Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2823–2831
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Generalized Newton Raphson's method free from second derivative

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Communicated by Y. J. Cho

Abstract

In this paper, we suggest and analyze two new iterative methods for solving nonlinear scalar equations namely: the modified generalized Newton Raphson's method and generalized Newton Raphson's method free from second derivative are having convergence of order six and five respectively. We also give several examples to illustrate the efficiency of these methods.

Keywords: Nonlinear equations, Newton's method, generalized Newton Raphson's method, Halley's method
2010 MSC: 65H05.

1. Introduction

Finding roots of nonlinear equations efficiently has widespread applications in numerical mathematics. Due to their importance and significant applications in various branches of science, several methods are being developed for solving $f(x) = 0$ using different techniques such as Taylor series, quadrature formulas, homotopy perturbation method, Adomian decomposition and variational iteration technique [7, 8, 9, 10, 13, 14, 15, 17, 21, 23, 26, 27]. Newton method is an important and basic method [26], which converges quadratically. To improve the local order of convergence, many modified methods have been proposed. See [8, 9] and [21, 23]. Some basic iterative methods are given in literature [1, 2, 3, 4, 5, 18, 19, 20, 22, 25]

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Received 2015-11-09

and the references therein. Success of Newton's method and similar second or less order methods have led to the wrong idea that the higher order iterative methods are meaningless. But the reality is that some of the higher order iterative methods have vast applications and have best performance as compared to those which have low order of convergence. No doubt, higher order iterative methods require more functional evaluations which is the main drawback of these methods [11, 16, 24]. We are interested in finding higher order iterative method free from second derivative.

In this paper, we suggest modified generalized Newton Raphson's method and generalized Newton Raphson's method free from second derivative. Unlike other higher order iterative methods, generalized Newton Raphson's method free from second derivative requires only three evaluations and has fast convergence. We proved that modified generalized Newton Raphson's method has sixth order of convergence and generalized Newton Raphson's method free from second derivative has fifth order convergence. Some examples are given which show the performance of this method as compared to other methods.

2. Iterative methods

Consider the nonlinear algebraic equation

$$f(x) = 0, \tag{2.1}$$

we assume that α is a simple zero of (2.1) and γ is an initial guess sufficiently close to α. Using the Taylor's series around γ for (2.1), we have

$$f(\gamma) + (x-\gamma)f'(\gamma) + \frac{1}{2!}(x-\gamma)^2 f''(\gamma) + \cdots = 0. \tag{2.2}$$

If $f'(\gamma) \neq 0$, we can evaluate the above expression as follows:

$$f(x_k) + (x - x_k)f'(x_k) = 0.$$

If we choose x_{k+1} the root of equation, then we have

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{2.3}$$

This is so-called the Newton's method [26] for root-finding of nonlinear functions, which converges quadratically. From (2.2) one can evaluate

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2f'^2(x_k) - f(x_k)f''(x_k)}. \tag{2.4}$$

This is so-called the Halley's method [6, 11, 12] for root-finding of nonlinear functions, which converges cubically. Simplification of (2.2) yields another iterative method as follows:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f^2(x_k)f''(x_k)}{2f'^3(x_k)}. \tag{2.5}$$

This is known as HouseHölder's method [16] for solving non linear equations in one variable and converges cubically. Again from (2.2), we have

$$x_{k+1} = x_k - \frac{f'(x_k) - \sqrt{f'^2(x_k) - 2f(x_k)f''(x_k)}}{f''(x_k)}, \tag{2.6}$$

which is known as generalize Newton Raphson's method [24]. The order of convergence of generalize Newton Raphson's method (GNR) is three and requires three functional evaluations to solve the nonlinear equations.

