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Generalized $\alpha - \psi$ contractive mappings in partial b-metric spaces and related fixed point theorems

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Abstract

In this paper, we introduce some concepts in partial b-metric spaces. We establish fixed point theorems for some new generalized $\alpha - \psi$ type contractive mappings in the setting of partial b-metric spaces. Some examples are presented to illustrate our obtained results. Finally, we show that the results generalize some recent results.

Keywords: Contractive mapping, partial b-metric space, fixed point theorem.
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1. Introduction and Preliminaries

In the last few decades, fixed point theory was one of the most interesting research fields in nonlinear functional analysis. Fixed point theory results are widely used in the economy, computer science, engineering etc. The most remarkable result is the Banach Contraction Principle [8] in this direction.

Fixed points theorems for $\alpha - \psi$ type contractive mappings in metric spaces were firstly obtain by Samet *et al.* [26] in 2012, and then by Karapinar and Samet [15]. In this direction several authors obtained further results (see, e.g., [3, 4, 9, 16, 24]).

Let Ψ be family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

It is easy to show that $\lim_{n\to+\infty} \psi^n(t) = 0$ and this implies $\psi(t) < t$.

Definition 1.1 ([26]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

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Definition 1.2 ([26])**.** Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X.$$

Remark 1.3. We easily see any $\alpha - \psi$ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt, k \in (0, 1)$ satisfies the Banach contraction.

The concept of b-metric space was introduced by Baktin [7] and by Czerwik in [12, 13]. After that, several interesting results about the existence of fixed point in b-metric spaces have been obtained (see, e.g.,[1, 2, 5, 6, 10, 11, 14, 17, 18, 19, 20, 21, 22, 23, 25, 27]). Very recently, Shukla [27] and Mustafa [17] obtained fixed point theorems in partial b-metric spaces.

Definition 1.4 ([17])**.** Let X be a nonempty set and the mapping $b : X \times X \to R^+$ satisfies:
(b1) $b(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(b2) $b(x, y) = b(y, x)$ for all $x, y \in X$;
(b3) there exists a real number $s \geq 1$ such that $b(x, y) \leq s[b(x, z) + b(z, y)]$ for all $x, y, z \in X$.
Then b is called a b-metric on X and (X, b) is called a b-metric space with coefficient s.

Definition 1.5 ([17])**.** Let X be a nonempty set and the mapping $p : X \times X \to R^+$, for all $x, y, z \in X$ satisfies:
(p1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
Then p is called a partial metric on X and (X, p) is called a partial metric space.

Definition 1.6 ([17])**.** Let X be a nonempty set and the mapping $p_b : X \times X \to R^+$, for all $x, y, z \in X$ satisfies:
(p_b1) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_b2) $p_b(x, x) \leq p_b(x, y)$;
(p_b3) $p_b(x, y) = p_b(y, x)$;
(p_b4) there exists a real number $s \geq 1$ such that $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.
Then p_b is called a partial b-metric on X and (X, p_b) is called a partial b-metric space.

Remark 1.7. Any metric is a partial metric, b-metric and partial b-metric, but the converse is not true in general.

Remark 1.8. It is clear that every b-metric space is a partial b-metric with coefficient $s = 1$ and zero self-distance, and every partial metric space is a partial b-metric with coefficient $s = 1$. However, the converse of this fact need not hold.

Example 1.9 ([27])**.** Let $X = R^+, q > 1$ be a constant and $p_b : X \times X \to R^+$ be defined by

$$p_b(x, y) = [\max\{x, y\}]^q + |x - y|^q, \quad \text{for all} \quad x, y \in X.$$

Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial space.

Now, we present some definitions and propositions in partial b-metric space.

Definition 1.10 ([17])**.** Let (X, p_b) be a partial b-metric space, then for $x \in X$ and $\epsilon > 0$, the p_b-ball with center x and radius ϵ is

$$B_{p_b}(x, \epsilon) = \{y \in X | p_b(x, y) < p_b(x, x) + \epsilon\}.$$

For example, let (X, p_b) be the partial b-metric space from Example 1.9 (with $q = 2$). Then

$$B_{p_b}(1, 4) = \{y \in X | p_b(1, y) < p_b(1, 1) + 4\} = (0, 2).$$

Proposition 1.11 ([17])**.** Let (X, p_b) be a partial b-metric space, for all $x \in X$ and $r > 0$, if $y \in B_{p_b}(x, r)$, then there exist $\delta > 0$ such that $B_{p_b}(y, \delta) \subseteq B_{p_b}(x, r)$.

Thus, from the above proposition the family of all p_b-balls

$$\triangle = \{B_{p_b}(x, r) | x \in X, r > 0\}$$

is a base of a T_0 topology τ_{p_b} on X which we call the p_b-metric topology. It is T_0, but need not be T_1.

Definition 1.12 ([17])**.** A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is said to be:

(i) p_b-convergent to a $x \in X$ if $\lim_{n\to\infty} p_b(x, x_n) = p_b(x, x)$;
(ii) A p_b-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists (and is finite);
(iii) A partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b-converges to a point $x \in X$ such that $\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n,m\to\infty} p_b(x_n, x) = p_b(x, x)$.

Note that in a partial b-metric space the limit of convergent sequence may not be unique.

Proposition 1.13 ([17])**.**

(1) A sequence $\{x_n\}$ is A p_b-Cauchy sequence in a partial b-metric space (X, p_b) if and only if it is A b-Cauchy sequence in a b-metric space (X, b).
(2) A partial b-metric space (X, p_b) is p_b-complete if and only if b-metric space (X, b) is b-complete.

Definition 1.14 ([17])**.** Let (X, p_b) and (X', p'_b) be two partial b-metric spaces, let $T : (X, p_b) \to (X', p'_b)$ be a mapping. Then T is said to be p_b-continuous at a point $a \in X$ if for a given $\epsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $pb(a, x) < \delta + p_b(a, a)$ imply $pb(Ta, Tx) < \epsilon + p_b(Ta, Ta)$. The mapping T is p_b-continuous on X if it is p_b-continuous at all $a \in X$.

Lemma 1.15 ([17])**.** *Let (X, p_b) and (X', p'_b) be two partial b-metric spaces. Then $T : X \to X'$ is p_b-continuous at $x \in X$ if and only if it is p_b-sequentially continuous at x, that is, whenever $\{x_n\}$ is p_b-convergent to x, then $\{Tx_n\}$ is p_b-convergent to Tx.*

2. Main results

Since that $lim_{n\to\infty}\psi^n(t) = 0$, for all $t > 0$, this implies each $\epsilon > 0$, there exist $N(\epsilon) \in N$, $n \geq N(\epsilon)$ such that $\psi^n(\epsilon) < \frac{\epsilon}{2s}$. We use n_0 note that $N(\epsilon)$ with $\psi^{n_0}(\epsilon) < \frac{\epsilon}{2s}$.

Lemma 2.1. *$\{x_n\}$ is a sequence in partial b-metric space. Then*

$$p_b(x_{n+p}, x_n) \leq \sum_{i=1}^{i=p} s^i p_b(x_{n+i}, x_{n+i-1}) \text{ for all } p, n \in N, p \geq 1.$$

Proof. Using the triangular inequality, we get

$$\begin{aligned} p_b(x_{n+p}, x_n) &\leq s[p_b(x_{n+p}, x_{n+1}) + p_b(x_{n+1}, x_n)] - p_b(x_{n+1}, x_{n+1}) \\ &\leq s[p_b(x_{n+p}, x_{n+1}) + p_b(x_{n+1}, x_n)], \end{aligned}$$

recursively, we can obtain

$$p_b(x_{n+p}, x_n) \leq \sum_{i=1}^{i=p} s^i p_b(x_{n+i}, x_{n+i-1}).$$

□

Definition 2.2. Let (X, p_b) be a partial b-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ such that

$$\alpha(x, y)p_b(Tx, Ty) \leq \psi(p_b(x, y)), \quad \forall x, y \in X. \tag{2.1}$$

Theorem 2.3. *Let (X, p_b) be a complete partial b-metric space. Suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by (2.1) which satisfies:*

(i) T is α-admissible;
(ii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^{n_0}x_0) \geq 1$, there n_0 satisfies $\psi^{n_0}(\epsilon) < \frac{\epsilon}{2s}, \epsilon > 0$;
(iii) T is continuous.

