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The stability of sextic functional equation in fuzzy modular spaces

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Abstract

By using the fixed point technique, we prove the stability of sixtic functional equations. Our results are studied and proved in the framework of fuzzy modular spaces (briefly, $\mathcal{FM}$-spaces). The lower semi continuous (briefly, l.s.c.) and β-homogeneous are necessary conditions for this work. ©2016 All rights reserved.

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1. Introduction

In 1940 during a conference at Wisconsin University, S. M. Ulam [16] presented the following question concerning stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $g : G_1 \to G_2$ with $d(f(x), g(x)) < \epsilon$ for all $x \in G_1$?

When the homomorphisms are stable? So, we are interested in this question, that is, if a mapping is almost a homomorphism, then there exists an exact homomorphism that must be close. In following year, Hyers [7] was the first to give an affirmative answer to Ulam's question for the case where G_1 and G_2 are

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Banach spaces. After that, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [14]. Later, the stability problems of various functional equation have been extensively investigated by many authors [3, 4].

One of the interesting functional equations studied is the system of additive-quadratic-cubic functional equations [6]:

$$\begin{cases} f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)=2af(x_1,y,z), \\ f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)=2a^2f(x,y_1,z)+2b^2f(x,y_2,z), \\ f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2)=ab^2(f(x,y,z_1+z_2) \\ \qquad\qquad +f(x,y,z_1-z_2))+2a(a^2-b^2)f(x,y,z_1), \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{Z}\setminus\{0\}$ with $a \neq \pm 1, \pm b$.

The function $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $f(x, y, z) = cxy^2z^3$ is a solution of the system (1.1). In particular, letting $y = z = x$, we get a sextic function $h : \mathbb{R} \to \mathbb{R}$ in one variable given by $h(x) := f(x, x, x) = cx^6$.

The concept of modular spaces was introduced by Nakano [12]. Soon after, the notation of modular spaces was redefined and generalized by Musielak and Orlicz [11]. In 2007, Nourouzi [13] presented probabilistic modular spaces related to the theory of modular spaces.

After that, Shen and Chen [15] following the idea of probabilistic modular spaces and the definition of fuzzy metric spaces based on George and Veeramani 's sense [5], applied fuzzy concept to the classical notions of modular and modular spaces, and in 2013, Shen and Chen [15] presented the concept of a fuzzy modular space. After that, Kumam [9, 10], Wongkum and et al [18] studied fixed points and some properties in modular or fuzzy modular spaces.

In this paper, we investigate the generalized Ulam-Hyers-Rassias (briefly, UHR) stability of a sextic functional equations from linear spaces into $\mathcal{FM}$-spaces, by using some ideas of [2, 18].

2. Preliminaries

In this section, conventionally, we write throughout the paper $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ to denote respectively the set of all reals, complexes, and nonnegative integers.

Moreover, we recall some basic definitions and properties of a fuzzy modular space.

Definition 2.1 ([17])**.** A *fuzzy set* A in X is a function with domain X and value in $[0, 1]$.

Definition 2.2 ([1])**.** A *triangular norm* (briefly, t-norm) is a function $* : [0, 1] \times [0, 1] \to [0, 1]$ satisfying, for each a, b, c, d $\in [0, 1]$, the following conditions:

(1) $a * 1 = a$;

(2) $a * b \le c * d$ whenever $a \le c, b \le d$;

(3) $a * b = b * a$; and $(a * b) * c = a * (b * c)$.

Definition 2.3. Let X be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized functional $\rho : X \to [0,\infty]$ is called a *modular* if for arbitrary $x, y \in X$,

(m1) $\rho(x) = 0$ if and only if $x = 0$,

(m2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(m3) $\rho(z) \le \rho(x) + \rho(y)$,whenever z is a convex combination of x and y.