3. New iterative methods

Let $f : X \to R$, $X \subset R$ is a scalar function then by using Taylor series expansion one can obtain generalized Newton Raphson's method:

$$x_{n+1} = x_n - \frac{f'(x_n) - \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)}.$$

Algorithm 3.1. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:*

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f'(y_n) - \sqrt{f'^2(y_n) - 2f(y_n)f''(y_n)}}{f''(y_n)}, \end{aligned} \tag{3.1}$$

which is our modified generalized Newton Raphson's method.

In order to find the solution of the given nonlinear equation, we have to calculate the first as well as second derivative of the function $f(x)$, but in several cases, we face such a situation where the second derivative of the function does not exist and our method fails to find the solution. To overcome this difficulty, we use the finite difference approximation of the second derivative as follows:

$$f''(y_n) \approx \frac{f'(x_n) - f'(y_n)}{y_n - x_n} \tag{3.2}$$

Using the above idea, we derive the generalized Newton Raphson's method free from second derivative as follows:

Algorithm 3.2. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:*

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)f'(y_n) - \sqrt{f^2(x_n)f'^2(y_n) - 2f(x_n)f(y_n)f'(x_n)[f'(x_n) - f'(y_n)]}}{f'(x_n)[f'(x_n) - f'(y_n)]}. \end{aligned} \tag{3.3}$$

Algorithm 3.2 is called the generalized Newton Raphson's method free from second derivative. With the help of this method, we can solve such type of non linear equations in which second derivative does not exist. Also this requires only two evaluations of the function and one of its derivatives which shows that the efficiency index of this method is greater as compared to those methods which require second derivative. Several examples are given which shows the best performance of this method as compared to other well known iterative methods which need second derivative.

4. Convergence analysis

In this section, we will show that the convergence order of modified generalized Newton Raphson's method (Algorithm 3.1) is at least six and that of generalized Newton Raphson's method free from second derivative (Algorithm 3.2) is at least five.

Theorem 4.1. *Suppose that α is a root of the equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in the neighborhood of α, then the convergence order of the modified generalized Newton Raphson's method (*Algorithm 3.1*) is six.*

Proof. To analysis the convergence of Algorithm 3.1 , suppose that α is a root of the equation $f(x) = 0$ and e_n be the error at nth iteration, than $e_n = x_n - \alpha$ then by using Taylor series expansion, we have

$$\begin{aligned}f(x) =& f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^2 + \frac{1}{3!}f'''(\alpha)e_n^3 + \frac{1}{4!}f^{(iv)}(\alpha)e_n^4 + \frac{1}{5!}f^{(v)}(\alpha)e_n^5\\ &+ \frac{1}{6!}f^{(vi)}(\alpha)e_n^6 + \frac{1}{7!}f^{(vii)}(\alpha)e_n^7 + O(e_n^8),\end{aligned}$$

$$f(x) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + O(e_n^8)], \tag{4.1}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)], \tag{4.2}$$

where

$$c_n = \frac{1}{n!}\frac{f^{(n)}(\alpha)}{f'(\alpha)}.$$

With the help of (4.1) and (4.2), we get

$$\begin{aligned}y_n =& f'(\alpha)[\alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + (-6c_3^2 + 20c_3c_2^2 - 10c_2c_4\\ &+ 4c_5 - 8c_2^4)e_n^5 + (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 5c_6 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6\\ &+ (-22c_3c_5 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 + 92c_3c_2c_4 - 72c_2^3c_4 + 18c_3^3 - 126c_3^2c_2^2\\ &+ 128c_3c_2^4 - 32c_2^6)e_n^7 + O(e_n^8)],\end{aligned} \tag{4.3}$$

$$\begin{aligned}f(y_n) =& f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + (24c_3c_2^2 - 12c_2^4 - 6c_3^2 - 10c_2c_4\\ &+ 4c_5)e_n^5 + (34c_4c_2^2 - 73c_3c_2^3 + 28c_2^5 + 37c_2c_3^2 - 17c_4c_3 - 13c_2c_5 + 5c_6)e_n^6 + (-160c_3^2c_2^2\\ &+ 206c_3c_2^4 + 44c_5c_2^2 - 104c_2^3c_4 - 64c_2^6 + 104c_3c_2c_4 - 22c_3c_5 + 6c_7 - 16c_2c_6 - 12c_4^2\\ &+ 18c_3^3)e_n^7 + O(e_n^8)],\end{aligned} \tag{4.4}$$