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, T^{n_0}x_0) \geq 1$, there is n_0 satisfying $\forall \epsilon > 0, \psi^n(\epsilon) < \frac{\epsilon}{2s}$ as $n \geq n_0$. Take $F = T^{n_0}$ and $x_{k+1} = Fx_k, \forall k \in N$. By condition (i), we can easily show that F is α-admissible, then for all $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Fx, Fy) \geq 1. \tag{2.2}$$

Since T is α-admissible, for all $n \in N$ we easily obtain

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(T^n x, T^n y) \geq 1. \tag{2.3}$$

So, by (2.1) and (2.3), for all $\alpha(x, y) \geq 1$ we have

$$\begin{aligned} p_b(Fx, Fy) &= p_b(T^{n_0}x, T^{n_0}y) \\ &\leq \alpha(T^{n_0-1}x, T^{n_0-1}y) p_b(TT^{n_0-1}x, TT^{n_0-1}y) \\ &\leq \psi(p_b(T^{n_0-1}x, T^{n_0-1}y)), \end{aligned}$$

recursively, it implies that

$$p_b(Fx, Fy) \leq \psi^{n_0}(p_b(x, y)). \tag{2.4}$$

Also, From (2.2), we have

$$\alpha(x_0, x_1) = \alpha(x_0, T^{n_0}x_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Fx_0, Fx_1) \geq 1,$$

by indiction, we get

$$\alpha(x_k, x_{k+1}) \geq 1, \quad \forall k \in N. \tag{2.5}$$

Using (2.4), we have

$$p_b(x_k, x_{k+1}) = p_b(Fx_{k-1}, Fx_k) \leq \psi^{n_0} p_b(x_{k-1}, x_k). \tag{2.6}$$

Recursively, we get

$$p_b(x_k, x_{k+1}) = p_b(Fx_{k-1}, Fx_k) \leq \psi^{n_0 k}(p_b(x_0, x_1)). \tag{2.7}$$

Let $k \to \infty$ in the above inequality, we have $p_b(x_k, x_{k+1}) \to 0$.

Now we choose $k_0 \in N$, for all $\epsilon > 0, k \geq k_0$ such that

$$p_b(x_k, x_{k+1}) \leq \frac{\epsilon}{2s}. \tag{2.8}$$

According to (2.6), (2.8) and condition(ii), we have

$$p_b(x_{k+1}, x_{k+2}) \leq \psi^{n_0}(p_b(x_k, x_{k+1})) \leq \psi^{n_0}(\frac{\epsilon}{2s}) < \frac{\epsilon}{(2s)^2},$$

by indiction, for all $p \in N, p \geq 1$, we find

$$p_b(x_{k+p-1}, x_{k+p}) < \frac{\epsilon}{(2s)^p}. \tag{2.9}$$

Using Lemma 2.1 and (2.9), for all $k, p \in N, p \geq 1, k \geq k_0$, we derive

$$p_b(x_{k+p}, x_k) < \sum_{i=1}^{p} \frac{\epsilon}{2^i} < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon. \tag{2.10}$$

Hence, $\{x_k\}$ is Cauchy sequence in complete partial b-metric space (X, p_b). It implies that there exists $x^* \in X$ such that $x_n \to x^*$, then

$$\lim_{n\to\infty} p_b(x_n, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x^*, x^*) = 0. \tag{2.11}$$

Since T is continuous, by Lemma 1.15, then $Tx_n \to Tx^*$, i.e.

$$\lim_{n\to\infty} p_b(Tx_n, Tx^*) = p_b(Tx^*, Tx^*). \tag{2.12}$$

Next we show x^* is a fixed point of T. By condition (ii), we have

$$\alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, T^2x_0) \geq 1,$$

for all $n \in N$, by indiction, we get

$$\alpha(T^n x_0, T^{n+1}x_0) \geq 1. \tag{2.13}$$

So, using (2.13) and (2.1), then

$$\begin{aligned} p_b(x_k, Tx_k) &= p_b(TT^{kn_0-1}x_0, TT^{kn_0}x_0) \\ &\leq \alpha(T^{kn_0-1}x_0, T^{kn_0}x_0)p_b(TT^{kn_0-1}x_0, TT^{kn_0}x_0) \\ &\leq \psi(p_b(T^{kn_0-1}x_0, T^{kn_0}x_0)), \end{aligned}$$

recursively, we get

$$p_b(x_k, Tx_k) \leq \psi^{kn_0}((p_b(x_0, Tx_0)). \tag{2.14}$$

Let $k \to \infty$, we have

$$p_b(x_k, Tx_k) \to 0. \tag{2.15}$$

For all $k \geq k_0, p \geq 1$, from (2.1), (2.5) and (2.9), we derive

$$\begin{aligned} p_b(Tx_{k+p-1}, Tx_{k+p}) &\leq \alpha(x_{k+p-1}, x_{k+p})p_b(Tx_{k+p-1}, Tx_{k+p}) \\ &\leq \psi(p_b(x_{k+p-1}, x_{k+p})) \leq \psi(\frac{\epsilon}{(2s)^p}) < \frac{\epsilon}{(2s)^p}. \end{aligned} \tag{2.16}$$

Using Lemma 2.1, similarly, we can obtain

$$p_b(Tx_{k+p}, Tx_k) < \epsilon.$$

Which implies $\{Tx_k\}$ is also a cauchy sequence, so by (2.12), we have

$$\lim_{n\to\infty} p_b(Tx_n, Tx^*) = \lim_{n,m\to\infty} p_b(Tx_n, Tx_m) = p_b(Tx^*, Tx^*) = 0. \tag{2.17}$$

Using the triangle inequality, we obtain

$$\begin{aligned} p_b(x^*, Tx^*) &\leq s(p_b(x_n, x^*) + p_b(x_n, Tx^*)) - p_b(x_n, x_n) \\ &\leq sp_b(x_n, x^*) + s^2p_b(x_n, Tx_n)) + s^2p_b(Tx_n, Tx^*). \end{aligned}$$

Let $n \to \infty$, by (2.11), (2.15), (2.17), we have $p_b(x^*, Tx^*) = 0$, then $x^* = Tx^*$, Therefore x^* is a fixed point of T. □

Theorem 2.4. *Let (X, p_b) be a complete partial b-metric space. Suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping which satisfies:*

(i) T is $\alpha -$ admissible;

(ii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^{n_0}x_0) \geq 1$, there n_0 satisfies $\psi^{n_0}(\epsilon) < \frac{\epsilon}{2s}, \varepsilon > 0$;

(iii) if $\{x_n\}$ is a sequence in (X, p_b) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a fixed point.

Proof. Following the proof of Theorem 2.3, we know $\{x_n\}$ satisfying (2.11), (2.15) and the condition (iii), i.e $\alpha(x_n, x*) \geq 1$. Then by (2.1), we have

$$p_b(Tx_n, Tx*) \leq \alpha(x_n, x*)p_b(Tx_n, Tx*) \leq \psi(p_b(x_n, x*)) \leq p_b(x_n, x*). \tag{2.18}$$

Let $n \to \infty$, from (2.11) we get

$$\lim_{n\to\infty} p_b(Tx_n, Tx^*) = 0. \tag{2.19}$$

Also, using the triangle inequality, we have

$$\begin{aligned} p_b(x^*, Tx^*) &\leq s(p_b(x_n, x^*) + p_b(x_n, Tx^*)) - p_b(x_n, x_n) \\ &\leq sb(x_n, x^*) + s^2 b(x_n, Tx_n) + s^2 b(Tx_n, Tx^*). \end{aligned}$$

Let $n \to \infty$, hence, by (2.11), (2.15), (2.19), we obtain $p_b(x^*, Tx^*) = 0$, then $x^* = Tx^*$, therefore x^* is a fixed point of T. □

Example 2.5. Let $X = R^+$, endowed with the partial b-metric $p_b(x, y) = |x - y|^2 + (\max\{x, y\})^2$(with $s = 2$) for all $x, y \in R^+$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2x - \frac{3}{2}, & x > 1; \\ \frac{x}{2}, & 0 \leq x \leq 1. \end{cases}$$

We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & if \quad x \in [0, 1]; \\ 0, & otherwise. \end{cases}$$

Clearly T is α-admissible and an $\alpha - \psi$ contractive mapping with $\psi(t) = \frac{t}{4}$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have

$$\alpha(x, y)p_b(Tx, Ty) \leq \frac{1}{4}p_b(x, y).$$

Moreover, there exists $x_0 = 1 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(1, \frac{1}{2}) = 1$$

and

$$\alpha(x_0, T^n x_0) = \alpha(1, \frac{1}{2^n}) = 1.$$

Obviously T is continuous.

Now, all the hypotheses of Theorem 2.3 are satisfied. T has a fixed point. In this example, 0 and $\frac{3}{2}$ are two fixed point of T.