The corresponding *modular space*, denoted by X_ρ, is then defined by

$$X_\rho := \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Remark 2.4. Note that for a fixed $x \in X_\rho$, the valuation $\gamma \in \mathbb{K} \mapsto \rho(\gamma x)$ is increasing.

Unlike a norm, a modular needs not be continuous or convex in general. However, it often occurs that some weaker form of them are assumed.

Remark 2.5. In case a modular ρ is convex, one has $\rho(x) \leq \delta\rho(\frac{1}{\delta}x)$ for all $x \in X_\rho$, provided that $0 < \delta \leq 1$.

Definition 2.6. Let X_ρ be a modular space and $\{x_n\}$ be a sequence in X_ρ. Then,

(i) $\{x_n\}$ is *ρ-convergent* to a point $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is called *ρ-Cauchy* if for all $\epsilon > 0$, we have $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$.

(iii) A subset $K \subset X_\rho$ is called *ρ-complete* if any ρ-Cauchy sequence is ρ-convergent.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to x does not imply that $\{cx_n\}$ converges to cx, where c is chosen from the corresponding scalar field. Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the Δ_2-conditions.

A modular ρ is said to satisfy the Δ_2-condition if there exists $\kappa \geq 2$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in X_\rho$. Some authors varied the notion so that only $\kappa > 0$ is required and called it the Δ_2-type condition. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the Δ_2-conditions.

Remark 2.7. We have to be very careful about the convergence behaviors on multiples and sums of ρ-convergent sequences. In general, we suppose that $\{x_n^1\}, \{x_n^2\}, \cdots, \{x_n^{2k}\}$, for some $k \in \mathbb{N}$, are sequences in X_ρ in which they ρ-converge to the points $x^1, x^2, \cdots, x^{2k} \in X_\rho$, respectively. Then, the averaged sequence $\{\frac{1}{2^k}\sum_{i=1}^{2k} x_n^i\}$ ρ-converges to $\frac{1}{2^k}\sum_{i=1}^{2k} x^i$.

In [8], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy Δ_2- conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Definition 2.8. Given a modular space X_ρ, a nonempty subset $C \subset X_\rho$, and a mapping $T : C \to C$. The *orbit* of T around a point $x \in X_\rho$ is the set

$$\mathcal{O}(x) := \{x, Tx, T^2x, \cdots\}.$$

The quantity $\delta_\rho(x) := \sup\{\rho(u - v) : u, v \in \mathcal{O}(x)\}$ is then associated and is called the *orbital diameter* of T at x. In particular, if $\delta_\rho(x) < \infty$, we say that T has a bounded orbit at x.

Lemma 2.9 ([8])**.** *Let X_ρ be a modular space whose the induced modular is l.s.c. and $C \subset X_\rho$ be a ρ-complete subset. If $T : C \to C$ is a ρ-contraction, i.e., there is a constant $k \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq k\rho(x - y), \quad \forall x, y \in X_\rho,$$

and T has a bounded orbit at a point $x_0 \in X_\rho$, then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $w \in C$.

Definition 2.10 ([15])**.** Let V be a real or complex vector space with a zero θ, $*$ a continuous triangular norm, and μ a fuzzy set on the product $V \times \mathbb{R}^+$. Suppose that the following properties hold for $x, y \in V$ and $s, t > 0$:

(FM1) $\mu(x, t) > 0$;

(FM2) $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = \theta$;

(FM3) $\mu(x, t) = \mu(-x, t)$;

(FM4) $\mu(z, s+t) \geq \mu(x, s) * \mu(y, t)$ whenever z is the convex combination between x and y;

(FM5) the mapping $t \mapsto \mu(x, t)$ is continuous at each fixed $x \in V$.

Then, we write $(V, \mu, *)$ to represent the space with the pre-defined properties. In particular, we call μ a *fuzzy modular* and the triple $(V, \mu, *)$ a *fuzzy modular space* (briefly, $\mathcal{FM}$-space).

It is worth noting that every fuzzy modular is non-decreasing with respect to $t > 0$.

