Journal of Nonlinear Science and Applications

On the new fractional derivative and application to nonlinear Baggs and Freedman model

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Communicated by R. Saadati

Abstract

We presented the nonlinear Baggs and Freedman model with new fractional derivative. We derived the special solution using an iterative method. The stability of the iterative method was presented using the fixed point theory. The uniqueness of the special solution was presented in detail using some properties of the inner product and the Hilbert space. We presented some numerical simulations to underpin the effectiveness of the used derivative and semi-analytical method. ©2016 All rights reserved.

Keywords: Nonlinear Baggs and Freedman model, special solution, fixed point theorem, iterative method.
2010 MSC: 34A34, 47H10, 65L07.

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1. Introduction

The notion of language is as old as the creation itself. Language is the facility to develop and use multifaceted systems of interaction, exceptionally the human capability to do so. The scientific study of language is called linguistics. Nowadays in the world, the number of languages is estimated to vary between 5000 and 7000. However, any piece estimate depends on a partly arbitrary distinction between languages and dialects. However, monoglottism or unilingualism is the condition of being able to speak only a single language, in contrast to multilingualism is the use of two or more languages either by an individual speaker or by community speakers. Interactions between groups that speak different languages

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are occurring continuously in all nations in the world due to globalization. Throughout history, in countries and communities all over the world, populations from different language groups have been constantly coming into contact, for many reasons including tourism, colonisation, religions, trades, marriages, sport activities, researcher collaborations and other. This contact can be between, Cameroonian and Turkish as it is now in some research collaborations, Chinese, English and French as it is observed in Cameroon and other part of Africa. In order to perfect the interaction between these groups, the need for common language of communication has led in each case to some more degree of bilingualism or even multilingualism. Nowadays, there is a whole course about languages interpretation. Top factors that increase the need of bilingualism are perhaps industrialization, migration, research, sport and urbanization.

A mathematical model portraying the dynamics of interactions between bilingual components and a monolingual component of a population in a particular environment was proposed in [2, 10]. In [2], the authors investigated the condition under which both components of population will approach a unique and steady state. Also they presented the condition under which the bilingual component could persist and conditions under which it could become extinct. Authors have generalized this model within the scope of derivative with fractional order [8]. However, these models do not take into account the degree of interest in time and also the memory of the interaction, meaning the recall of the original meeting or interaction or contact up to a particular period of time in the present. The weakness of the proposed model using the concept of derivative with fractional order is the memory effect of the interaction, because of the design of the derivative, which faces singularity at the end of interval when tracing the history. Recently, Caputo and Fabrizio proposed a derivative with fractional order with no singular kernel. This derivative has therefore an advantage over the old Caputo derivative because the full effect of the memory can be portrayed [4, 5]. In this paper, we further investigate the dynamics of interactions between bilingual components and a monolingual component of a population in a particular environment using the newly proposed derivative.

The dynamics of interactions between a bilingual component and a monolingual component of a population in a particular environment is presented in mathematical formulas as follow

$$
\begin{aligned}
\frac{dx_1}{dt} &= (B_1 - D_1 - H_1)x_1(t) - L_1 x_1^2(t) - \alpha_1 \frac{x_1(t)x_2(t)}{1 + x_1(t)} + P_1 B_2 x_2(t), \\
\frac{dx_2}{dt} &= (B_2 - D_2 - H_2)x_2(t) - L_2 x_2^2(t) + \alpha_1 \frac{x_1(t)x_2(t)}{1 + x_1(t)} - P_1 B_2 x_2(t).
\end{aligned}
\tag{1.1}
$$

In this system, B_i and $D_i + L_i x_i$ are the specific birth rate and death rate of x_i. $0 < |H_i| \leq 1$ is emigration parameter for $i = 1, 2$. $0 < P_1 \leq 1$ is infant language acquisition parameter, that proportion of births in the x_2 population raised unilingually. $0 < \alpha_1 \leq 1$ is the non-infant language acquisition rate, the proportion of x_1 learning the x_2 language per unit time after infancy.

2. Information about the Caputo-Fabrizio derivative with fractional order

Definition 2.1 ([4]). Let $f \in L^1(a, b)$, $b > a, \alpha \in [0, 1]$ then the new Caputo time derivative of fractional order is defined as:

$$
D_t^\alpha(f(t)) = \frac{M(\alpha)}{1 - \alpha} \int_a^t f'(x) \exp\left[-\alpha \frac{t - x}{1 - \alpha}\right] dx,
\tag{2.1}
$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. But, if the function does not belongs to $L^1(a, b)$ then, the derivative can be reformulated as

$$
D_t^\alpha(f(t)) = \frac{\alpha M(\alpha)}{1 - \alpha} \int_a^t (f(t) - f(x)) \exp\left[-\alpha \frac{t - x}{1 - \alpha}\right] dx.
\tag{2.2}
$$

Remark 2.2. The instigators observed that, if $\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty)$, $\alpha = \frac{1}{1+\sigma} \in [0,1]$, then new Caputo derivative of fractional order assumes the form

$$D_t^\sigma(f(t)) = \frac{N(\sigma)}{\sigma} \int_a^t f'(x) \exp\left[-\frac{t-x}{\sigma}\right] dx, \ N(0) = N(\infty) = 1. \tag{2.3}$$

In addition,

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t).$$

At this instant subsequent to the preface of the novel derivative, the connected anti-derivative turns out to be imperative; the connected integral of the derivative was proposed by Nieto and Losada.

Definition 2.3 ([5])**.** Let $0 < \alpha < 1$. The fractional integral of order α of a function f is defined by

$$^{CF}I_\alpha^t(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s)ds, \ t \geq 0. \tag{2.4}$$

We have the following relation.