Example 2.6. Let $X = R^+$, endowed with the partial b-metric $p_b(x, y) = |x - y|^2 + (\max\{x, y\})^2$(with $s = 2$) for all $x, y \in R^+$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2x - \frac{3}{2}, & x > 1; \\ \frac{x}{4}, & 0 \leq x \leq 1. \end{cases}$$

It is clear that T is not continuous at 1. We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & if \quad x \in [0, 1]; \\ 0, & otherwise. \end{cases}$$

Clearly T is α-admissible and an $\alpha - \psi$ contractive mapping with $\psi(t) = \frac{t}{16}$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have

$$\alpha(x,y)p_b(Tx,Ty) \leq \frac{1}{16}p_b(x,y).$$

Moreover, there exists $x_0 = 1 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(1, \frac{1}{4}) = 1$$

and

$$\alpha(x_0, T^n x_0) = \alpha(1, \frac{1}{4^n}) = 1.$$

Finally, let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$, we have $x_n \in [0,1]$ for all $n \in N$ and $x \in [0,1]$. Then $\alpha(x_n, x) \geq 1$.

Now, all the hypotheses of Theorem 2.4 are satisfied. T has a fixed point. In this example, 0 and $\frac{3}{2}$ are two fixed point of T.

To the uniqueness of a fixed point of a generalized $\alpha - \psi$ contractive mapping, we will consider the following hypothesis.

(H): For all $x, y \in X$ there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Theorem 2.7. *Adding condition (H) to the hypotheses of Theorem* 2.3 *(resp. Theorem* 2.4*) we obtain uniqueness of the fixed point of* T*.*

Proof. Suppose that $x*$ and $y*$ are two fixed point of T. By condition (H), there exists $z \in X$ such that

$$\alpha(x*, z) \geq 1 \text{ and } \alpha(y*, z) \geq 1.$$

Since T is α-admissible, from the above inequalities, for all $n \in N$, we have

$$\alpha(x*, T^n z) \geq 1 \text{ and } \alpha(y*, T^n z) \geq 1. \tag{2.20}$$

Using (2.1) and (2.20), we get

$$\begin{aligned} p_b(x*, T^n z) &= p_b(Tx*, T^n z) \\ &\leq \alpha(x*, T^{n-1}z)p_b(Tx*, T^n z) \\ &\leq \psi(p_b(x*, T^{n-1}z)), \end{aligned}$$

recursively, for all $n \in N$, we obtain

$$p_b(x*, T^n z) \leq \psi^n(p_b(x*, z)),$$

let $n \to \infty$, then

$$\lim_{n\to\infty} p_b(x*, T^n z) = 0. \tag{2.21}$$

Similarly, we can get

$$\lim_{n\to\infty} p_b(y*, T^n z) = 0. \tag{2.22}$$

Also, using the triangle inequality, we have

$$p_b(x*, y*) \leq sp_b(x*, T^n z) + sp_b(x*, T^n z).$$

Let $n \to \infty$, using (2.21) and (2.22), we get $p_b(x*, y*) = 0$, then $x* = y*$. □

Definition 2.8. Let (X, p_b) be a partial b-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$, for all $x, y \in X, s \geq 1$ such that

$$\alpha(x,y)p_b(Tx,Ty) \leq \psi(\max\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{1}{2s^2}(p_b(x,Ty) + p_b(y,Tx))\}). \tag{2.23}$$

Theorem 2.9. *Let (X, p_b) be a complete partial b-metric space. Suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by (2.23) which satisfies:*

(i) T is α-admissible;

(ii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^{n_0}x_0) \geq 1$, there n_0 satisfies $\psi^{n_0}(\epsilon) < \frac{\epsilon}{2s}, \epsilon > 0$;

(iii) T is continuous.

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^{n_0}x_0) \geq 1$, there n_0 satisfying $\forall \epsilon > 0, \psi^n(\epsilon) < \frac{\epsilon}{2s}$ as $n \geq n_0$. Take $x_{n+1} = Tx_n, \quad n \in N$, if $x_{n+1} = x_n$ for some $n \in N$, then $x^* = x_n$ is a fixed point of T. Assumed that $x_{n+1} \neq x_n$, take $y_{k+1} = Fy_k$, for all $k \in N$, $y_0 = x_0$ and $F = T^{n_0}$, then we have $y_k = x_{n_0 k}$ and we may easily show that F is α-admissible, then for all $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Fx, Fy) \geq 1. \tag{2.24}$$

Since T and F are $\alpha - admissible$, by condition (ii), we get

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1,$$

$$\alpha(x_0, T^{n_0}x_0) = \alpha(x_0, x_{n_0}) \geq 1 \Rightarrow \alpha(Tx_0, Tx_{n_0}) = \alpha(x_1, x_{n_0+1}) \geq 1$$

and

$$\alpha(y_0, y_1) = \alpha(x_0, T^{n_0}x_0) \geq 1 \Rightarrow \alpha(Fy_0, Fy_1) = \alpha(y_1, y_2) \geq 1,$$

by indiction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{2.25}$$

$$\alpha(x_n, x_{n+n_0}) \geq 1 \tag{2.26}$$

and

$$\alpha(y_k, y_{k+1}) \geq 1. \tag{2.27}$$

From (2.23), (2.25) and the triangle inequality, we have

$$\begin{aligned}
p_b(x_n, x_{n+1}) &= p_b(Tx_{n-1}, Tx_n) \\
&\leq \alpha(x_{n-1}, x_n) p_b(Tx_{n-1}, Tx_n) \\
&\leq \psi(\max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, Tx_{n-1}), p_b(x_n, Tx_n), \\
&\qquad \frac{1}{2s^2}(p_b(x_{n-1}, Tx_n) + p_b(x_n, Tx_{n-1}))\}) \\
&= \psi(\max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \frac{1}{2s^2}(p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n))\}) \\
&\leq \psi(\max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \frac{1}{2s}(p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}))\}) \\
&= \psi(\max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\}).
\end{aligned} \tag{2.28}$$

If $p_b(x_{n-1}, x_n) < p_b(x_n, x_{n+1})$, by (2.28), then

$$p_b(x_n, x_{n+1}) \leq \psi(p_b(x_n, x_{n+1})) < p_b(x_n, x_{n+1}).$$

It is a contradiction, hence

$$p_b(x_n, x_{n+1}) \leq p_b(x_{n-1}, x_n). \tag{2.29}$$

Then, by (2.28) and (2.29), we get

$$p_b(x_n, x_{n+1}) \leq \psi(p_b(x_{n-1}, x_n)),$$

recursively, we have

$$p_b(x_n, x_{n+1}) \leq \psi^n(p_b(x_0, x_1)). \tag{2.30}$$

Hence, from (2.23), (2.26), (2.29) and using the triangle inequality, we obtain

$$\begin{aligned}
&p_b(y_k, y_{k+1})\\
&\quad = p_b(x_{n_0k}, x_{n_0(k+1)})\\
&\quad = p_b(Tx_{n_0k-1}, Tx_{n_0(k+1)-1})\\
&\quad \leq \alpha(x_{n_0k-1}, x_{n_0(k+1)-1})p_b(Tx_{n_0k-1}, Tx_{n_0(k+1)-1})\\
&\quad \leq \psi(\max\{p_b(x_{n_0k-1}, x_{n_0(k+1)-1}), p_b(x_{n_0k-1}, Tx_{n_0k-1}), p_b(x_{n_0(k+1)-1}, Tx_{n_0(k+1)-1}),\\
&\qquad \frac{1}{2s^2}(p_b(x_{n_0k-1}, Tx_{n_0(k+1)-1}) + p_b(x_{n_0(k+1)-1}, Tx_{n_0k-1}))\})\\
&\quad = \psi(\max\{p_b(x_{n_0k-1}, x_{n_0(k+1)-1}), p_b(x_{n_0k-1}, x_{n_0k}), p_b(x_{n_0(k+1)-1}, x_{n_0(k+1)}),\\
&\qquad \frac{1}{2s^2}(p_b(x_{n_0k-1}, x_{n_0(k+1)}) + p_b(x_{n_0(k+1)-1}, x_{n_0k}))\})\\
&\quad \leq \psi(\max\{p_b(x_{n_0k-1}, x_{n_0(k+1)-1}), p_b(x_{n_0k-1}, x_{n_0k}),\\
&\qquad \frac{1}{2s^2}(sp_b(x_{n_0k-1}, x_{n_0k}) + sp_b(x_{n_0k}, x_{n_0(k+1)})\\
&\qquad + sp_b(x_{n_0k-1}, x_{n_0k)}) + sp_b(x_{n_0k-1}, x_{n_0(k+1)-1}))\})\\
&\quad \leq \psi(\max\{p_b(x_{n_0k-1}, x_{n_0(k+1)-1}), p_b(x_{n_0k-1}, x_{n_0k}),\\
&\qquad \frac{1}{2s}(2p_b(x_{n_0k-1}, x_{n_0k}) + p_b(y_k, y_{k+1}) + p_b(x_{n_0k-1}, x_{n_0(k+1)-1}))\}),\\
&\quad \leq \psi(\max\{p_b(x_{n_0k-1}, x_{n_0(k+1)-1}), p_b(x_{n_0k-1}, x_{n_0k}),\\
&\qquad \frac{1}{2s-1}(2p_b(x_{n_0k-1}, x_{n_0k}) + p_b(x_{n_0k-1}, x_{n_0(k+1)-1}))\}).
\end{aligned} \tag{2.31}$$