Example 2.11. Let X be a real or complex vector space and ρ be a modular on X. Take the t-norm $a * b$ = min {a,b}. For every $t \in (0, \infty)$, define $\mu(x, t) = \frac{t}{t+\rho(x)}$ for all $x \in X$. Then $(X, \mu, *)$ is a $\mathcal{FM}$-space.

Remark 2.12. Note that the above conclusion still holds even if the t-norm is replaced by $a * b = a \cdot b$ and $a * b = max\{a + b - 1, 0\}$, respectively.

Definition 2.13. Let (X, μ) be a $\mathcal{FM}$-space, $\{x_n\}$ be a sequence in X and $x \in X$.

1. The sequence $\{x_n\}$ with $x_n \in (X, \mu)$ is said to be μ - convergent to x (write $x_n \overset{\mu}{\to} x$) if, for any $t > 0$ and λ $(0, 1)$, there exists a positive integer n_0 such that
$$\mu(x_n - x, t) > 1 - \lambda$$
for all $n \geq n_0$
2. The sequence $\{x_n\}$ with $x_n \in (X, \mu)$ is called a μ - Cauchy sequence if, for any $t > 0$ and $\lambda \in (0, 1)$, there exists a positive integer n_0 such that
$$\mu(x_n - x_m, t) > 1 - \lambda$$
for all $n, m \geq n_0$.
3. Every μ - convergent sequence in $\mathcal{FM}$ - space is μ - Cauchy sequence. If each μ - Cauchy sequence is μ - convergent sequence in a $\mathcal{FM}$ - space (X, μ), then (X, μ) is called a μ - complete $\mathcal{FM}$ - space.

Shen and Chen [15] also studied the topological properties of a fuzzy modular space with a special property that for every $x \in V$ and a non-zero real λ, the equality
$$\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^\beta}\right)$$
holds for some fixed $\beta \in (0, 1]$. If the fuzzy modular μ has this property, we shall say that it is β-*homogeneous*.

The μ-*ball* in $(V, \mu, *)$ is the set of the form
$$B(x, r, t) := \{y \in V | \mu(x - y, t) > 1 - r\},$$
where $r \in (0, 1)$ and $t > 0$.

Now, suppose that μ is β-homogeneous for some $\beta \in (0, 1]$. According to Shen and Chen [15], the family $\mathfrak{B}$ of all μ-balls forms a base for a first-countable Hausdorff topology, written as $\mathfrak{T}_\mu$. With the notion of the μ-balls, it is easy to see that a sequence (x_n) in V μ-*converges* (i.e. it converges in the topology $\mathfrak{T}_\mu$) to its μ-*limit* $x \in V$ if and only if $\mu(x - x_n, t) \to 1$ as $n \to \infty$ for all $t > 0$. Note here that the μ-limit is unique if it does exists after all. It is then natural to say that (x_n) is μ-*Cauchy* if for any given $\varepsilon \in (0, 1)$ and $t > 0$, there exists $N \in \mathbb{N}$ with $\mu(x_m - x_n, t) > 1 - \varepsilon$ whenever $m, n > N$. We say that μ-*complete* if every μ-Cauchy sequence converge.

From here, let us turn to a typical example of a triangular norm which is defined by $(a * b) = min\{a, b\}$. This triangular norm has a very special property that if $*'$ be an arbitrary triangular norm, then $(a *' b) \leq (a * b)$ for all $a, b \in [0, 1]$. With this property, it is suitable to call this $*$ a *strongest triangular norm.* As is claimed by Shen and Chen [15], if V is a real vector space equipped with a β-homogeneous fuzzy modular μ and a strongest triangular norm $*$, then a μ-convergent sequence is μ-Cauchy. The authors also mentioned that if $*$ is not the strongest one, such implementation is not always true.