Remark 2.4. Note that, according to the above definition, the fractional integral of Caputo type of function of order $0 < \alpha < 1$ is an average between function f and its integral of order one [5]. This therefore imposes

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1. \tag{2.5}$$

The above expressions yields an explicit formula for

$$M(\alpha) = \frac{2}{(2-\alpha)}, \ 0 \leq \alpha \leq 1.$$

Because of the above, Nieto and Losada proposed that the new Caputo derivative of order $0 < \alpha < 1$ can be reformulated as

$$^{CF}D_*^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s)ds, \ t \geq 0. \tag{2.6}$$

3. Equilibrium points of system and asymptotic stability

Let consider the model with Caputo-Fabrizio derivative and α satisfying $0 < \alpha \leq 1$ below:

$$\begin{aligned}
{}_0^{CF}D_t^\alpha X_1(t) &= (B_1 - D_1 - H_1)X_1(t) - L_1 X_1^2(t) \\
&\quad -\alpha_1 \frac{X_1(t)X_2(t)}{1 + X_1(t)} + P_1 B_2 X_2(t), \\
{}_0^{CF}D_t^\alpha X_2(t) &= (B_2 - D_2 - H_2)X_2(t) - L_2 X_2^2(t) \\
&\quad +\alpha_1 \frac{X_1(t)X_2(t)}{1 + X_1(t)} - P_1 B_2 X_2(t),
\end{aligned} \tag{3.1}$$

where $0 < B_i \leq 1$, $0 < D_i \leq 1$, $0 < |H_i| \leq 1$ are the birth, death and emigration parameters for $i = 1, 2$. The model describes the interaction of a majority unilingual population with X_1 and a bilingual population with X_2. The term $\alpha_1 \frac{X_1(t)X_2(t)}{1+X_1(t)}$ describes that part of population X_1 lost to X_2 due to virtual predation on the part of X_2.

Let $\alpha \in (0, 1]$ and consider the system

$$\begin{cases} {}^{CF}_{0}D^{\alpha}_t X_1(t) = f_1(X_1, X_2), \\ {}^{CF}_{0}D^{\alpha}_t X_2(t) = f_2(X_1, X_2), \end{cases}$$

with the initial values $X_1(0) = X_{01}$ and $X_2(0) = X_{02}$.

Here $f_1(X_1, X_2) = (B_1 - D_1 - H_1)X_1(t) - L_1 X_1^2(t) - \alpha_1 \frac{X_1(t)X_2(t)}{1+X_1(t)} + P_1 B_2 X_2(t)$ and

$$f_2(X_1, X_2) = (B_2 - D_2 - H_2)X_2(t) - L_2 X_2^2(t) + \alpha_1 \frac{X_1(t)X_2(t)}{1 + X_1(t)} - P_1 B_2 X_2(t).$$

To evaluate the equilibrium points let $f_i(X_1, X_2) = 0$, $i = 1, 2$ then the equilibrium points are $E_0(0, 0)$ and $E_1(X_1^*, X_2^*)$, where

$$X_1^* = \frac{(B_2 - D_2 - H_2)X_2^* - L_2 X_2^{2*} - P_1 B_2 X_2^*}{L_2 X_2^{2*} + P_1 B_2 X_2^* - \alpha_1 X_2^* - (B_2 - D_2 - H_2)X_2^*},$$

$$X_2^* = \frac{(B_1 - D_1 - H_1)X_1^* - L_1 X_1^{2*}}{\alpha_1 \frac{X_1^*}{1+X_1^*} - P_1 B_2}.$$

Lack of population equilibrium for the system is $E_0 = (0, 0)$ and the Jacobian matrix at this point is given below:

$$J(E_0) = \begin{bmatrix} (B_1 - D_1 - H_1) & P_1 B_2 \\ 0 & (B_2 - D_2 - H_2) - P_1 B_2 \end{bmatrix}.$$

Theorem 3.1. *Lack of population equilibrium of the system* (3.1) *is asymptotic stable if*

$$(B_1 - D_1 - H_1)\left((B_2 - D_2 - H_2) - P_1 B_2\right) > 0.$$

Proof. The equilibrium point of the system (3.1) is asymptotic stable if all the eigenvalues, λ_i, $i = 1, 2$ of $J(E_0)$ satisfy the following conditions [1]-[6]

$$|\arg \lambda_i| > \frac{\alpha \pi}{2}. \tag{3.2}$$

To find the eigenvalues of system, let solve the following characteristic equation below

$$\det(J(E_0) - \lambda I) = 0.$$

Thus, we have the following equation

$$\lambda^2 - (A + B)\lambda + AB = 0,$$

where

$$A = (B_1 - D_1 - H_1),$$
$$B = (B_2 - D_2 - H_2) - P_1 B_2.$$

If $AB > 0$, then the condition given by (3.2) is satisfied. This completes the proof of Theorem 3.1. $\qquad\square$

We now discuss the asymptotic stability of the $E_1(X_1^*, X_2^*)$ equilibrium of the system given by (3.1). The Jacobian matrix $J(E_1)$ evaluated at the equilibrium is given by

$$J(E_1) = \begin{bmatrix} (B_1 - D_1 - H_1) \\ -2L_1 X_1^* - \alpha_1 \frac{X_2^*}{(1+X_1^*)^2} & -\alpha_1 \frac{X_1^*}{1+X_1^*} + P_1 B_2 \\ \alpha_1 \frac{X_2^*}{(1+X_1^*)^2} & (B_2 - D_2 - H_2) \\ & -2L_2 X_2^* + \alpha_1 \frac{X_1^*}{1+X_1^*} - P_1 B_2 \end{bmatrix},$$

so the characteristic equation of the linearized system is of the form

$$\lambda^2 - (c+d)\lambda + cd - kl = 0,$$

where

$$
\begin{aligned}
c &= (B_1 - D_1 - H_1) - 2L_1 X_1^* - \alpha_1 \frac{X_2^*}{(1+X_1^*)^2}, \\
d &= (B_2 - D_2 - H_2) - 2L_2 X_2^* + \alpha_1 \frac{X_1^*}{1+X_1^*} - P_1 B_2, \\
k &= \alpha_1 \frac{X_2^*}{(1+X_1^*)^2}, \\
l &= -\alpha_1 \frac{X_1^*}{1+X_1^*} + P_1 B_2.
\end{aligned}
\tag{3.3}
$$