By (2.31), we have

$$\begin{aligned}
p_b(x_{n_0k-1}, x_{n_0(k+1)-1}) \leq \psi(\max\{&p_b(x_{n_0k-2}, x_{n_0(k+1)-2}), p_b(x_{n_0k-2}, x_{n_0k-1}),\\
&\frac{1}{2s-1}(2p_b(x_{n_0k-2}, x_{n_0k-1}) + p_b(x_{n_0k-2}, x_{n_0(k+1)-2}))\}),
\end{aligned} \tag{2.32}$$

recursively, and using (2.30), since ψ is nondecreasing, we can obtain

$$\begin{aligned}
p_b(y_k, y_{k+1}) \leq \max\{&\psi^{n_0k}(p_b(x_0, y_1)), \psi^{n_0k}(p_b(x_0, x_1)),\\
&\frac{1}{2s-1}(2\psi^{n_0k}(p_b(x_0, x_1)) + \psi^{n_0k}(p_b(x_0, y_1)))\}.
\end{aligned} \tag{2.33}$$

Let $k \to \infty$ in (2.33), then

$$\begin{aligned}
p_b(y_k, y_{k+1}) \leq \max\{&\psi^{n_0k}(p_b(x_0, y_1)), \psi^{n_0k}(p_b(x_0, x_1)),\\
&\frac{1}{2s-1}(2\psi^{n_0k}(p_b(x_0, x_1)) + \psi^{n_0k}(p_b(x_0, y_1)))\} \to 0.
\end{aligned} \tag{2.34}$$

Now we choose $k_0, \forall \epsilon > 0$, for all $k \geq k_0$ such that

$$\begin{aligned}
p_b(y_k, y_{k+1}) \leq \max\{&\psi^{n_0k}(p_b(x_0, y_1)), \psi^{n_0k}(p_b(x_0, x_1)),\\
&\frac{1}{2s-1}(2\psi^{n_0k}(p_b(x_0, x_1)) + \psi^{n_0k}(p_b(x_0, y_1)))\} \leq \frac{\epsilon}{2s}.
\end{aligned} \tag{2.35}$$

Since $\psi^{n_0}(\epsilon) < \frac{\epsilon}{2s}$, from (2.35), we have

$$\begin{aligned} p_b(y_{k+1}, y_{k+2}) \leq \max\{&\psi^{n_0(k+1)}(p_b(x_0, y_1)), \psi^{n_0(k+1)}(p_b(x_0, x_1)), \\ &\frac{1}{2s-1}(2\psi^{n_0k}(p_b(x_0, x_1)) + \psi^{n_0k}(p_b(x_0, y_1)))\} \\ < &\frac{\epsilon}{(2s)^2}, \end{aligned}$$

by induction, for all $p \geq 1, p \in N$, we get

$$p_b(y_{k+p-1}, y_{k+p}) < \frac{\epsilon}{(2s)^p}. \tag{2.36}$$

Using Lemma 2.1 and (2.36), we derive

$$p_b(y_{k+p}, y_k) < \sum_{i=1}^{p} \frac{\epsilon}{2^i} < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon. \tag{2.37}$$

Hence, $\{y_k\}$ is Cauchy sequence in complete partial b-metric space (X, p_b). It implies that there exists $y^* \in X$ such that $y_n \to y^*$, then

$$\lim_{n\to\infty} p_b(y_n, y^*) = \lim_{n,m\to\infty} p_b(y_n, y_m) = p_b(y^*, y^*) = 0. \tag{2.38}$$

Since T is continuous, then $Ty_n \to Ty^*$, so we have

$$\lim_{n\to\infty} p_b(Ty_n, Ty^*) = p_b(Ty^*, Ty^*). \tag{2.39}$$

Finally, we show y^* is a fixed point of T. By (2.30), we have

$$p_b(Ty_k, y_k) = p_b(x_{n_0k+1}, x_{n_0k}) \leq \psi^{n_0k}(p_b(x_1, x_0)).$$

Let $k \to \infty$ in the above inequality, we have

$$p_b(Ty_k, y_k) \to 0. \tag{2.40}$$

Analogous inequality (2.37) of the acquisition process, we can obtain

$$p_b(Ty_{k+p}, Ty_k) < \epsilon. \tag{2.41}$$

Then, $\{Ty_n\}$ is a Cauchy sequence in complete b-metric space (X, p_b), by (2.39), for all $m, n \in N$, we get

$$\lim_{n\to\infty} p_b(Ty_n, Ty^*) = \lim_{n,m\to\infty} p_b(Ty_n, Ty_m) = b(Ty^*, Ty^*) = 0. \tag{2.42}$$

Using triangle inequality, we have

$$\begin{aligned} p_b(y^*, Ty^*) &\leq s(p_b(y_n, y^*) + p_b(y_n, Ty^*)) - p_b(y_n, y_n) \\ &\leq sp_b(y_n, y^*) + s^2 p_b(y_n, Ty_n)) + s^2 p_b(Ty_n, Ty^*). \end{aligned}$$

Let $n \to \infty$, from (2.38), (2.40) and (2.42), which implies that $p_b(y^*, Ty^*) = 0$, then $y^* = Ty^*$, therefore y^* is a fixed point of T. □

To the uniqueness of a fixed point of a generalized $\alpha - \psi$ contractive mapping, we will consider the following hypothesis.

(H'): For all $x, y \in X$ there exists $z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$. and $\alpha(z, Tz) \geq 1$.

Theorem 2.10. *Adding condition (H') to the hypotheses of Theorem 2.9 we obtain uniqueness of the fixed point of T.*

Proof. Suppose that $x*$ and $y*$ are two fixed point of T. By condition (H'), there exists $z \in X$ such that

$$\alpha(x*, z) \geq 1, \alpha(y*, z) \geq 1 \text{ and } \alpha(z, Tz) \geq 1,$$

since T is α-admissible, from the above inequalities, for all $n \in N$, we have

$$\alpha(x*, T^n z) \geq 1, \alpha(y*, T^n) \geq 1 \text{ and } \alpha(T^{n-1}z, T^n z) \geq 1. \tag{2.43}$$

By (2.23), (2.43), and (p_b2), we get

$$\begin{aligned} p_b(x*, T^n z) &= p_b(Tx*, T^n z) \\ &\leq \alpha(x*, T^{n-1}z) p_b(Tx*, T^n z) \\ &\leq \psi(\max\{p_b(x*, T^{n-1}z), p_b(x*, Tx*), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s^2}(p_b(x*, T^n z) + p_b(T^{n-1}z, Tx*))\}) \\ &= \psi(\max\{p_b(x*, T^{n-1}z), p_b(x*, x*), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s^2}(p_b(x*, T^n z) + p_b(T^{n-1}z, x*))\}) \\ &= \psi(\max\{p_b(x*, T^{n-1}z), p_b(T^{n-1}z, T^n z)\}). \end{aligned} \tag{2.44}$$

Also, from (2.23), (2.43), and using the triangle inequality, we have

$$\begin{aligned} p_b(T^{n-1}z, T^n z) &\leq \alpha(T^{n-2}z, T^{n-1}z) p_b(T^{n-1}z, T^n z) \\ &\leq \psi(\max\{p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s^2}(p_b(T^{n-2}z, T^n z) + p_b(T^{n-1}z, T^{n-1}z))\}) \\ &\leq \psi(\max\{p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s}(p_b(T^{n-2}z, T^{n-1}z) + p_b(T^{n-1}z, T^n z))\}) \\ &\leq \psi(\max\{p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-1}z, T^n z)\}), \end{aligned} \tag{2.45}$$

if $p_b(T^{n-2}z, T^{n-1}z) < p_b(T^{n-1}z, T^n z)$, then by (2.45), we get

$$p_b(T^{n-1}z, T^n z) \leq \psi(p_b(T^{n-1}z, T^n z)) < p_b(T^{n-1}z, T^n z).$$

It is a contraction. So, by (2.45), we have

$$p_b(T^{n-1}z, T^n z) \leq \psi(p_b(T^{n-2}z, T^{n-1}z)).$$

Recursively, this implies that

$$p_b(T^{n-1}z, T^n z) \leq \psi^{n-1}(p_b(z, Tz)). \tag{2.46}$$

Moreover, from (2.44), we can obtain

$$p_b(x*, T^{n-1}z) \leq \psi(\max\{p_b(x*, T^{n-2}z), p_b(T^{n-2}z, T^{n-1}z)\}),$$

recursively, for all $n \in N$, and by (2.46), since ψ is nondecreasing, then

$$p_b(x*, T^n z) \leq \max\{\psi^n(p_b(x*, z)), \psi^n(p_b(z, Tz))\}. \tag{2.47}$$