We say that $\mathcal{FM}$-space $(X, \mu, *)$ satisfies the lower semi continuous if, for any sequence x_n of X and μ-converging to a point $x \in X$,

$$\mu(x,t) \leq \liminf_{n\to\infty} \mu(x_n,t)$$

for all $t > 0$.

Theorem 2.14 ([8])**.** *Let X_ρ be a modular space satisfying l.s.c. property. Let $\mathcal{C}$ be a ρ-complete nonempty subset of X_ρ and $T : \mathcal{C} \to \mathcal{C}$ be a quasi-contraction, that is, there exists $K < 1$ such that*

$$\rho(T(x) - T(y)) \leq K max\{\rho(x-y), \rho(x-T(x)), \rho(y-T(y)), \rho(x-T(y)), \rho(y-T(x))\}.$$

Let $X \in \mathcal{C}$ such that

$$\delta_\rho(x) := sup\{\rho(T^n(x) - T^m(x)) : m,n \in \mathbb{N}\} < \infty.$$

Then $\{T^n(x)\}$ ρ-converges to a point $w \in \mathcal{C}$. Moreover, if $\rho(w - T(w)) < \infty$ and $\rho(x - T(w)) < \infty$, then the ρ-limit of $T^n(x)$ is a fixed point of T. Furthermore, if w^* is any fixed point of T in $\mathcal{C}$ such that $\rho(w - w^*) < \infty$, then one has $w - w^*$.

In this section, we assume that μ is a fuzzy modular on V with the l.s.c. (in the fuzzy modular sense) and $(V, \mu, *)$ is a μ-complete β-homogeneous $\mathcal{FM}$-space with $\beta \in (0, 1]$ and $*$ is defined by minimum t-norm. Also, we establish the conditional UHR stability of sextic functional equations in a $\mathcal{FM}$-space.

Theorem 2.15. *Let E be a linear space and $(V, \mu, *)$ be a μ-complete β-homogeneous $\mathcal{FM}$-space and $p \in \{-1, 1\}$ be fixed. Suppose that $f : E \times E \times E \to (V, \mu, *)$ satisfies the condition $f(x, 0, z) = 0$ and the inequalities of the form:*

$$\begin{aligned}&\mu(f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)-2af(x_1,y,z),t)\\ &\geqslant \tau(x_1,x_2,y,z,t),\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\mu(f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)-2a^2f(x,y_1,z)\\ &-2b^2f(x,y_2,z),t)\\ &\geqslant \varsigma(x,y_1,y_2,z,t),\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\mu(f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2)-ab^2f(x,y,z_1+z_2)\\ &+f(x,y,z_1-z_2)-2a(a^2-b^2)f(x,y,z_1),t)\\ &\geqslant \upsilon(x,y,z_1,z_2,t),\end{aligned} \tag{2.3}$$

where $\tau, \varsigma, \upsilon : E^4 \to \triangle$, and $\triangle$ is the set of all non-decreasing functions, are given functions such that

$$\begin{aligned}&\lim_{n\to\infty} \tau(a^nx_1, a^nx_2, a^ny, a^nz, a^{6\beta pn}t) = 1,\\ &\lim_{n\to\infty} \varsigma(a^nx, a^ny_1, a^ny_2, a^nz, a^{6\beta pn}t) = 1,\\ &\lim_{n\to\infty} \upsilon(a^nx, a^ny, a^nz_1, a^nz_2, a^{6\beta pn}t) = 1\end{aligned}$$

for all $x, x_i, y, y_i, z, z_i \in E, i = 1, 2$. Assume that

$$\begin{aligned}\Phi(x,y,z,t) :=&\upsilon(a^{\frac{p+1}{2}}x, a^{\frac{p+1}{2}}y, a^{\frac{p-1}{2}}z, 0, a^{(9-3p)\beta}t/2^{\beta+2})\\ &* \varsigma(a^{\frac{p+1}{2}}x, a^{\frac{p-1}{2}}y, 0, a^{\frac{p-1}{2}}z, a^{(6-3p)\beta}t/2^{\beta+2})\\ &* \tau(a^{\frac{p-1}{2}}x, 0, a^{\frac{p-1}{2}}y, a^{\frac{p-1}{2}}z, a^{(4-3p)\beta}t/2)\end{aligned} \tag{2.4}$$