Theorem 3.2. *Let c, d, k and l be as given in (3.3). If $(c+d) < 0$ and $cd < kl$ is satisfied then the equilibrium point $E_1(X_1^*, X_2^*)$ of the system (3.1) is unstable.*

Proof. If $(c+d) < 0$ and $cd < kl$ is satisfied, from Descartes rule of signs, it is clear that the characteristic equation has at least one positive real root. So, the equilibrium point $E_1(X_1^*, X_2^*)$ of the system (3.1) is unstable. □

4. Solution of nonlinear Baggs and Freedman model with Caputo-Fabrizio derivative

First we give the following useful definition of Sumudu transform. The Sumudu transform, an integral transform similar to the Laplace transform, introduced in the early 1990s by Watugala [9] to solve differential equations and control engineering problems is given below:

Definition 4.1. The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by

$$S(f(t)) = F_s(u) = \int_0^\infty \frac{1}{u} \exp[\frac{-t}{u}] f(t) dt.$$

4.1. Derivation of the special solution

The following model is considered using the Caputo-Fabrizio fractional derivative below

$$
\begin{aligned}
{}_0^{CF}D_t^\alpha X_1(t) &= (B_1 - D_1 - H_1) X_1(t) - L_1 X_1^2(t) \\
&\quad - \alpha_1 \frac{X_1(t)X_2(t)}{1+X_1(t)} + P_1 B_2 X_2(t), \\
{}_0^{CF}D_t^\alpha X_2(t) &= (B_2 - D_2 - H_2) X_2(t) - L_2 X_2^2(t) \\
&\quad + \alpha_1 \frac{X_1(t)X_2(t)}{1+X_1(t)} - P_1 B_2 X_2(t).
\end{aligned}
\tag{4.1}
$$

The aim of this section is to provide a special solution of the above equation applying the Sumudu transform on both sides of equation (4.1) together with an iterative method. We shall give the Sumudu transform in the following theorem.

Theorem 4.2. *Let $f(t)$ be a function for which the Caputo-Fabrizio exists, then the Sumudu transform of the Caputo-Fabrizio fractional derivative of $f(t)$ is given as:*

$$ST\left({}_0^{CF}D_t^\alpha\right)((f(t)) = M(\alpha)\frac{S(f(t)) - f(0)}{1 - \alpha + \alpha u}.$$

To solve the above equation (4.1), we apply the Sumudu transform on both sides of equation (4.1), we obtain

$$M(\alpha)\frac{S(X_1(t)) - X_1(0)}{1 - \alpha + \alpha s} = S\left(\begin{array}{c}(B_1 - D_1 - H_1)X_1 - L_1X_1^2 \\ -\alpha_1\frac{X_1X_2}{1+X_1} + P_1B_2X_2\end{array}\right),$$

$$M(\alpha)\frac{S(X_2(t)) - X_2(0)}{1 - \alpha + \alpha s} = S\left(\begin{array}{c}(B_2 - D_2 - H_2)X_2 - L_2X_2^2 \\ +\alpha_1\frac{X_1X_2}{1+X_1} - P_1B_2X_2\end{array}\right). \tag{4.2}$$

Rearranging, we obtain

$$S(X_1(t)) = X_1(0)$$
$$+ \frac{(1 - \alpha + \alpha s)}{M(\alpha)}S\left(\begin{array}{c}(B_1 - D_1 - H_1)X_1 - L_1X_1^2 \\ -\alpha_1\frac{X_1X_2}{1+X_1} + P_1B_2X_2\end{array}\right),$$

$$S(X_2(t)) = X_2(0)$$
$$+ \frac{(1 - \alpha + \alpha s)}{M(\alpha)}S\left(\begin{array}{c}(B_2 - D_2 - H_2)X_2 - L_2X_2^2 \\ +\alpha_1\frac{X_1X_2}{1+X_1} - P_1B_2X_2\end{array}\right). \tag{4.3}$$

Now applying the inverse Sumudu transform on both sides of equation (4.3), we obtain;

$$X_1(t) = X_1(0)$$
$$+ S^{-1}\left(\frac{(1 - \alpha + \alpha s)}{M(\alpha)}S\left(\begin{array}{c}(B_1 - D_1 - H_1)X_1 - L_1X_1^2 \\ -\alpha_1\frac{X_1X_2}{1+X_1} + P_1B_2X_2\end{array}\right)\right),$$

$$X_2(t) = X_2(0)$$
$$+ S^{-1}\left(\frac{(1 - \alpha + \alpha s)}{M(\alpha)}S\left(\begin{array}{c}(B_2 - D_2 - H_2)X_2 - L_2X_2^2 \\ +\alpha_1\frac{X_1X_2}{1+X_1} - P_1B_2X_2\end{array}\right)\right). \tag{4.4}$$

We next obtain the following recursive formula;

$$X_{1(n+1)}(t) = X_{1(n)}(0)$$
$$+ S^{-1}\left(\frac{(1 - \alpha + \alpha s)}{M(\alpha)}S\left(\begin{array}{c}(B_1 - D_1 - H_1)X_{1(n)} - L_1X_{1(n)}^2 \\ -\alpha_1\frac{X_{1(n)}X_{2(n)}}{1+X_{1(n)}} + P_1B_2X_{2(n)}\end{array}\right)\right),$$

$$X_{2(n+1)}(t) = X_{2(n)}(0)$$
$$+ S^{-1}\left(\frac{(1 - \alpha + \alpha s)}{M(\alpha)}S\left(\begin{array}{c}(B_2 - D_2 - H_2)X_{2(n)} - L_2X_{2(n)}^2 \\ +\alpha_1\frac{X_{1(n)}X_{2(n)}}{1+X_{1(n)}} - P_1B_2X_{2(n)}\end{array}\right)\right), \tag{4.5}$$

and the solution of (4.1) is provided by

$$X_1(t) = \lim_{n\to\infty} X_{1(n)}(t),$$
$$X_2(t) = \lim_{n\to\infty} X_{2(n)}(t).$$