$$\begin{aligned}f'(y_n) =& f'(\alpha)[1 + 2c_2^2e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^4)e_n^4 + (28c_3c_2^3 - 20c_4c_2^2\\ &+ 8c_2c_5 - 16c_2^5)e_n^5 + (-16c_4c_2c_3 - 68c_3c_2^4 + 12c_3^3 + 60c_4c_2^3 - 26c_5c_2^2 + 10c_2c_6 + 32c_2^6)e_n^6\\ &+ (-20c_2c_5c_3 + 112c_4c_3c_2^2 - 84c_2c_3^3 + 160c_3c_2^5 + 36c_4c_3^2 + 72c_2^3c_5 + 12c_2c_7 - 32c_2^2c_6\\ &- 24c_2c_4^2 - 168c_4c_2^4 - 64c_2^7)e_n^7 + O(e_n^8)],\end{aligned} \tag{4.5}$$

$$\begin{aligned}f''(y_n) =& f'(\alpha)[2c_2 + 6c_2c_3e_n^2 + (12c_3^2 - 12c_3c_2^2)e_n^3 + (-42c_2c_3^2 + 18c_4c_3 + 24c_3c_2^3 + 12c_4c_2^2)e_n^4\\ &+ (-12c_2c_4c_3 + 24c_5c_3 - 36c_3^3 + 120c_3^2c_2^2 - 48c_3c_2^4 - 48c_4c_2^3)e_n^5 + (-78c_3c_2c_5 + 30c_3c_6\\ &- 54c_4c_3^2 - 96c_3c_4c_2^2 + 198c_2c_3^3 - 312c_3^2c_2^3 + 96c_3c_2^5 + 72c_2c_4^2 + 144c_4c_2^4 + 20c_5c_2^3)e_n^6\\ &+ (72c_3c_4^2 + 72c_4c_2c_3^2 + 576c_3c_4c_2^3 - 384c_2^2c_4^2 - 384c_4c_2^5 + 96c_2c_5c_4 - 132c_5c_3^2 + 336c_3c_5c_2^2\\ &+ 36c_3c_7 - 96c_3c_2c_6 + 108c_3^4 - 756c_3^3c_2^2 + 768c_3^2c_2^4 - 192c_3c_2^6 - 120c_5c_2^4)e_n^7 + O(e_n^8)].\end{aligned} \tag{4.6}$$

Using equations (4.3), (4.4), (4.5), (4.6) in Algorithm 3.1, we get

$$x_{n+1} = \alpha - c_3c_2^3e_n^6 + (-6c_3^2c_2^2 + 6c_3c_2^4)e_n^7 + O(e_n^8),$$

which implies that

$$e_{n+1} = -c_3c_2^3e_n^6 + (-6c_3^2c_2^2 + 6c_3c_2^4)e_n^7 + O(e_n^8).$$

The above equation shows that the order of convergence of modified generalized Newton Raphson's method (Algorithm 3.1) is six. □

Theorem 4.2. *Suppose that α is a root of the equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in the neighborhood of α, then the convergence order of the generalized Newton Raphson's method free from second derivative (*Algorithm 3.2*) is five.*