has the property:

$$\Phi(a^p x, a^p y, a^p z, a^{6\beta p} Lt) \geqslant \Phi(x,y,z,t) \tag{2.5}$$

for all $x, y, z \in E$ with a constant $0 < L < \frac{1}{2^\beta}$. Then there exists a unique sextic function $s : E \times E \times E \to (V, \mu, *)$ satisfying the system (1.1) such that

$$\mu(s(x,y,z) - f(x,y,z), \frac{2^\beta}{1-2^\beta L} t) \geqslant \Phi(x,y,z,t). \tag{2.6}$$

Proof. Let $x_1 = 2x$ and $x_2 = 0$ and replacing y, z by $2y, 2z$ in (2.1), respectively, we get

$$\mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), t) \geqslant \tau(2x, 0, 2y, 2z, t) \tag{2.7}$$

for all $x, y, z \in E$.

Let $y_1 = 2y$ and $y_2 = 0$ and replacing x, z by $2ax, 2z$ in (2.2), respectively, we have

$$\mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), t) \geqslant \varsigma(2ax, 2y, 0, 2z, t) \tag{2.8}$$

for all $x, y, z \in E$.

Let $z_1 = 2z$ and $z_2 = 0$ and replacing x, y by $2ax, 2ay$ in (2.3), respectively, we obtain

$$\mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z), t) \geqslant \upsilon(2ax, 2ay, 2z, 0, t) \tag{2.9}$$

for all $x, y, z \in E$. Since μ is β-homogeneous. We note that, since

$$\begin{aligned}&\mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z), t)\\&\geq \mu(\frac{1}{a^3}(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z)), t).\end{aligned}$$

Hence, since μ is β-homogeneous, it follows from (2.8) and (2.9) that

$$\begin{aligned}
&\mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z)\\
&\quad + 2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z), t)\\
&\geq \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z)\\
&\quad + 2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), t)\\
&= \mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), t)\\
&\geq \mu(a^{-3} f(2ax, 2ay, 2az) - a^2 f(2ax, 2y, 2z), t)\\
&= \mu(\frac{2}{2} a^{-3} f(2ax, 2ay, 2az) - \frac{2}{2} a^2 f(2ax, 2y, 2z), t)\\
&= \mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), 2^\beta t)\\
&= \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z) + 2f(2ax, 2ay, 2z)\\
&\quad - 2a^2 f(2ax, 2y, 2z), 2^\beta t)\\
&= \mu(2(a^{-3} f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z)\\
&\quad - a^2 f(2ax, 2y, 2z)), 2^\beta t)\\
&= \mu((a^{-3} f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z)) + (f(2ax, 2ay, 2z)\\
&\quad - a^2 f(2ax, 2y, 2z)), t)\\
&= \mu(\frac{1}{2}(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z)) + \frac{1}{2}(2f(2ax, 2ay, 2z)\\
&\quad - 2a^2 f(2ax, 2y, 2z)), t/2 + t/2)\\
&\geqslant \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), t/2)
\end{aligned}$$