4.2. Application of fixed-point theorem for stability analysis of iteration method

Let $(X, \|.\|)$ be a Banach space and H a self-map of X. Let $y_{n+1} = g(H, y_n)$ be particular recursive procedure. Suppose that, $F(H)$ the fixed-point set of H has at least one element and that y_n converges to a point $p \in F(H)$. Let $\{x_n\} \subseteq X$ and define $e_n = \|x_{n+1} - g(H, x_n)\|$. If $\lim_{n\to\infty} e_n = 0$ implies that $\lim_{n\to\infty} x_n = p$, then the iteration method $y_{n+1} = g(H, y_n)$ is said to be H-stable. Without any loss of generality, we

must assume that, our sequence $\{x_n\}$ has an upper boundary; otherwise we cannot expect the possibility of convergence. If all these conditions are satisfied for $y_{n+1} = Hy_n$ which is known as Picard's iteration, consequently the iteration is H-stable. We shall then state the following theorem.

Theorem 4.3 ([7]). *Let* $(X, \|.\|)$ *be a Banach space and* H *a self-map of* X *satisfying*

$$\|H_x - H_y\| \leq C \|x - H_x\| + c \|x - y\|,$$

for all x, y *in* X *where* $0 \leq C$, $0 \leq c < 1$. *Suppose that* H *is Picard* H-*Stable .*

Let us take into account the following recursive formula equation (4.5) connected to equation (3.1).

$$
\begin{aligned}
X_{1(n+1)}(t) =& X_{1(n)}(0) \\
&+ S^{-1}\left(\frac{1+(s-1)\alpha}{M(\alpha)} S\left(\begin{array}{c} (B_1 - D_1 - H_1)X_{1(n)} - L_1 \widetilde{X}^2_{1(n)} \\ -\alpha_1 \frac{X_{1(n)}X_{2(n)}}{1+X_{1(n)}} + P_1 B_2 X_{2(n)} \end{array}\right)\right), \\
X_{2(n+1)}(t) =& X_{2(n)}(0) \\
&+ S^{-1}\left(\frac{1+(s-1)\alpha}{M(\alpha)} S\left(\begin{array}{c} (B_2 - D_2 - H_2)X_{2(n)} - L_2 \widetilde{X}^2_{2(n)} \\ +\alpha_1 \frac{X_{1(n)}X_{2(n)}}{1+X_{1(n)}} - P_1 B_2 X_{2(n)} \end{array}\right)\right),
\end{aligned}
\tag{4.6}
$$

where $\frac{1+(s-1)\alpha}{M(\alpha)}$ is the fractional Lagrange multiplier and $\widetilde{X}^2_{1(n)}$, $\widetilde{X}^2_{2(n)}$ are restricted variation implying $\delta \widetilde{X}^2_{1(n)} = \delta \widetilde{X}^2_{2(n)}$.

Theorem 4.4. *Let* T *be a self-map defined as*

$$
\begin{aligned}
T(X_{1(n)}(t)) =& X_{1(n+1)}(t) = X_{1(n)}(t) \\
&+ S^{-1}\left(\frac{1+(s-1)\alpha}{M(\alpha)} S\left(\begin{array}{c} (B_1 - D_1 - H_1)X_{1(n)} - L_1 X^2_{1(n)} \\ -\alpha_1 \frac{X_{1(n)}X_{2(n)}}{1+X_{1(n)}} + P_1 B_2 X_{2(n)} \end{array}\right)\right), \\
T(X_{2(n)}(t)) =& X_{2(n+1)}(t) = X_{2(n)}(t) \\
&+ S^{-1}\left(\frac{1+(s-1)\alpha}{M(\alpha)} S\left(\begin{array}{c} (B_2 - D_2 - H_2)X_{2(n)} - L_2 X^2_{2(n)} \\ +\alpha_1 \frac{X_{1(n)}X_{2(n)}}{1+X_{1(n)}} - P_1 B_2 X_{2(n)} \end{array}\right)\right),
\end{aligned}
\tag{4.7}
$$

then the iteration is $T-$*stable in* $L^1(a,b)$ *if*

$$
\begin{aligned}
\left(\begin{array}{c} 1 + (B_1 - D_1 - H_1)f(\gamma) - L_1 \|X_{1(n)} + X_{1(m)}\| g(\gamma) \\ -\alpha_1 \frac{(CA+C+K)}{MN} h(\gamma) - P_1 B_2 k(\gamma) \end{array}\right) &< 1, \\
\left(\begin{array}{c} 1 + (B_2 - D_2 - H_2)j(\gamma) - L_2 \|X_{2(n)} + X_{2(m)}\| p(\gamma) \\ +\alpha_1 \frac{(CA+C+K)}{MN} s(\gamma) - P_1 B_2 L(\gamma) \end{array}\right) &< 1,
\end{aligned}
\tag{4.8}
$$

where f, g, h, k *are functions from* $S^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} S\right\}$.