Proof. To analyze the convergence of the generalized Newton Raphson's method free from second derivative (Algorithm 3.2), suppose that α is a root of the equation $f(x) = 0$ and e_n be the error at nth iteration, than $e_n = x_n - \alpha$ then by using Taylor series expansion, we have

$$f(x) = f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^2 + \frac{1}{3!}f'''(\alpha)e_n^3 + \frac{1}{4!}f^{(iv)}(\alpha)e_n^4 + \frac{1}{5!}f^{(v)}(\alpha)e_n^5 + \frac{1}{6!}f^{(vi)}(\alpha)e_n^6 + \frac{1}{7!}f^{(vii)}(\alpha)e_n^7 + O(e_n^8),$$

$$f(x) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + O(e_n^8)], \tag{4.7}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)], \tag{4.8}$$

where

$$c_n = \frac{1}{n!}\frac{f^{(n)}(\alpha)}{f'(\alpha)}.$$

With the help of (4.7) and (4.8), we get

$$\begin{aligned} y_n =& f'(\alpha)[\alpha + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + (-6c_3^2 + 20c_3c_2^2 - 10c_2c_4 + 4c_5 \\ &- 8c_2^4)e_n^5 + (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 5c_6 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6 + (-22c_3c_5 \\ &+ 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 + 92c_3c_2c_4 - 72c_2^3c_4 + 18c_3^3 - 126c_3^2c_2^2 + 128c_3c_2^4 - 32c_2^6)e_n^7 \\ &+ O(e_n^8)], \end{aligned} \tag{4.9}$$

$$\begin{aligned} f(y_n) =& f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7_c2c_3 + 5c_2^3)e_n^4 + (24c_3c_2^2 - 12c_2^4 - 6c_3^2 - 10c_2c_4 \\ &+ 4c_5)e_n^5 + (34c_4c_2^2 - 73c_3c_2^3 + 28c_2^5 + 37c - 2c_3^2 - 17c_4c_3 - 13c_2c_5 + 5c_6)e_n^6 \\ &+ (-160c_3^2c_2^2 + 206c_3c_2^4 + 44c_5c_2^2 - 104c_2^3c_4 - 64c_2^6 + 104c_3c_2c_4 - 22c_3c_5 + 6c_7 \\ &- 16c_2c_6 - 12c_4^2 + 18c_3^3)e_n^7 + O(e_n^8)], \end{aligned} \tag{4.10}$$

$$\begin{aligned} f'(y_n) =& f'(\alpha)[1 + 2c_2^2e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^4)e_n^4 + (28c_3c_2^3 - 20c_4c_2^2 \\ &+ 8c_2c_5 - 16c_2^5)e_n^5 + (-16c_4c_2c_3 - 68c_3c_2^4 + 12c_3^3 + 60c_4c_2^3 - 26c_5c_2^2 + 10c_2c_6 + 32c_2^6)e_n^6 \\ &+ (-20c_2c_5c_3 + 112c_4c_3c_2^2 - 84c_2c_3^3 + 160c_3c_2^5 + 36c_4c_3^2 + 72c_2^3c_5 + 12c_2c_7 - 32c_2^2c_6 \\ &- 24c_2c_4^2 - 168c_4c_2^4 - 64c_2^7)e_n^7 + O(e_n^8)]. \end{aligned} \tag{4.11}$$

Using equations (4.9), (4.10) and (4.11) in Algorithm 3.2, we get

$$x_{n+1} = \alpha - \frac{3}{2}c_3c_2^2e_n^5 + O(e^6),$$

which implies that

$$e_{n+1} = -\frac{3}{2}c_3c_2^2e_n^5 + O(e^6).$$

This shows that the generalized Newton Raphson's method free from second derivative is of fifth order of convergence. □