$$
\begin{aligned}
&\quad * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), t/2)\\
&= \mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z), a^{3\beta} t/2)\\
&\quad * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), t/2)\\
&\geqslant \upsilon(2ax, 2ay, 2z, 0, a^{3\beta} t/2) * \varsigma(2ax, 2y, 0, 2z, t/2)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\quad \mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), t)\\
&\geq \mu(\frac{1}{a^2}(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z)), t)\\
&= \mu(2a^{-5} f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z), t)\\
&= \mu((2a^{-5})\frac{a^2}{a^2} f(2ax, 2ay, 2az) - 2\frac{a^2}{a^2} f(2ax, 2y, 2z), t)\\
&= \mu(\frac{1}{a^2}(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z)), t)\\
&= \mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t)\\
&= \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z) + 2f(2ax, 2ay, 2z)\\
&\quad - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t)\\
&= \mu(2(a^{-3} f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z)\\
&\quad - a^2 f(2ax, 2y, 2z)), a^{2\beta} t)\\
&= \mu(a^{-3} f(2ax, 2ay, 2az) - f(2ax, 2ay, 2z) + f(2ax, 2ay, 2z)\\
&\quad - a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^{\beta})\\
&= \mu(\frac{1}{2}(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z)) + \frac{1}{2}(2f(2ax, 2ay, 2z)\\
&\quad - 2a^2 f(2ax, 2y, 2z)), a^{2\beta} t/2^{\beta+1} + a^{2\beta} t/2^{\beta+1})\\
&\geq \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z), a^{2\beta} t/2^{\beta+1})\\
&\quad * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^{\beta+1})\\
&= \mu((2a^{-3})\frac{a^3}{a^3} f(2ax, 2ay, 2az) - 2\frac{a^3}{a^3} f(2ax, 2ay, 2z), a^{2\beta} t/2^{\beta+1})\\
&\quad * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^{\beta+1})\\
&= \mu(\frac{1}{a^3}(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z)), a^{2\beta} t/2^{\beta+1})\\
&\quad * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^{\beta+1})\\
&= \mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z)), a^{5\beta} t/2^{\beta+1})\\
&\quad * \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{2\beta} t/2^{\beta+1})\\
&\geq \upsilon(2ax, 2ay, 2z, 0, a^{5\beta} t/2^{\beta+1}) * \varsigma(2ax, 2y, 0, 2z, a^{2\beta} t/2^{\beta+1})
\end{aligned}
$$

for all $x, y, z \in E$. By (2.7) and the last inequality, we get

$$
\begin{aligned}
&\quad \mu(a^{-5} f(2ax, 2ay, 2az) - af(2x, 2y, 2z), t)\\
&= \mu(a^{-5} f(2ax, 2ay, 2az) - f(2ax, 2y, 2z)\\
&\quad + f(2ax, 2y, 2z) - af(2x, 2y, 2z), t)\\
&= \mu(\frac{1}{2}(2a^{-5} f(2ax, 2ay, 2az) - 2f(2ax, 2y, 2z))\\
&\quad + \frac{1}{2}(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z)), t/2 + t/2)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mu(2a^{-5}f(2ax,2ay,2az)-2f(2ax,2y,2z),t/2)\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),t/2)\\
&=\mu(a^{-5}f(2ax,2ay,2az)-f(2ax,2y,2z),t/2^{\beta+1})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),t/2)\\
&=\mu(\frac{1}{2}(2a^{-5}f(2ax,2ay,2az)-2a^{-2}f(2ax,2ay,2z))\\
&\quad +\frac{1}{2}(2a^{-2}f(2ax,2ay,2z)-2f(2ax,2y,2z)),t/2^{\beta+2}+t/2^{\beta+2})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),t/2)\\
&\geq \mu(2a^{-5}f(2ax,2ay,2az)-2a^{-2}f(2ax,2ay,2z),t/2^{\beta+2})\\
&\quad *\mu(2a^{-2}f(2ax,2ay,2z)-2f(2ax,2y,2z),t/2^{\beta+2})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),t/2)\\
&=\mu(2f(2ax,2ay,2az)-2a^{3}f(2ax,2ay,2z),a^{5\beta}t/2^{\beta+2})\\
&\quad *\mu(2f(2ax,2ay,2z)-2a^{2}f(2ax,2y,2z),a^{2\beta}t/2^{\beta+2})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),t/2)\\
&\geq \upsilon(2ax,2ay,2z,0,a^{5\beta}t/2^{\beta+2})*\varsigma(2ax,2y,0,2z,a^{2\beta}t/2^{\beta+2})\\
&\quad *\tau(2x,0,2y,2z,t/2)
\end{aligned}
$$