Let $n \to \infty$ in (2.47), we have

$$\lim_{n\to\infty} p_b(x*, T^n z) = 0. \tag{2.48}$$

Similarly, we can get

$$\lim_{n\to\infty} p_b(y*, T^n z) = 0. \tag{2.49}$$

Also, using the triangle inequality, we have

$$p_b(x*, y*) \le sp_b(x*, T^n z) + sp_b(y*, T^n z).$$

Let $n \to \infty$, using (2.48) and (2.49), we get $p_b(x*, y*) = 0$, then $x* = y*$. □

Example 2.11. Let $X = R^+$, endowed with the partial b-metric $p_b(x, y) = (\max\{x, y\})^2$(with $s = 2$) for all $x, y \in R_+$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2}, & x > 1; \\ \frac{x}{\sqrt{2}\sqrt{1+x}}, & 0 \le x \le 1. \end{cases}$$

We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & if \quad y \le x; \\ 0, & otherwise. \end{cases}$$

Clearly T is α-admissible and an $\alpha - \psi$ contractive mapping with $\psi(t) = \frac{t}{2}$ for all $t \ge 0$. In fact, for all $x, y \in X$, we have

$$\alpha(x, y)p_b(Tx, Ty) \le \frac{1}{2} \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{1}{2s^2}(p_b(x, Ty) + p_b(y, Tx))\}.$$

Moreover, there exists $x_0 = 1 \in X$ such that

$$\alpha(x_0, Tx_0) = 1$$

and

$$\alpha(x_0, T^n x_0) = 1.$$

Obviously T is continuous , condition (H') is satisfied.

Now, all the hypotheses of Theorem 2.10 are satisfied. T has a unique fixed point. In this example, 0 is the unique fixed point of T.

Definition 2.12. Let (X, p_b) be a partial b-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$, for all $x, y \in X, s \ge 1$ such that

$$\alpha(x, y)p_b(Tx, Ty) \le \frac{1}{s}\psi(\max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{1}{2s}(p_b(x, Ty) + p_b(y, Tx))\}). \tag{2.50}$$

Theorem 2.13. *Let (X, p_b) be a complete partial b-metric space. suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by* (2.50) *which satisfies:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

(iii) T is continuous.

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$, Take $x_{n+1} = Tx_n$, for all $n \in N$. If $x_{n+1} = x_n$ for some $n \in N$, then $x* = x_n$ is a fixed point of T. Assume that $x_{n+1} \ne x_n$, for all $n \in N$. Since T is $\alpha - admissible$, we get

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \ge 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \ge 1,$$

by indiction, we obtain

$$\alpha(x_n, x_{n+1}) \ge 1. \tag{2.51}$$

So, from (2.50), (2.51) and the triangle inequality, we have

$$\begin{aligned}
&p_b(x_n, x_{n+1}) \\
&\quad = p_b(Tx_{n-1}, Tx_n) \\
&\quad \leq \alpha(x_{n-1}, x_n) p_b(Tx_{n-1}, Tx_n) \\
&\quad \leq \frac{1}{s}\psi(\max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, Tx_{n-1}), p_b(x_n, Tx_n), \\
&\qquad \frac{1}{2s}(p_b(x_{n-1}, Tx_n) + p_b(x_n, Tx_{n-1}))\} \\
&\quad \leq \frac{1}{s}\psi(\max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \\
&\qquad \frac{1}{2s}(p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n))\} \\
&\quad \leq \frac{1}{s}\psi(\max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \\
&\qquad \frac{1}{2s}(sp_b(x_{n-1}, x_n) + sp_b(x_n, x_{n+1}))\} \\
&\quad \leq \frac{1}{s}\psi(\max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\}.
\end{aligned} \tag{2.52}$$

If $p_b(x_{n-1}, x_n) < p_b(x_n, x_{n+1})$, by (2.52), then

$$p_b(x_n, x_{n+1}) \leq \frac{1}{s}\psi(p_b(x_n, x_{n+1})) < \frac{1}{s}p_b(x_n, x_{n+1}),$$

it is a contradiction, hence

$$p_b(x_n, x_{n+1}) \leq p_b(x_{n-1}, x_n). \tag{2.53}$$

Then, by (2.52) and (2.53), we get

$$p_b(x_n, x_{n+1}) \leq \frac{1}{s}\psi(p_b(x_{n-1}, x_n)), \tag{2.54}$$

recursively, we can obtain

$$p_b(x_n, x_{n+1}) \leq \frac{1}{s^n}\psi^n(p_b(x_0, x_1)). \tag{2.55}$$

Also, fix $\epsilon > 0$ and $n(\epsilon) \in N$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(p_b(x_0, x_1)) < \epsilon. \tag{2.56}$$

Hence, by (2.55), (2.56) and Lemma 2.1, we get

$$\begin{aligned}
p_b(x_{n+p}, x_n) &\leq \sum_{i=1}^{p} s^i p_b(x_{n+i}, x_{n+1-1}) \\
&\leq \sum_{i=1}^{p} s^i \frac{1}{s^{n+i}}\psi^{n+i}(p_b(x_0, x_1)) \\
&\leq \sum_{i=1}^{p} \frac{1}{s^n}\psi^{n+i}(p_b(x_0, x_1)) \\
&\leq \sum_{n \geq n(\epsilon)} \psi^n(p_b(x_0, x_1)) < \epsilon.
\end{aligned} \tag{2.57}$$

Therefore $\{x_n\}$ is Cauchy sequence in complete partial b-metric space (X, p_b). It implies that there exists $x^* \in X$ such that $x_n \to x^*$, then

$$\lim_{n\to\infty} p_b(x_n, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x^*, x^*) = 0. \tag{2.58}$$

Since T is continue, then $x_{n+1} = Tx_n \to Tx^*$, so

$$\lim_{n\to\infty} p_b(x_n, Tx^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(Tx^*, Tx^*) = 0. \tag{2.59}$$

Also, using the triangle inequality, we have

$$\begin{aligned} p_b(x^*, Tx^*) &\le s(p_b(x_n, x^*) + p_b(x_n, Tx^*)) - b(x_n, x_n) \\ &\le s(p_b(x_n, x^*) + p_b(x_n, Tx^*)). \end{aligned}$$

Let $n \to \infty$, by (2.57) and (2.58), which implies $p_b(x^*, Tx^*) = 0$, then $x^* = Tx^*$, therefore x^* is a fixed point of T. □

Theorem 2.14. *Let (X, p_b) be a complete partial b-metric space. suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by* (2.50) *which satisfies:*

(i) T is $\alpha -$ admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
(iii) if$\{x_n\}$ is a consequence in (X, p_b) such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$.

Then T has a fixed point.

Proof. Following the proof of Theorem 2.13, we know $\{x_n\}$ satisfying (2.55), (2.57) and the condition (iii), i.e. $\alpha(x_n, x*) \ge 1$. If $p_b(x^*, Tx^*) \ne 0$, by (2.50), (2.55) and the triangle inequality, we can obtain

$$\begin{aligned} p_b(x^*, Tx^*) &\le sp_b(x_{n+1}, x^*) + sp_b(x_{n+1}, Tx^*) - p_b(x_{n+1}, x_{n+1}) \\ &\le sp_b(x_{n+1}, x^*) + p_b(Tx_n, Tx^*) \\ &\le sp_b(x_{n+1}, x^*) + s\alpha(x_n, x*)p_b(Tx_n, Tx*) \\ &\le sp_b(x_{n+1}, x^*) + \psi(\max\{p_b(x_n, x*), p_b(x_n, Tx_n), p_b(x*, Tx*), \\ &\quad \frac{1}{2s}(p_b(x_n, Tx*) + p_b(Tx_n, x*))\}) \\ &\le sp_b(x_{n+1}, x^*) + \psi(\max\{p_b(x_n, x*), p_b(x_n, x_{n+1}), p_b(x*, Tx*), \\ &\quad \frac{1}{2s}(sp_b(x_n, x*) + sp_b(x*, Tx*)) + \frac{1}{2s}p_b(x_{n+1}, x*)\}) \\ &\le sp_b(x_{n+1}, x^*) + \psi(\max\{p_b(x_n, x*), \frac{1}{s^n}\psi^n(p_b(x_0, x_1)), p_b(x*, Tx*), \\ &\quad \frac{1}{2}(p_b(x_n, x*) + p_b(x*, Tx*)) + \frac{1}{2s}p_b(x_{n+1}, x*)\}), \end{aligned} \tag{2.60}$$

take

$$\begin{aligned} M = &\max\{p_b(x_n, x*), \frac{1}{s^n}\psi^n(p_b(x_0, x_1)), p_b(x*, Tx*), \\ &\frac{1}{2}(p_b(x_n, x*) + p_b(x*, Tx*)) + \frac{1}{2s}p_b(x_{n+1}, x*)\}. \end{aligned}$$

There are three cases:

1. if $M = \max\{p_b(x_n, x*), \frac{1}{s^n}\psi^n(p_b(x_0, x_1))\}$, let $n \to \infty$ in (2.59), and by (2.57), we have $p_b(x^*, Tx^*) = 0$, it is a contradiction;

2. if $M = p_b(x*, Tx*)$, let $n \to \infty$ in (2.59), and by (2.57), we have $p_b(x^*, Tx^*) \leq \psi(p_b(x^*, Tx^*)) < p_b(x^*, Tx^*)$, it is a contradiction;
3. if $M = \frac{1}{2}(p_b(x_n, x*) + p_b(x*, Tx*)) + \frac{1}{2s}p_b(x_{n+1}, x*)$, let $n \to \infty$ in (2.59), and by (2.57), we have $p_b(x^*, Tx^*) \leq \frac{1}{2}p_b(x^*, Tx^*)$, it is a contradiction.