Proof. The fist step of the proof consists of showing that T has a fixed point. To achieve this, we evaluate the following for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

$$T(X_{1(n+1)}, X_{2(n+1)}) - T(X_{1(m+1)}, X_{2(m+1)})$$

$$= \left\{ \begin{array}{l} S^{-1}\left(\dfrac{1+(s-1)\alpha}{M(\alpha)}S\left(\begin{array}{c} X_{1(n+1)} - X_{1(m+1)} = X_{1(n)} - X_{1(m)} + \\ (B_1 - D_1 - H_1)\left\{X_{1(n)} - X_{1(m)}\right\} \\ -L_1\left\{X_{1(n)}^2 - X_{1(m)}^2\right\} \\ -\alpha_1 \dfrac{\begin{array}{c}X_{1(m)}X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\} + X_{2(m)}\left\{X_{1(n)} - X_{1(m)}\right\}\end{array}}{\left(1+X_{1(n)}\right)\left(1+X_{1(m)}\right)} \\ +P_1 B_2\left\{X_{2(n)} - X_{2(m)}\right\} \end{array} \right)\right) \\[3em] S^{-1}\left\{\dfrac{1+(s-1)\alpha}{M(\alpha)}S\left\{ \begin{array}{c} X_{2(n+1)} - X_{2(m+1)} = X_{2(n)} - X_{2(m)} + \\ (B_2 - D_2 - H_2)\left\{X_{2(n)} - X_{2(m)}\right\} \\ -L_2\left\{X_{2(n)}^2 - X_{2(m)}^2\right\} \\ +\alpha_1 \dfrac{\begin{array}{c}X_{1(m)}X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\} + X_{2(m)}\left\{X_{1(n)} - X_{1(m)}\right\}\end{array}}{\left(1+X_{1(n)}\right)\left(1+X_{1(m)}\right)} \\ -P_1 B_2\left\{X_{2(n)} - X_{2(m)}\right\} \end{array} \right\}\right\} \end{array} \right\}. \tag{4.9}$$

Now applying norm on both sides and without loss of generality

$$\left\|X_{1(n+1)} - X_{1(m+1)}\right\|$$

$$= \left\| S^{-1}\left(\dfrac{1+(s-1)\alpha}{M(\alpha)}S\left(\begin{array}{c} X_{1(n)} - X_{1(m)} + \\ (B_1 - D_1 - H_1)\left\{X_{1(n)} - X_{1(m)}\right\} \\ -L_1\left\{X_{1(n)}^2 - X_{1(m)}^2\right\} \\ -\alpha_1 \dfrac{\begin{array}{c}X_{1(m)}X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\} + X_{2(m)}\left\{X_{1(n)} - X_{1(m)}\right\}\end{array}}{\left(1+X_{1(n)}\right)\left(1+X_{1(m)}\right)} \\ +P_1 B_2\left\{X_{2(n)} - X_{2(m)}\right\} \end{array} \right)\right) \right\|, \tag{4.10}$$

using the properties of the norm in particular the triangular inequality, the right hand side of equation (4.10) is converted to

$$\left\| S^{-1}\left(\dfrac{1+(s-1)\alpha}{M(\alpha)}S\left(\begin{array}{c} X_{1(n)} - X_{1(m)} + \\ (B_1 - D_1 - H_1)\left\{X_{1(n)} - X_{1(m)}\right\} \\ -L_1\left\{X_{1(n)}^2 - X_{1(m)}^2\right\} \\ -\alpha_1 \dfrac{\begin{array}{c}X_{1(m)}X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{2(m)}\left\{X_{1(n)} - X_{1(m)}\right\}\end{array}}{\left(1+X_{1(n)}\right)\left(1+X_{1(m)}\right)} \\ +P_1 B_2\left\{X_{2(n)} - X_{2(m)}\right\} \end{array} \right)\right) \right\|$$

$$\left\|X_{1(n)} - X_{1(m)}\right\| +$$

$$\left\| S^{-1}\left(\dfrac{1+(s-1)\alpha}{M(\alpha)}S\left(\begin{array}{c} (B_1 - D_1 - H_1)\left\{X_{1(n)} - X_{1(m)}\right\} \\ -L_1\left\{X_{1(n)}^2 - X_{1(m)}^2\right\} \\ -\alpha_1 \dfrac{\begin{array}{c}X_{1(m)}X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{1(n)}\left\{X_{2(n)} - X_{2(m)}\right\}\\ +X_{2(m)}\left\{X_{1(n)} - X_{1(m)}\right\}\end{array}}{\left(1+X_{1(n)}\right)\left(1+X_{1(m)}\right)} \\ +P_1 B_2\left\{X_{2(n)} - X_{2(m)}\right\} \end{array} \right)\right) \right\|, \tag{4.11}$$

using further the linearity of the inverse Sumudu transform, we obtain the following

$$
\begin{aligned}
\leq & \left\| X_{1(n)} - X_{1(m)} \right\| \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| (B_1 - D_1 - H_1) \left\{ X_{1(n)} - X_{1(m)} \right\} \right\| \right) \right) \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| -L_1 \left\{ X_{1(n)}^2 - X_{1(m)}^2 \right\} \right\| \right) \right) \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| -\alpha_1 \frac{\begin{array}{c} X_{1(m)} X_{1(n)} \left\{ X_{2(n)} - X_{2(m)} \right\} \\ + X_{1(n)} \left\{ X_{2(n)} - X_{2(m)} \right\} \\ + X_{2(m)} \left\{ X_{1(n)} - X_{1(m)} \right\} \end{array}}{\left(1 + X_{1(n)} \right) \left(1 + X_{1(m)} \right)} \right\| \right) \right) \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| P_1 B_2 \left\{ X_{2(n)} - X_{2(m)} \right\} \right\| \right) \right).
\end{aligned}
\tag{4.12}
$$

Since both the solutions play the same role, we shall assume in this case that

$$
\left\| X_{2(n)} - X_{2(m)} \right\| \cong \left\| X_{1(n)} - X_{1(m)} \right\|.
$$