for all $x,y,z\in E$. Therefore, we get

$$
\begin{aligned}
&\mu(a^{-6}f(2ax,2ay,2az)-f(2x,2y,2z),t)\\
&=\mu((a^{-6})\frac{a}{a}f(2ax,2ay,2az)-\frac{a}{a}f(2x,2y,2z),t)\\
&=\mu(\frac{1}{a}(a^{-5}f(2ax,2ay,2az)-af(2x,2y,2z)),t)\\
&=\mu(a^{-5}f(2ax,2ay,2az)-af(2x,2y,2z),a^{\beta}t)\\
&=\mu(a^{-5}f(2ax,2ay,2az)-f(2ax,2y,2z)\\
&\quad +f(2ax,2y,2z)-af(2x,2y,2z),a^{\beta}t)\\
&=\mu(\frac{1}{2}(2a^{-5}f(2ax,2ay,2az)-2f(2ax,2y,2z))\\
&\quad +\frac{1}{2}(2f(2ax,2y,2z)-2af(2x,2y,2z)),a^{\beta}t/2+a^{\beta}t/2)\\
&\geq \mu(2a^{-5}f(2ax,2ay,2az)-2f(2ax,2y,2z),a^{\beta}t/2)\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),a^{\beta}t/2)\\
&=\mu(a^{-5}f(2ax,2ay,2az)-f(2ax,2y,2z),a^{\beta}t/2^{\beta+1})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),a^{\beta}t/2)\\
&=\mu(\frac{1}{2}(2a^{-5}f(2ax,2ay,2az)-2a^{-2}f(2ax,2ay,2z))\\
&\quad +\frac{1}{2}(2a^{-2}f(2ax,2ay,2z)-2f(2ax,2y,2z)),a^{\beta}t/2^{\beta+2}+a^{\beta}t/2^{\beta+2})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),a^{\beta}t/2)\\
&\geq \mu(2a^{-5}f(2ax,2ay,2az)-2a^{-2}f(2ax,2ay,2z),a^{\beta}t/2^{\beta+2})\\
&\quad *\mu(2a^{-2}f(2ax,2ay,2z)-2f(2ax,2y,2z),a^{\beta}t/2^{\beta+2})\\
&\quad *\mu(2f(2ax,2y,2z)-2af(2x,2y,2z),a^{\beta}t/2)\\
&=\mu(2f(2ax,2ay,2az)-2a^{3}f(2ax,2ay,2z),a^{6\beta}t/2^{\beta+2})
\end{aligned}
$$

$$
\begin{aligned}
&* \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z), a^{3\beta} t/2^{\beta+2}) \\
&* \mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z), a^{\beta} t/2) \\
\geq\ & \upsilon(2ax, 2ay, 2z, 0, a^{6\beta} t/2^{\beta+2}) * \varsigma(2ax, 2y, 0, 2z, a^{3\beta} t/2^{\beta+2}) \\
&* \tau(2x, 0, 2y, 2z, a^{\beta} t/2).
\end{aligned}
$$

Replacing x, y and z by $\frac{x}{2}, \frac{y}{2}$ and $\frac{z}{2}$ in the last inequality, respectively, we get

