geq \lambda \left\|\hat{u}^0\right\|_\infty ,$$

which completes the proof. □

**Proposition 4.2.** *Let* $\omega \in [0,1]$ *be such that*

$$w - v \leq \omega w, \ \forall w, v \in \mathbf{Q}. \tag{4.3}$$

*Then, under* Propositions 3.6 *and* Proposition 3.7*, the following holds:*

$$T_h v - T_h w \leq \omega(1-\lambda) T_h v. \tag{4.4}$$

*Proof.* By (4.3), we have

$$(1-\omega) w \leq v,$$

thus, using the fact that $T_h$ is increasing and concave, it follows that

$$(1-\omega) T_h v + \omega T_h(0) \leq T_h((1-\omega) v + \omega.0) \leq T_h w.$$

Finally, using Lemma 3.2 we get (4.4). □

From Propositions 3.5 and 4.2, we derive our main result.

**Theorem 4.3.** *The sequences* $\left(\hat{u}_h^k\right)$ *and* $\left(\check{u}_h^k\right)$ *are well defined in* $\mathbf{Q}$ *and converge, respectively, from above and below, to the unique solution of system of inequalities* (2.14).

*Proof.* The proof consists of four steps.

**Step 1.** We show that the sequence $\left(\hat{u}^k\right)$ is monotone decreasing. From (4.1) and (3.4), it is easy to see that, for all $k \geq 1$, $\hat{u}^k$ is a solution to

$$\begin{cases} b\left(\hat{u}_h^k, v_h - \hat{u}_h^k\right) \geq \left(F^{k-1}, v_h - \hat{u}_h^k\right), \ v_h \in V^h, \\ \hat{u}_h^k \leq r_h M \hat{u}_h^k. \end{cases} \tag{4.5}$$

Since $F^{i,0}$ and $\hat{u}^0$ are positive, combining comparison results in variational inequalities with a simple induction, it follows that

$$\hat{u}^k \geq 0. \tag{4.6}$$

Furthermore, by Proposition 3.5,

$$0 \leq \hat{u}^1 = T_h\left(\hat{u}^0\right) \leq \hat{u}^0,$$

thus we can deduce that

$$\hat{u}^1 \geq 0. \tag{4.7}$$

For $k \geq 2$, we know by Proposition 4.2 that $T_h$ increasing. Thus, inductively,

$$0 \leq \hat{u}^{k+1} = T_h\left(\hat{u}^k\right) \leq \hat{u}^k \leq \ldots \leq \hat{u}^1 \leq \hat{u}^0. \tag{4.8}$$

**Step 2.** We show that $\left(\hat{u}^k\right)$ converges to the solution of the system (2.14). From (4.6) and (4.8), it is clear that

$$\lim_{k \longrightarrow \infty} \hat{u}^k = \bar{u}, \quad x \in \Omega, \ \bar{u} \in H^1(\Omega). \tag{4.9}$$

Moreover, from (4.6) we have

$$r_h M \hat{u}^k \geq 0.$$

Then we can take $v_h = 0$ as a trial function in (4.5), which yields

$$\begin{aligned} \alpha \left\|\hat{u}_h^k\right\|_{V^h}^2 &\leq b\left(\hat{u}_h^k, \hat{u}_h^k\right) \leq \left(F^{k-1}, \hat{u}_h^k\right) \leq \left\|F^k\right\|_{L^2(\Omega)} \left\|\hat{u}_h^k\right\|_{V^h} \\ &\leq \left(f\left(\hat{u}_h^{k-1}\right) + \lambda \left\|\hat{u}_h^k\right\|_{L^2(\Omega)}\right) \left\|\hat{u}_h^k\right\|_{V^h}. \end{aligned}$$

Therefore

$$\alpha \left\|\hat{u}_h^k\right\|_{V^h} \leq \left\|F^{k-1}\left(\hat{u}_h^k\right)\right\|_{L^2(\Omega)} + \mu \left\|\hat{u}_h^k\right\|_{V^h},$$

or more simply

$$\left\|\hat{u}_h^k\right\|_{V^h} \leq C_{f,\alpha,\mu} \leq C,$$

where $C$ is a constant independent of $k$ and we choose $\Delta t$ such that $\frac{1}{\Delta t} < \alpha$. Hence, $\hat{u}_h^k$ stays bounded in $V^h \subseteq H^1(\Omega)$ and consequently we can complete (3.8) by

$$\lim_{k \longrightarrow \infty} \hat{u}^k = \bar{u} \text{ weakly in } H^1(\Omega). \tag{4.10}$$