5. Comparisons of efficiency index

The term "efficiency index" tells us how fast and efficient our method is. It is used to analyze the performance of different iterative methods. It depends upon the two factors, one of which is the order of convergence and the other is number of function evaluations and derivative evaluations of the iterative method. If the order of convergence is denoted by "r" and "N_f" denote the number of functional and derivative evaluations of an iterative method, then the efficiency index E.I is mathematically defined as:

$$E.I = r^{\frac{1}{N_f}}.$$

Since the order of convergence of Newton's method is two and it requires one function evaluation and one of its derivative, so the Newton's method has an efficiency of $2^{\frac{1}{2}} \approx 1.4142$, Similarly the efficiency index of Halley's method and HouseHölder's method is $3^{\frac{1}{3}} \approx 1.4422$, because both methods require one function evaluations and two derivative evaluations and these methods achieve cubic order of convergence. The generalized Newton Raphson's method has cubic convergence, requires one function evaluations and two derivative evaluations so that its efficiency index is $3^{\frac{1}{3}} \approx 1.4422$.

Our modified generalized Newton Raphson's method (Algorithm 3.1) developed in this paper has six order of convergence, requires two function evaluations and two of its derivative so that the efficiency index is $6^{\frac{1}{4}} \approx 1.5651$.

Now, we move to calculate the efficiency index of our generalized Newton Raphson's method free from second derivative (Algorithm 3.2) as follows:

The generalized Newton Raphson's method free from second derivative need two evaluations of the function and one of its first derivatives. So the total number of evaluations of this method is three. i.e

$$N_f = 3.$$

Also, in the earlier section, we have proved that the order of convergence of the generalized Newton Raphson's method free from second derivative is five. i.e

$$r = 5.$$

Thus the efficiency index of the generalized Newton Raphson's method free from second derivative is:

$$E.I = 5^{\frac{1}{3}} \approx 1.7100.$$

The efficiencies of the methods we have discussed are summarized in Table 1 given below.

Table 1: Comparison of efficiencies of various methods

Method	Number of function or derivative evaluations	Efficiency index
Newton, quadratic	2	$2^{\frac{1}{2}} \approx 1.4142$
Halley, Cubic	3	$3^{\frac{1}{3}} \approx 1.4422$
HouseHölder,Cubic	3	$3^{\frac{1}{3}} \approx 1.4422$
Generalized Newton Raphson, Cubic	3	$3^{\frac{1}{3}} \approx 1.4422$
Modified Generalized Newton Raphson's Method 6th order	4	$6^{\frac{1}{4}} \approx 1.5651$
Generalized Newton Raphson, free from second derivative 5th order	3	$5^{\frac{1}{3}} \approx 1.7100$

It can be seen from the above comparison table that the efficiency index of the generalized Newton Raphson's method free from second derivative is much higher as compare to other iterative methods.

6. Applications

In this section, we included following nonlinear test functions to illustrate the efficiency of our developed modified generalized Newton Raphson's method (MGNRM) and generalized Newton Raphson's method free from second derivative (GNRM(Free)) by comparing with the generalized Newton Raphson's method (GNRM), Newton Raphson's method (NR) , Halley's method (HM) and HouseHölder's method (HHM),

$$\begin{aligned} &f(x) = x^3 - x^2 + 3x\cos(x) - 1, && f(x) = (1+\cos(x))(e^x - 2), \\ &f(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}, && f(x) = \ln(x) - \cos(x), \\ &f(x) = x + \tan^{-1}(x), && f(x) = x - \ln(x+2). \end{aligned}$$

Table 2: Comparison of NR, HM, HHM, GNRM, MGNRM and GNRM(Free)($f(x) = x^3 - x^2 + 3x\cos(x) - 1$, $x_0 = 0.7$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NR	5	10	$2.160638e-16$	
HM	4	12	$1.071256e-38$	0.39532362298631518838
HHM	4	12	$2.201001e-15$	
GNRM	3	9	$8.759561e-34$	
MGNRM	2	8	$1.101228e-20$	
MGNRM(Free)	2	4	$2.813259e-15$	