Therefore there must be $p_b(x^*, Tx^*) = 0$, then $x^* = Tx^*$, therefore x^* is a fixed point of T. □

Theorem 2.15. *Adding condition (H') to the hypotheses of Theorem* 2.13 *(resp. Theorem* 2.14*) we obtain uniqueness of the fixed point of* T.

Proof. Suppose that $x*$ and $y*$ are two fixed point of T. By condition (H'), there exists $z \in X$ such that

$$\alpha(x*, z) \geq 1, \alpha(y*, z) \geq 1 \text{ and } \alpha(z, Tz) \geq 1.$$

Since T is α-admissible, from the above inequalities, for all $n \in N$, we have

$$\alpha(x*, T^n z) \geq 1, \alpha(y*, T^n) \geq 1 \text{ and } \alpha(T^{n-1}z, T^n z) \geq 1. \tag{2.61}$$

By (2.50), (2.60), and (p_b2), we get

$$\begin{aligned} p_b(x*, T^n z) &= p_b(Tx*, T^n z) \\ &\leq \alpha(x*, T^{n-1}z)p_b(Tx*, T^n z) \\ &\leq \frac{1}{s}\psi(\max\{p_b(x*, T^{n-1}z), p_b(x*, Tx*), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s}(p_b(x*, T^n z) + p_b(T^{n-1}z, Tx*))\}) \\ &= \frac{1}{s}\psi(\max\{p_b(x*, T^{n-1}z), p_b(x*, x*), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s}(p_b(x*, T^n z) + p_b(T^{n-1}z, x*))\}) \\ &\leq \frac{1}{s}\psi(\max\{p_b(x*, T^{n-1}z), p_b(T^{n-1}z, T^n z)\}). \end{aligned} \tag{2.62}$$

Also, from (2.50), (2.60), and using the triangle inequality, we have

$$\begin{aligned} p_b(T^{n-1}z, T^n z) &\leq \alpha(T^{n-2}z, T^{n-1}z)p_b(T^{n-1}z, T^n z) \\ &\leq \frac{1}{s}\psi(\max\{p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2s}(p_b(T^{n-2}z, T^n z) + p_b(T^{n-1}z, T^{n-1}z))\}) \\ &\leq \psi(\max\{p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-1}z, T^n z), \\ &\qquad \frac{1}{2}(p_b(T^{n-2}z, T^{n-1}z) + p_b(T^{n-1}z, T^n z))\}) \\ &\leq \psi(\max\{p_b(T^{n-2}z, T^{n-1}z), p_b(T^{n-1}z, T^n z)\}). \end{aligned} \tag{2.63}$$

If $p_b(T^{n-2}z, T^{n-1}z) < p_b(T^{n-1}z, T^n z)$, by (2.62), we get

$$p_b(T^{n-1}z, T^n z) \leq \psi(p_b(T^{n-1}z, T^n z)) < p_b(T^{n-1}z, T^n z).$$

It is a contraction. So, by (2.62), we have

$$p_b(T^{n-1}z, T^n z) \leq \psi(p_b(T^{n-2}z, T^{n-1}z)).$$

Recursively, this implies that

$$p_b(T^{n-1}z, T^n z) \leq \psi^{n-1}(p_b(z, Tz)). \tag{2.64}$$

Moreover, from (2.61), we can obtain

$$p_b(x*, T^{n-1}z) \leq \psi(\max\{p_b(x*, T^{n-2}z), p_b(T^{n-2}z, T^{n-1}z)\}),$$

recursively, for all $n \in N$, and by (2.63), since ψ is nondecreasing, then

$$p_b(x*, T^n z) \leq \max\{\psi^n(p_b(x*, z)), \psi^n(p_b(z, Tz))\}. \tag{2.65}$$

Let $n \to \infty$ in (2.64), we have

$$\lim_{n\to\infty} p_b(x*, T^n z) = 0. \tag{2.66}$$

Similarly, we can get

$$\lim_{n\to\infty} p_b(y*, T^n z) = 0. \tag{2.67}$$

Also, using the triangle inequality, we have

$$p_b(x*, y*) \leq sp_b(x*, T^n z) + sp_b(y*, T^n z).$$

Let $n \to \infty$, using (2.65) and (2.66), we get $p_b(x*, y*) = 0$, then $x* = y*$. □

Definition 2.16. Let $f, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that a pair (f, g) of mappings is α-admissible if for all $x, y \in K$, and we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, gy) \geq 1 \text{ and } \alpha(gx, fy) \geq 1.$$

Definition 2.17. Let (X, p_b) be a partial b-metric space and $f, g : X \to X$ be a given mapping. We say that a pair (f, g) of self-mappings is a generalized $\alpha - \psi$ contractive pair if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$, for all $x, y \in X, s \geq 1$ such that

$$\alpha(x, y)p_b(Tx, Ty) \leq \frac{1}{s}\psi(\max\{p_b(x, y), p_b(x, fx), p_b(y, gy), \frac{1}{2s}(p_b(x, gy) + p_b(y, fx))\}). \tag{2.68}$$

Theorem 2.18. *Let (X, p_b) be a complete partial b-metric space. suppose that $f, g : X \to X$, and (f, g) is a generalized $\alpha - \psi$ contractive pair defined by* (2.67) *which satisfies:*

(i) (f, g) is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;
(iii) f and g are continuous.

Then f and g have a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, We construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ $\forall n \in N$. Since (f, g) is α-admissible, then

$$\begin{aligned}\alpha(x_0, fx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(fx_0, gx_1) \geq 1 = \alpha(x_1, x_2) \geq 1 \\ \Rightarrow \alpha(gx_1, fx_2) = \alpha(x_2, x_3) \geq 1,\end{aligned}$$

by indiction, we have

$$\alpha(x_n, x_{n+1}) \geq 1. \tag{2.69}$$

If $x_{2n+1} = x_{2n}$ for some $n \in N$, then $x_{2n} = fx_{2n}$. Thus $x_{2n+1} = x_{2n}$ is a fixed point of f. This must $x_{2n} = x_{2n+1}$ is fixed point of g, i.e. $x_{2n+1} = gx_{2n+1}$. Indeed, if $x_{2n+1} \neq gx_{2n+1}$, then $p_b(x_{2n+1}, x_{2n+2}) \neq 0$, so by (2.67), (2.68), (p_b2) and the triangle inequality, we get

$$\begin{aligned}
p_b(x_{2n+1}, x_{2n+2}) &= p_b(fx_{2n}, gx_{2n+1}) \\
&\leq \alpha(x_{2n}, x_{2n+1}) p_b(fx_{2n}, gx_{2n+1}) \\
&\leq \frac{1}{s}\psi(\max\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n}, fx_{2n}), p_b(x_{2n+1}, gx_{2n+1}), \\
&\quad \frac{1}{2s}(p_b(x_{2n}, gx_{2n+1}) + p_b(x_{2n+1}, fx_{2n}))\}) \\
&= \frac{1}{s}\psi(\max\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}), \\
&\quad \frac{1}{2s}(p_b(x_{2n}, x_{2n+2}) + p_b(x_{2n+1}, x_{2n+1}))\}) \\
&\leq \frac{1}{s}\psi(\max\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}), \\
&\quad \frac{1}{2s}(sp_b(x_{2n}, x_{2n+1}) + sp_b(x_{2n+1}, x_{2n+2}))\}) \\
&= \frac{1}{s}\psi(\max\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2})\}) \\
&= \frac{1}{s}\psi(p_b(x_{2n+1}, x_{2n+2})) \\
&< \frac{1}{s}p_b(x_{2n+1}, x_{2n+2}),
\end{aligned} \tag{2.70}$$

which gives a contradiction. Therefore $p_b(x_{2n+1}, x_{2n+2}) = p_b(x_{2n+1}, gx_{2n+1}) = 0$, then $x_{2n} = x_{2n+1}$ is a fixed point of g.