**Step 3.** We prove that $\bar{u}^k$ coincides with the solution of system (2.5). Indeed, since

$$\hat{u}_h^k \leq r_h M \hat{u}_h^k,$$

relation (4.10) implies

$$\bar{u}_h^k \leq r_h M \bar{u}_h^k.$$

Now, let $v_h \leq r_h M \bar{u}_h^k$. Then $v_h \leq r_h M \hat{u}_h^k$, for all $k = 1, \ldots, n$. We can, therefore, take $v_h$ as a trial function for the system (4.5). Consequently, combining (4.9) and (4.10) we have

$$\underline{\lim}_{k \longrightarrow \infty} b\left(\hat{u}_h^k, \hat{u}_h^k\right) \leq \underline{\lim}_{k \longrightarrow \infty} \left[b\left(\hat{u}_h^k, v_h\right) - \left(F^{k-1}, v_h - \hat{u}_h^k\right)\right], v_h \in V^h.$$

The continuous system of $b\left(v_h, v_h\right)$ is a weak lower semicontinuity, then

$$\underline{\lim}_{k \longrightarrow \infty} b\left(\hat{u}_h^k, \hat{u}_h^k\right) \leq b\left(\bar{u}_h, v_h\right) - \left(F^{k-1}, v_h - \bar{u}_h\right), v_h \in V^h.$$

But

$$0 \leq b\left(\hat{u}_h^k - \bar{u}_h^k, \hat{u}_h^k - \bar{u}_h^k\right) \leq b\left(\hat{u}_h^k, \hat{u}_h^k\right) - b\left(\hat{u}_h^k, \bar{u}_h^k\right) - -b\left(\bar{u}_h^k, \hat{u}_h^k\right) + b\left(\bar{u}_h^k, \bar{u}_h^k\right), \tag{4.11}$$

whence

$$b\left(\hat{u}_h^k, \hat{u}_h^k\right) \geq b\left(\hat{u}_h^k, \bar{u}_h^k\right) + b\left(\bar{u}_h^k, \hat{u}_h^k\right) - b\left(\bar{u}_h^k, \bar{u}_h^k\right). \tag{4.12}$$

Passing to the limit in problem (4.12), we obtain

$$b\left(\bar{u}_h^k, \bar{u}_h^k\right) \leq \underline{\lim}_{k \longrightarrow \infty} b\left(\hat{u}_h^k, \hat{u}_h^k\right) \leq b\left(\bar{u}_h^k, v_h\right) - \left(F^{k-1}, v_h - \bar{u}^k\right),$$

which yields

$$\begin{cases} b\left(\bar{u}_h^k, v_h - \bar{u}_h^k\right) \geq \left(F^{k-1}, v_h - \bar{u}_h^k\right), \; v_h \in V^h \\ \\ \bar{u}_h^k \leq r_h M \bar{u}_h^k. \end{cases}$$

Thus $\bar{u}_h^k$ is the solution of system (4.5).

**Step 4.** The monotonicity of the sequence $\left(\check{u}_h^k\right)$ can be shown similarly to that of sequence $\left(\hat{u}_h^k\right)$. Let us prove its convergence to the solution of system (4.5). Indeed, we use (4.4) together with

$$v = \hat{u}_h^0, \quad \tilde{v} = \check{u}_h, \; \gamma = 1,$$

and obtain

$$T_h \hat{u}^0 - T_h \check{u}^0 \leq (1 - \lambda) T_h \hat{u}^0,$$

so

$$\hat{u}_h^1 - \check{u}_h^1 \leq (1 - \lambda) \hat{u}_h^1.$$

Applying (4.4) again, this yields

$$\hat{u}_h^2 - \check{u}_h^2 \leq (1 - \lambda)^2 \hat{u}_h^2$$

and generally

$$\hat{u}_h^k - \check{u}_h^k \leq (1 - \lambda)^k \hat{u}_h^k,$$

or

$$\hat{u}_h^k - \check{u}_h^k \leq (1 - \lambda)^k \hat{u}_h^0 \leq (1 - \lambda)^k \left\|\hat{u}_h^0\right\|_\infty.$$

We can prove that $\check{u}_h^k \underset{k \longrightarrow \infty}{\longrightarrow} \underline{u}_h$ similarly as in the case of sequence $\left(\hat{u}_h^k\right)$ in Step 3. Since $(1 - \lambda)^k \longrightarrow 0$, after passing to the limit, we get

$$\hat{u}_h \leq \check{u}_h.$$

Interchanging the roles of $\hat{u}_h^k$ and $\check{u}_h^k$ we also get

$$\check{u}_h \leq \hat{u}_h.$$

Finally, we deduce that

$$\check{u}_h = \hat{u}_h = u_h,$$

i.e. the solution of (4.5) is unique. □

*Remark* 4.4. From the above proposition, one can see that the solution of system (2.14) or (4.5) is a fixed point of $T_h$. i.e.,

$$T_h u = u_h.$$

## 5. Conclusion

In this paper, we presented a new proof for the existence and uniqueness of solutions of PQVIs, based on some properties of the discrete iterative algorithm using the semi-implicit scheme with respect to the $t$-variable combined with a finite element spatial approximation, and which has been used for proving the asymptotic behavior in uniform norm in the previous paper [4]. As further development of this work, the convergence of discrete iterative schemes for the sequences defined in Theorem 4.3 will be proved, and we will see that this result plays a major role in the finite element error analysis section.


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