Replacing this in equation (4.12), we obtain the following relation

$$
\begin{aligned}
\leq & \left\| X_{1(n)} - X_{1(m)} \right\| \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| (B_1 - D_1 - H_1) \left\{ X_{1(n)} - X_{1(m)} \right\} \right\| \right) \right) \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| -L_1 \left\{ X_{1(n)}^2 - X_{1(m)}^2 \right\} \right\| \right) \right) \\
& + S^{-1} \left(\frac{1 + (s-1)\alpha}{M(\alpha)} S \left(\left\| -\alpha_1 \frac{\begin{array}{c} X_{1(m)} X_{1(n)} \left\{ X_{1(n)} - X_{1(m)} \right\} \\ + X_{1(n)} \left\{ X_{1(n)} - X_{1(m)} \right\} \\ + X_{2(m)} \left\{ X_{1(n)} - X_{1(m)} \right\} \end{array}}{\left(1 + X_{1(n)} \right) \left(1 + X_{1(m)} \right)} \right\| \right) \right) \\
& + S^{-1} \left\{ \frac{1 + (s-1)\alpha}{M(\alpha)} S \left\{ \left\| P_1 B_2 \left\{ X_{1(n)} - X_{1(m)} \right\} \right\| \right\} \right\}.
\end{aligned}
\tag{4.13}
$$

Since $X_{1(n)}$, $X_{1(m)}$ are bounded, we can find five different positive constants, C, M, A, N, K such that for all t.

$$
\begin{aligned}
\left\| X_{1(n)} \right\| < & C, \ \left\| X_{1(m)} \right\| < A, \ \left\| 1 + X_{1(n)} \right\| < M, \\
\left\| 1 + X_{1(m)} \right\| < & N, \ \left\| X_{2(m)} \right\| < K, \ (n, m) \in \mathbb{N} \times \mathbb{N}.
\end{aligned}
\tag{4.14}
$$

Now considering equation (4.13) with (4.14), we obtain the following

$$
\begin{aligned}
& \left\| T \left(X_{1(n)} \right) - T \left(X_{1(m)} \right) \right\| \\
& \leq \left(\begin{array}{c} 1 + (B_1 - D_1 - H_1) f(\gamma) - L_1 \left\| X_{1(n)} + X_{1(m)} \right\| g(\gamma) \\ -\alpha_1 \frac{(CA + C + K)}{MN} h(\gamma) - P_1 B_2 k(\gamma) \end{array} \right) \left\| X_{1(n)} - X_{1(m)} \right\|,
\end{aligned}
\tag{4.15}
$$

where f, g, h, k are functions from $S^{-1}\left(\frac{1+(s-1)\alpha}{M(\alpha)}S\right)$. In the same way, we get

$$
\begin{aligned}
&\left\| T\left(X_{2(n)}\right) - T\left(X_{2(m)}\right)\right\| \\
&\leq \left(\begin{array}{c} 1 + (B_2 - D_2 - H_2)j\left(\gamma\right) - L_2\left\|X_{2(n)} + X_{2(m)}\right\|p\left(\gamma\right) \\ +\alpha_1\frac{(CA+C+K)}{MN}s(\gamma) - P_1 B_2 L(\gamma) \end{array}\right)\left\|X_{2(n)} - X_{2(m)}\right\|
\end{aligned}
\tag{4.16}
$$

for

$$
\left(\begin{array}{c} 1 + (B_1 - D_1 - H_1)f\left(\gamma\right) - L_1\left\|X_{1(n)} + X_{1(m)}\right\|g\left(\gamma\right) \\ -\alpha_1\frac{(CA+C+K)}{MN}h(\gamma) - P_1 B_2 k(\gamma) \end{array}\right) < 1
$$

and

$$
\left(\begin{array}{c} 1 + (B_2 - D_2 - H_2)j(\gamma) - L_2\left\|X_{2(n)} + X_{2(m)}\right\|p\left(\gamma\right) \\ +\alpha_1\frac{(CA+C+K)}{MN}s(\gamma) - P_1 B_2 L(\gamma) \end{array}\right) < 1.
$$

Then the nonlinear T-self mapping has a fixed point. We next show that, T satisfies the conditions in Theorem 4.3. Let (4.15)-(4.16) be held thus putting

$$
c = (0,0),\ C = \left\{ \begin{array}{c} 1 + (B_1 - D_1 - H_1)f\left(\gamma\right) - L_1\left\|X_{1(n)} + X_{1(m)}\right\|g\left(\gamma\right) \\ -\alpha_1\frac{(CA+C+K)}{MN}h(\gamma) - P_1 B_2 k(\gamma), \\ 1 + (B_2 - D_2 - H_2)j(\gamma) - L_2\left\|X_{2(n)} + X_{2(m)}\right\|p\left(\gamma\right) \\ +\alpha_1\frac{(CA+C+K)}{MN}s(\gamma) - P_1 B_2 L(\gamma), \end{array} \right.
\tag{4.17}
$$

then the above shows that the inequality of Theorem 4.3 holds for the nonlinear mapping T. Therefore since all conditions in Theorem 4.3 hold for the defined non-linear mapping T, then T is Picards $T-$stable. This completes the proof of Theorem 4.4. □

5. Uniqueness of the special solution

In this section, we show that the special solution of equation (1.1) is unique using the iteration method. We shall first assume that, equation (1.1) has an exact solution via which, the special solution converges for a large number m. We consider the following Hilbert space $H = L^2\left((a,b)\times(0,T)\right)$ that can be defined as the set of those functions.

$$
v : (a,b)\times[0,T] \to \mathbb{R},\ \iint uvdudv < \infty.
$$

We now, consider the following operator

$$
T(X_1, X_2) = \left\{ \begin{array}{c} (B_1 - D_1 - H_1)X_1(t) - L_1 X_1^2(t) - \alpha_1\frac{X_1(t)X_2(t)}{1+X_1(t)} + P_1 B_2 X_2(t), \\ (B_2 - D_2 - H_2)X_2(t) - L_2 X_2^2(t) + \alpha_1\frac{X_1(t)X_2(t)}{1+X_1(t)} - P_1 B_2 X_2(t). \end{array} \right.
\tag{5.1}
$$