Table 3: Comparison of NR, HM, HHM, GNRM, MGNRM and GNRM(Free) ($f(x) = (1+\cos(x))(e^x - 2)$, $x_0 = 1.3$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NR	5	10	$6.460582e-29$	
HM	4	12	$3.838955e-37$	0.69314718055994530942
HHM	4	12	$7.023596e-31$	
GNRM	4	12	$2.683726e-42$	
MGNRM	2	8	$2.269527e-26$	
GNRM(Free)	2	4	$9.880439e-16$	

Table 4: Comparison of NR, HM, HHM, GNRM, MGNRM and GNRM(Free) ($f(x) = x^2 + \sin(\frac{x}{5}) - \frac{1}{4}$, $x_0 = 1$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NR	6	12	$1.214687e-28$	
HM	4	12	$4.877772e-36$	0.40999201798913713162
HHM	4	12	$7.597933e-29$	
GNRM	3	9	$1.851166e-44$	
MGNRM	2	8	$2.278813e-35$	
GNRM(Free)	2	4	$2.300783e-26$	

Table 5: Comparison of NR, HM, HHM, GNRM, MGNRM and GNRM(Free) ($f(x) = \ln(x) - \cos(x)$, $x_0 = 2.4$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NR	5	10	$5.932950e-22$	
HM	4	12	$4.424728e-40$	1.30296400121601255250
HHM	4	12	$1.464880e-22$	
GNRM	3	9	$8.042814e-19$	
MGNRM GNRM(Free)	2	4	$2.432996e-16$	

Table 6: Comparison of NR, HM, HHM, GNRM, MGNRM and GNRM(Free) ($f(x) = x + \tan^{-1}(x)$, $x_0 = 1$)

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
NR	4	8	$3.492065e-26$	
HM	3	9	$4.604066e-23$	1.36523001341409684576
HHM	3	9	$1.374106e-15$	
GNRM	3	9	$2.568603e-19$	
MGNRM	2	8	$1.143491e-29$	
GNRM(Free)	2	6	$1.689295e-17$	

Table 7: Comparison of NR, HM, HHM, GNRM, MGNRM and GNRM(Free) ($f(x) = x - \ln(x+2)$, $x_0 = 3$)

Method	N	N_f	$\|f(x_{n+1})\|$	x_{n+1}
NR	4	8	$2.049559e-16$	
HM	3	9	$1.475989e-20$	1.14619322062058258520
HHM	3	9	$8.065461e-20$	
GNRM	3	9	$1.644856e-21$	
MGNRM	2	8	$5.396657e-34$	
GNRM(Free)	2	4	$4.017479e-22$	

Tables 2–7 show the numerical comparisons of Newton's method, Halley's method, HouseHölder's method, generalized Newton Raphson's method, modified generalized Newton Raphson's method and the generalized Newton Raphson's method free from second derivative. The columns represent the number of iterations N and the number of function or derivative evaluations N_f required to meet the stopping criteria, and the magnitude $|f(x)|$ of $f(x)$ at the final estimate x_n.

7. Conclusions

The modified generalized Newton Raphson's method (Algorithm 3.1) and the generalized Newton Raphson's method free from second derivative (Algorithm 3.2) for solving non linear functions have been established. We can conclude from Tables 1–7 that

1. The efficiency index of the modified generalized Newton Raphson's method is 1.5651 and the efficiency index of the generalized Newton Raphson's method free from second derivative is 1.7100.
2. The order of convergence of the modified generalized Newton Raphson's method is six and the order of convergence of the generalized Newton Raphson's method free from second derivative is five.
3. By using some examples the performance of modified generalized Newton Raphson's method and generalized Newton Raphson's method free from second derivative is also discussed. The modified generalized Newton Raphson's method and generalized Newton Raphson's method free from second derivative are performing fast as compared to Newton Raphson's method (NR), Halley's method (HM), HouseHölder's method (HHM) and generalized Newton Raphson's method (GNRM) as discussed in Tables 2–7.

Acknowledgement

We are thankful to the editor and the referees for valuable comments which helps us to improve this article. This study was supported by research funds from Dong-A University.

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