Similarly, if $x_{2n+2} = x_{2n+1}$ for some $n \in N$, we obtain x_{2n+1} is fixed point of g and f. Therefore we assume that $x_n \neq x_{n+1}$. If $p_b(x_{2n+1}, x_{2n+2}) > p_b(x_{2n}, x_{2n+1})$, from (2.69) we get

$$p_b(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s}\psi(p_b(x_{2n+1}, x_{2n+2})) < \frac{1}{s}p_b(x_{2n+1}, x_{2n+2}),$$

it is a contradiction. Hence,

$$p_b(x_{2n+1}, x_{2n+2}) \leq p_b(x_{2n}, x_{2n+1}).$$

Moreover, from (2.69), we have

$$p_b(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s}\psi(p_b(x_{2n}, x_{2n+1})).$$

Similarly, we can show that

$$p_b(x_{2n}, x_{2n+1}) \leq \frac{1}{s}\psi(p_b(x_{2n-1}, x_{2n})).$$

Recursively, we get

$$p_b(x_{2n+1}, x_{2n+2}) \leq \frac{1}{s^{2n+1}}\psi^{2n+1}(p_b(x_0, x_1))$$

and

$$p_b(x_{2n}, x_{2n+1}) \leq \frac{1}{s^{2n}}\psi^{2n}(p_b(x_0, x_1)).$$

Then, by the above two inequalities, which imply that

$$p_b(x_n, x_{n+1}) \leq \frac{1}{s^n}\psi^n(p_b(x_0, x_1)). \tag{2.71}$$

Also, fix $\epsilon > 0$ and $n(\epsilon) \in N$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(p_b(x_0, x_1)) < \epsilon. \tag{2.72}$$

Hence, by (2.70), (2.71) and Lemma 2.1, we can obtain

$$\begin{aligned} p_b(x_{n+p}, x_n) &\leq \sum_{i=1}^{p} s^i p_b(x_{n+i}, x_{n+i+1}) \\ &\leq \sum_{i=0}^{p} s^i \frac{1}{s^{n+i}} \psi^{n+i}(p_b(x_0, x_1)) \\ &\leq \sum_{n \geq n(\epsilon)} \psi^n(p_b(x_0, x_1)) < \epsilon. \end{aligned} \tag{2.73}$$

It shows that $\{x_n\}$ is Cauchy sequence in complete partial b-metric space (X, p_b). Which implies that there exists $x^* \in X$ such that $x_n \to x^*$, then

$$\lim_{n\to\infty} p_b(x_n, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x^*, x^*) = 0. \tag{2.74}$$

Moreover

$$\lim_{n\to\infty} p_b(x_{2n+1}, x^*) = \lim_{n,m\to\infty} p_b(x_{2n+1}, x_{2m+1}) = p_b(x^*, x^*) = 0, \tag{2.75}$$

and

$$\lim_{n\to\infty} p_b(x_{2n}, x^*) = \lim_{n,m\to\infty} p_b(x_{2n}, x_{2m}) = p_b(x^*, x^*) = 0. \tag{2.76}$$

Since f is continuous, then $x_{2n+1} = fx_{2n} \to fx^*$ as $n \to \infty$.

$$\lim_{n\to\infty} p_b(x_{2n+1}, fx^*) = \lim_{n,m\to\infty} p_b(x_{2n+1}, x_{2m+1}) = p_b(fx^*, fx^*) = 0. \tag{2.77}$$

Using the triangle inequality, we have

$$\begin{aligned} p_b(x^*, fx^*) &\leq s(p_b(x_{2n+1}, x^*) + p_b(x_{2n+1}, fx^*)) - b(x_{2n+1}, x_{2n+1}) \\ &\leq s(p_b(x_{2n+1}, x^*) + p_b(x_{2n+1}, fx^*)). \end{aligned}$$

Let $n \to \infty$, by (2.74) and (2.76), then $p_b(x^*, fx^*) = 0$, it implies $x^* = fx^*$, therefore x^* is a fixed point of f. Similarly, we can obtain x^* is a fixed point of g. □

Theorem 2.19. *Let (X, p_b) be a complete partial b-metric space. suppose that $f, g : X \to X$, and (f, g) is a generalized $\alpha - \psi$ contractive pair defined by* (2.67) *which satisfies:*

(i) (f, g) is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;
(iii) if $\{x_n\}$ is a consequence in (X, p_b) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.

Then f and g have a fixed point.

Proof. Following the proof of Theorem 2.18, we know $\{x_n\}$ satisfying (2.70), (2.74), (2.75) and the condition (iii), i.e. $\alpha(x_n, x*) \geq 1$, if $p_b(x^*, fx^*) \neq 0$, then by (2.67), (2.70) and using the triangle inequality, we obtain

$$\begin{aligned} p_b(x^*, fx^*) &\leq sp_b(gx_{2n+1}, x^*) + sp_b(gx_{2n+1}, fx^*) - p_b(x_{2n+1}, x_{2n+1}) \\ &\leq sp_b(x_{n+2}, x^*) + sp_b(gx_{2n+1}, fx^*) \\ &\leq sp_b(x_{2n+2}, x^*) + s\alpha(x_{2n+1}, x*)p_b(gx_{2n+1}, fx*) \\ &\leq sp_b(x_{2n+2}, x^*) + \psi(\max\{p_b(x_{2n+1}, x*), p_b(x_{2n+1}, gx_{2n+1}), p_b(x*, fx*), \\ &\quad \frac{1}{2s}(p_b(x_{2n+1}, fx*) + p_b(gx_{2n+1}, x*))\}) \\ &\leq sp_b(x_{2n+2}, x^*) + \psi(\max\{p_b(x_{2n+1}, x*), p_b(x_{2n+1}, x_{2n+2}), p_b(x*, fx*), \\ &\quad \frac{1}{2s}(p_b(x_{2n+1}, fx*) + p_b(x_{2n+2}, x*))\}) \\ &\leq sp_b(x_{2n+2}, x^*) + \psi(\max\{p_b(x_{2n+1}, x*), p_b(x_{2n+1}, x_{2n+2}), p_b(x*, fx*), \\ &\quad \frac{1}{2s}(sp_b(x_{2n+1}, x*) + sp_b(x*, fx*) + p_b(x_{2n+2}, x*))\}) \\ &\leq sp_b(x_{2n+2}, x^*) + \psi(\max\{p_b(x_{2n+1}, x*), \frac{1}{s^{2n+1}}\psi^{2n+1}(p_b(x_0, x_1)), p_b(x*, fx*), \\ &\quad \frac{1}{2}(p_b(x_{2n+1}, x*) + p_b(x*, fx*)) + \frac{1}{2s}p_b(x_{2n+2}, x*)\}), \end{aligned} \tag{2.78}$$

take

$$\begin{aligned} N = &\max\{p_b(x_{2n+1}, x*), \frac{1}{s^{2n+1}}\psi^{2n+1}(p_b(x_0, x_1)), p_b(x*, fx*), \\ &\frac{1}{2}(p_b(x_{2n+1}, x*) + p_b(x*, fx*)) + \frac{1}{2s}p_b(x_{2n+2}, x*)\}. \end{aligned}$$

There are three cases:

1. if $N = \max\{p_b(x_{2n+1}, x*), \frac{1}{s^{2n+1}}\psi^{2n+1}(p_b(x_0, x_1))\}$, let $n \to \infty$ in (2.77), by (2.74) and (2.75), we have $p_b(x*, fx*) = 0$, it is a contradiction;
2. if $N = p_b(x*, fx*)$, let $n \to \infty$ in (2.77), by (2.74) and (2.75), we have $p_b(x*, fx*) \leq \psi(p_b(x^*, fx^*)) < p_b(x^*, fx^*)$, it is a contradiction;
3. if $N = \frac{1}{2}(p_b(x_{2n+1}, x*) + p_b(x*, fx*)) + \frac{1}{2s}p_b(x_{2n+2}, x*)$, let $n \to \infty$ in (2.77), by (2.74) and (2.75), we have $p_b(x^*, fx^*) \leq \frac{1}{2}p_b(x^*, fx^*)$, it is a contradiction.

Hence, $p_b(x^*, fx^*) = 0$, then $x^* = fx^*$, therefore x^* is a fixed point of f. Similarly, we can obtain x^* is a fixed point of g. □

Theorem 2.20. *Adding condition (H') to the hypotheses of Theorem* 2.18 *(resp. Theorem* 2.19*) we obtain uniqueness of the fixed point of* T.