The aim of this part is to prove that the inner product of

$$
\left(T(X_{11} - X_{12}, X_{21} - X_{22}), (w_1, w_2)\right),
\tag{5.2}
$$

where $\left(X_{11} - X_{12}\right),\ \left(X_{21} - X_{22}\right)$ are special solution of system. However,

$$
\begin{aligned}
&\left(T(X_{11} - X_{12}, X_{21} - X_{22}), (w_1, w_2)\right) \\
&= \left\{ \begin{array}{c} \left(\begin{array}{c} (B_1 - D_1 - H_1)\left(X_{11} - X_{12}\right) - L_1\left(X_{11} - X_{12}\right)^2 \\ -\alpha_1\frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})} + P_1 B_2\left(X_{21} - X_{22}\right), w_1 \end{array}\right), \\ \left(\begin{array}{c} (B_2 - D_2 - H_2)\left(X_{21} - X_{22}\right) - L_2\left(X_{21} - X_{22}\right)^2 \\ +\alpha_1\frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})} - P_1 B_2\left(X_{21} - X_{22}\right), w_2 \end{array}\right). \end{array} \right.
\end{aligned}
\tag{5.3}
$$

We shall evaluate the first equation in the system without loss of generality

$$
\left(
\begin{array}{c}
(B_1 - D_1 - H_1)(X_{11} - X_{12}) - L_1(X_{11} - X_{12})^2 \\
-\alpha_1 \frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})} + P_1 B_2(X_{21} - X_{22}), w_1
\end{array}
\right)
$$
$$
= ((B_1 - D_1 - H_1)(X_{11} - X_{12}), w_1) + \left(-L_1(X_{11} - X_{12})^2, w_1 \right) \tag{5.4}
$$
$$
+ \left(-\alpha_1 \frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})}, w_1 \right) + (P_1 B_2(X_{21} - X_{22}), w_1).
$$

Since both solutions play almost the same role, we can assume that,

$$
X_{11} - X_{12} \cong X_{21} - X_{22},
$$

then the equation (5.3) becomes

$$
\left(
\begin{array}{c}
(B_1 - D_1 - H_1)(X_{11} - X_{12}) - L_1(X_{11} - X_{12})^2 \\
-\alpha_1 \frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})} + P_1 B_2(X_{21} - X_{22}), w_1
\end{array}
\right)
$$
$$
\cong ((B_1 - D_1 - H_1)(X_{11} - X_{12}), w_1) + \left(L_1(X_{11} - X_{12})^2, w_1 \right) \tag{5.5}
$$
$$
+ \left(\alpha_1 \frac{(X_{11} - X_{12})^2}{1 + (X_{11} - X_{12})}, w_1 \right) + (P_1 B_2(X_{11} - X_{12}), w_1).
$$

Nevertheless, using the relationship between the norm and the inner function, we obtain the following inequality

$$
\left(
\begin{array}{c}
(B_1 - D_1 - H_1)(X_{11} - X_{12}) - L_1(X_{11} - X_{12})^2 \\
-\alpha_1 \frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})} + P_1 B_2(X_{21} - X_{22}), w_1
\end{array}
\right)
$$
$$
\leq (B_1 - D_1 - H_1) \|X_{11} - X_{12}\| \|w_1\| + L_1 \|X_{11} - X_{12}\|^2 \|w_1\| \tag{5.6}
$$
$$
+ \alpha_1 \frac{\|X_{11} - X_{12}\|^2}{\|1 + X_{11} - X_{12}\|} \|w_1\| + P_1 B_2 \|X_{11} - X_{12}\| \|w_1\|
$$
$$
= \left((B_1 - D_1 - H_1) + L_1 \overline{w} + \alpha \overline{k} + P_1 B_2 \right) \|X_{11} - X_{12}\| \|w_1\|.
$$

Above in equality (5.6),

$$
\overline{w} = \|X_{11} - X_{12}\|, \quad \overline{k} = \alpha_1 \frac{\|X_{11} - X_{12}\|}{\|1 + X_{11} - X_{12}\|}.
$$

Using the same routine, the second equation of the system can be evaluated as follows

$$
\left(
\begin{array}{c}
(B_2 - D_2 - H_2)(X_{21} - X_{22}) - L_2(X_{21} - X_{22})^2 \\
+\alpha_1 \frac{(X_{11}-X_{12})(X_{21}-X_{22})}{1+(X_{11}-X_{12})} + P_1 B_2(X_{22} - X_{21}), w_2
\end{array}
\right)
$$
$$
\leq (H_2 + D_2 - B_2) \|X_{22} - X_{22}\| \|w_2\| + L_2 \|X_{22} - X_{21}\|^2 \|w_2\| \tag{5.7}
$$
$$
+ \alpha_1 \frac{\|X_{22} - X_{21}\|^2}{\|1 - (X_{22} - X_{21})\|} \|w_2\| + P_1 B_2 \|X_{22} - X_{21}\| \|w_2\|
$$
$$
= \left((H_2 + D_2 - B_2) + L_2 \overline{p} + \alpha \overline{f} + P_1 B_2 \right) \|X_{22} - X_{21}\| \|w_2\|.
$$

Above in equality (5.7),

$$
\overline{p} = \|X_{22} - X_{21}\|, \quad \overline{f} = \alpha_1 \frac{\|X_{22} - X_{21}\|}{\|1 - (X_{22} - X_{21})\|}.
$$

Replacing equation (5.6) and (5.7) into equation (5.3), we obtain

$$
(T(X_{11} - X_{12}, X_{21} - X_{22}), (w_1, w_2))
$$
$$
\leq
\begin{cases}
\left(\begin{array}{c} (B_1 - D_1 - H_1) \\ +L_1\overline{w} + \alpha_1\overline{k} + P_1B_2 \end{array} \right) \|X_{11} - X_{12}\| \, \|w_1\| \\
\left(\begin{array}{c} (H_2 + D_2 - B_2) \\ +L_2\overline{p} + \alpha_1\overline{f} + P_1B_2 \end{array} \right) \|X_{22} - X_{21}\| \, \|w_2\| .
\end{cases}
\tag{5.8}
$$

But, for large number m and n, both solution converge to the exact solution, using the topology concept; we can find two very small positive parameters l_n, l_m such that

$$
\|X_1 - X_{11}\|, \|X_1 - X_{12}\| < \frac{l_n}{2 \left(\begin{array}{c} (B_1 - D_1 - H_1) \\ +L_1\overline{w} + \alpha_1\overline{k} + P_1B_2 \end{array} \right) \|w_1\|},
$$
$$
\|X_2 - X_{21}\|, \|X_2 - X_{22}\| < \frac{l_m}{2 \left(\begin{array}{c} (H_2 + D_2 - B_2) \\ +L_2\overline{p} + \alpha_1\overline{f} + P_1B_2 \end{array} \right) \|w_2\|}.
\tag{5.9}
$$

Thus introducing the exact solution in the right hand side of equation (5.8), using the triangular inequality and finally taking $M = max(n, m), l = max(l_n, l_m)$.

$$
\begin{cases}
\left(\begin{array}{c} (B_1 - D_1 - H_1) + L_1\overline{w} \\ +\alpha_1\overline{k} + P_1B_2 \end{array} \right) \|X_{12} - X_{11}\| \, \|w_1\| \\
\left(\begin{array}{c} (H_2 + D_2 - B_2) + L_2\overline{p} \\ +\alpha_1\overline{f} + P_1B_2 \end{array} \right) \|X_{22} - X_{21}\| \, \|w_2\|
\end{cases}
< \begin{cases} l \\ l \end{cases}
\tag{5.10}
$$

Since l is a very small positive parameter, we conclude on the based of the topology idea that

$$
\begin{cases}
\left(\begin{array}{c} (B_1 - D_1 - H_1) + L_1\overline{w} \\ +\alpha_1\overline{k} + P_1B_2 \end{array} \right) \|X_{12} - X_{11}\| \, \|w_1\| = 0 \\
\left(\begin{array}{c} (H_2 + D_2 - B_2) + L_2\overline{p} \\ +\alpha_1\overline{f} + P_1B_2 \end{array} \right) \|X_{22} - X_{21}\| \, \|w_2\| = 0
\end{cases}
< \begin{cases} 0 \\ 0 \end{cases},
$$

But

$$
\big((B_1 - D_1 - H_1) + L_1\overline{w} + \alpha_1\overline{k} + P_1B_2\big) \neq ,
$$
$$
\big((H_2 + D_2 - B_2) + L_2\overline{p} + \alpha_1\overline{f} + P_1B_2\big) \neq 0,
\tag{5.11}
$$
$$
\|w_1\|, \|w_2\| \neq 0 \Rightarrow
$$

$$
\|X_{11} - X_{12}\| = \|X_{21} - X_{22}\| = 0 \Rightarrow X_{11} = X_{12}, X_{21} = X_{22}.
$$

This completes the proof of uniqueness.

6. Numerical solution

We present here the numerical simulation of the special solution of our model as function of time for different values of alpha. The simulations are depicted in Figures 1–6. Here the blue line represents X_1, the yellow line represents X_2. It is worth noting that, the prediction depends on the value of alpha also beside the theoretical parameters.
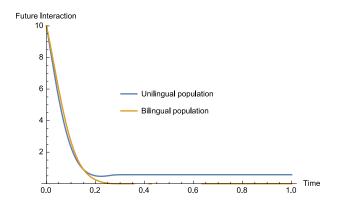
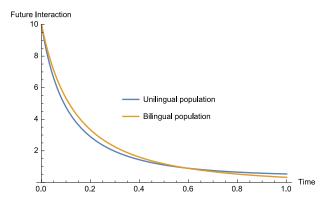Figure 1: Prediction between the two population for alpha = 0.95

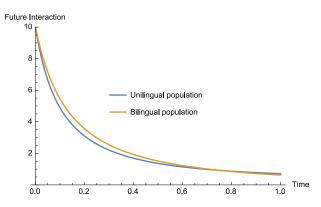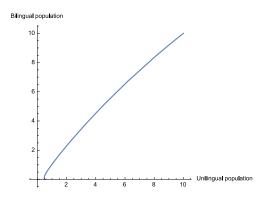Figure 2: Interaction between the two population for alpha = 0.95

Figure 3: Prediction between the two population for alpha = 0.65

Figure 4: Interaction between the two population for alpha = 0.65

Figure 5: Prediction between the two population for alpha = 0.25

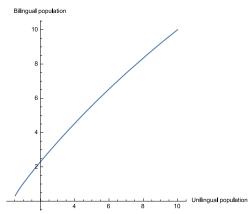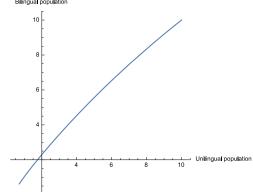Figure 6: Interaction between the two population for alpha = 0.25

7. Conclusion

In order to fully include effect of memory into the model portraying the dynamics of interactions between a bilingual component and a monolingual component of a population in a particular environment, we used the newly proposed derivative with fractional order. The design of the definition allows the description of the memory without any singularity like in the case of Caputo and Riemann-Liouville derivative. With the benefits of the new derivative, we presented the stability of the equilibrium points. We derived the solution of the new equation using an iterative scheme. In order to show the efficiency of the used method, we employed the fixed-point theorem to study the stability analysis of the method for solving the new equation.

We presented in detail the uniqueness of the special derived solution. Some numerical simulations were presented.

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