Proof. In the proof of Theorem 2.15, take $T = f$ to get the result. □

Example 2.21. Let $X = R^+$, endowed with the partial b-metric $p_b(x, y) = (\max\{x, y\})^2$ (with $s = 2$) for all $x, y \in R_+$. Define the mapping $f, g : X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & x > 1; \\ \frac{x}{\sqrt{2}\sqrt{1+x}}, & 0 \leq x \leq 1. \end{cases}$$

$$gx = \frac{x}{4}, x \in [0, \infty).$$

We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & if \quad y \leq x; \\ 0, & otherwise. \end{cases}$$

Clearly (f, g) is α-admissible pair and an $\alpha - \psi$ contractive pair with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. In fact, for all $x, y \in X$, we have

$$\alpha(x,y)p_b(Tx,Ty) \leq \frac{1}{2}\max\{p_b(x,y), p_b(x,fx), p_b(y,gy), \frac{1}{2s^2}(p_b(x,gy) + p_b(y,fx))\}.$$

Obviously, f and g are continuous and condition (H') is satisfied. Moreover, there exists $x_0 = 1 \in X$ such that

$$\alpha(x_0, fx_0) = 1.$$

Hence, all conditions of Theorem 2.20 are satisfied. f and g have a unique fixed point (which is $z = 0$).

3. Consequences

We will show that many latest existing Theorems in the literature can be deduced easily from our results Firstly, the following results from our Theorem 2.7.

Let $\psi(t) = \lambda t, \lambda \in [0, 1)$, we obtain the following result.

Corollary 3.1 ([27])**.** *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given mapping, for all $x, y \in X$ and $\lambda \in [0, 1)$ such that*

$$p_b(Tx,Ty) \leq \lambda p_b(x,y). \tag{3.1}$$

Then T has a unique fixed point.

In Corollary 3.1 take $s = 1$ and for all $x, y \in X$, $p_b(x, y) = 0$ if only if $x = y$, we obtain the following result.

Corollary 3.2 ([26])**.** *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by (2.1) for all $x, y \in X$. Which satisfies:*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous or $\{x_n\}$ is a sequence in (X, d) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a fixed point, if condition (H) is satisfied, one has uniqueness of the fixed point.

Secondly, the following results from our Theorem 2.10.

Let $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain,

Corollary 3.3. *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given continuous mapping, suppose there exists a function $\psi \in \Psi$, for all $x, y \in X$ such that*

$$p_b(Tx,Ty) \leq \psi(\max\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{1}{2s^2}(p_b(x,Ty) + p_b(y,Tx))\}). \tag{3.2}$$

Then T has a unique fixed point.

The following results follow immediately from Corollary 3.3.

Corollary 3.4. *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given continuous mapping, for all $x, y \in X$ and $\lambda \in [0, \frac{1}{2})$ such that*

$$p_b(Tx,Ty) \leq \lambda[p_b(x,Tx) + p_b(y,Ty)]. \tag{3.3}$$

Then T has a unique fixed point.

Corollary 3.5. *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given continuous mapping, for all $x, y \in X$ and $\lambda \in [0, 1)$ such that*

$$p_b(Tx, Ty) \leq \lambda \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty)\}. \tag{3.4}$$

Then T has a unique fixed point.

Corollary 3.6. *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given continuous mapping, for all $x, y \in X$, $A, B, C \geq 0$, $A + B + C \in [0, 1)$ such that*

$$p_b(Tx, Ty) \leq Ap_b(x, y) + Bp_b(x, Tx) + Cp_b(y, Ty). \tag{3.5}$$

Then T has a unique fixed point.

Corollary 3.7. *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given continuous mapping, for all $x, y \in X$ and $\lambda \in [0, 1)$ such that*

$$p_b(Tx, Ty) \leq \lambda \max\{p_b(x, y), \frac{1}{2}(p_b(x, Tx) + p_b(y, Ty)), \frac{1}{2s^2}(p_b(x, Ty) + p_b(y, Tx))\}. \tag{3.6}$$

Then T has a unique fixed point.

Corollary 3.8. *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given continuous mapping, for all $x, y \in X$ and $\lambda \in [0, 1)$ such that*

$$p_b(Tx, Ty) \leq \lambda \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{1}{2s^2}(p_b(x, Ty) + p_b(y, Tx))\}. \tag{3.7}$$

Then T has a unique fixed point.

From Theorem 2.10, we will deduce very easily the following results on a partial b-metric space endowed with a partial ordered.

Corollary 3.9. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ and satisfying (3.1) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Corollary 3.10. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$, suppose there exists a function $\psi \in \Psi$ satisfying (3.2) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Corollary 3.11. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ and satisfying (3.3) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Corollary 3.12. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ and satisfying (3.4) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Corollary 3.13. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ and satisfying (3.5) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Corollary 3.14. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ and satisfying (3.6) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Corollary 3.15. *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$ and satisfying (3.7) for all $x, y \in X$ with $y \preceq x$. If there exists x_0 such that $x_0 \preceq Tx_0$.*

Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Corollary 3.16 ([17]). *Let $(X, \preceq p_b)$ be a complete ordered partial b-metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$, for all $x, y \in X$ with $y \preceq x$ such that*

$$p_b(Tx, Ty) \le \frac{k}{s} \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{1}{2s}(p_b(x, Ty) + p_b(y, Tx))\}. \tag{3.8}$$

If there exists x_0 such that $x_0 \preceq Tx_0$. Then T has a fixed point. If for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

Thirdly, the following results from our Theorem 2.15.

Let $\psi(t) = \lambda st, \lambda s < 1$, we obtain the following results.

Corollary 3.17 ([27]). *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given mapping, for all $x, y \in X$ and $\lambda \in [0, \frac{1}{2}), \lambda s < 1$ such that*

$$p_b(Tx, Ty) \le \lambda[p_b(x, Tx) + p_b(y, Ty)]. \tag{3.9}$$

Then T has a unique fixed point.

Corollary 3.18 ([27]). *Let (X, p_b) be a complete partial b-metric space. $T : X \to X$ be a given mapping, for all $x, y \in X$ and $\lambda \in [0, 1), \lambda s < 1$ such that*

$$p_b(Tx, Ty) \le \lambda \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty)\}. \tag{3.10}$$

Then T has a unique fixed point.

Let $s = 1$ and for all $x, y \in X$, $p_b(x, y) = 0$ if and only if $x = y$, we obtain the following results.

Corollary 3.19. *Let (X, d) be a complete metric space. suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by the following inequality*

$$\alpha(x, y)d(Tx, Ty) \le \psi(\max\{d(x, y), d(x, Tx), p_b(y, Ty)), \frac{1}{2}d(x, Ty) + d(y, Tx))\}), \tag{3.11}$$

for all $x, y \in X$, and which satisfies:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

(iii) *T is continuous or $\{x_n\}$ is a sequence in (X, d) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point, if condition (H') is satisfied, one has uniqueness of the fixed point.

Corollary 3.20 ([15])**.** *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized $\alpha - \psi$ contractive mapping defined by the following inequality*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(\max\{d(x, y), \frac{1}{2}d(x, Tx), p_b(y, Ty)), \frac{1}{2}d(x, Ty) + d(y, Tx))\}), \tag{3.12}$$

for all $x, y \in X$, and which satisfies:

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous or $\{x_n\}$ is a sequence in (X, d) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point, if condition (H') is satisfied, one has uniqueness of the fixed point.

Finally, the following results from our Theorem 2.20. Let $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain the following results.

Corollary 3.21. *Let (X, p_b) be a complete partial b-metric space. $f, g : X \to X$ be two given mappings, suppose that there exists a function $\psi \in \Psi$, for all $x, y \in X$ such that*

$$p_b(Tx, Ty) \leq \frac{1}{s}\psi(\max\{p_b(x, y), p_b(x, fx), p_b(y, gy), \frac{1}{2s}(p_b(x, gy) + p_b(y, fx))\}). \tag{3.13}$$

Then f and g have a unique fixed point.

Corollary 3.22 ([17])**.** *Let $(X, \preceq, p_b)$ be a complete ordered partial b-metric space. Also $f, g : X \to X$ be two given mappings with $fx \leq gfx, gx \leq fgx, \forall x \in X$, and for all $x, y \in X$ with $y \preceq x$ such that*

$$p_b(Tx, Ty) \leq \frac{k}{s}\max\{p_b(x, y), p_b(x, fx), p_b(y, gy), \frac{1}{2s}(p_b(x, gy) + p_b(y, fx))\}, \tag{3.14}$$

if f is continuous or $\{x_n\}$ is a nondecreasing sequence in $(X, \preceq, p_b)$ such that $x_n \to x \in X$ as $n \to \infty$, then $(x_n \preceq x)$ for all $n \in N$.

Then f and g have a fixed point, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$, $y \preceq z$ and $z \preceq Tz$, one has uniqueness of the fixed point.

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