$$
\begin{aligned}
&\mu\Big(\frac{f(ax, ay, az)}{a^6} - f(x, y, z), t\Big) \\
&= \mu\Big(\frac{f(ax, ay, az)}{a^6} - \frac{f}{a}(ax, y, z) + \frac{f}{a}(ax, y, z) - f(x, y, z), t\Big) \\
&= \mu\Big(\frac{1}{2}\Big(\frac{2f(ax, ay, az)}{a^6} - \frac{2f}{a}(ax, y, z)\Big) + \frac{1}{2}\Big(\frac{2f}{a}(ax, y, z) - 2f(x, y, z)\Big), t/2 + t/2\Big) \\
&\geq \mu\Big(\frac{2f(ax, ay, az)}{a^6} - \frac{2}{a} f(ax, y, z), t/2\Big) * \mu\Big(\frac{2}{a} f(ax, y, z) - 2f(x, y, z), t/2\Big) \\
&= \mu\Big(\frac{2f(ax, ay, az)}{a^6} - \frac{2}{a^3} f(ax, ay, z) + \frac{2}{a^3} f(ax, ay, z) - \frac{2}{a} f(ax, y, z), t/2\Big) \\
&\quad * \mu\Big(\frac{2}{a} f(ax, y, z) - 2f(x, y, z), t/2\Big) \\
&= \mu\Big(\frac{1}{2}(\frac{2 \cdot 2f(ax, ay, az)}{a^6} - \frac{2 \cdot 2f(ax, ay, z)}{a^3}) \\
&\quad + \frac{1}{2}(\frac{2 \cdot 2f(ax, ay, z)}{a^3} - \frac{2 \cdot 2f(ax, y, z)}{a}), t/2 \cdot 2 + t/2 \cdot 2\Big) \\
&\quad * \mu\Big(\frac{2}{a} f(ax, y, z) - 2f(x, y, z), t/2\Big) \\
&\geq \mu\Big(\frac{2 \cdot 2f(ax, ay, az)}{a^6} - \frac{2 \cdot 2f(ax, ay, z)}{a^3}, t/2 \cdot 2\Big) \\
&\quad * \mu\Big(\frac{2 \cdot 2f(ax, ay, z)}{a^3} - \frac{2 \cdot 2f(ax, y, z)}{a}), t/2 \cdot 2\Big) \\
&\quad * \mu\Big(\frac{2}{a} f(ax, y, z) - 2f(x, y, z), t/2\Big) \\
&= \mu(2f(ax, ay, az) - 2a^3 f(ax, ay, z), a^{6\beta} t/2^{\beta+2}) \\
&\quad * \mu(2f(ax, ay, z) - 2a^2 f(ax, y, z), a^{3\beta} t/2^{\beta+2}) \\
&\quad * \mu(2f(ax, y, z) - 2af(x, y, z), a^{\beta} t/2) \\
&\geq \upsilon(ax, ay, z, 0, a^{6\beta} t/2^{\beta+2}) * \varsigma(ax, y, 0, z, a^{3\beta} t/2^{\beta+2}) \\
&\quad * \tau(x, 0, y, z, a^{\beta} t/2)
\end{aligned} \tag{2.10}
$$

for all $x, y, z \in E$. Replacing x, y, z by $a^{-1}x, a^{-1}y, a^{-1}z$ in (2.10), we get

$$
\begin{aligned}
&\mu\Big(\frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z), t\Big) \\
&\geq \mu\Big(\frac{1}{a^6}\Big(\frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z)\Big), t\Big) \\
&= \mu\Big(\frac{f(x, y, z)}{a^6} - f(a^{-1}x, a^{-1}y, a^{-1}z), a^{6\beta} t\Big) \\
&= \mu\Big(\frac{1}{2}\Big(\frac{2f}{a^6}(x, y, z) - \frac{2f}{a}(x, a^{-1}y, a^{-1}z)\Big) \\
&\quad + \frac{1}{2}\Big(\frac{2f}{a}(x, a^{-1}y, a^{-1}z) - 2f(a^{-1}x, a^{-1}y, a^{-1}z)\Big), a^{6\beta} t/2 + a^{6\beta} t/2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mu\Big(\frac{2f}{a^6}(x,y,z) - \frac{2f}{a}(x,a^{-1}y,a^{-1}z), a^{6\beta}t/2\Big)\\
&\quad * \mu\Big(\frac{2f}{a}(x,a^{-1}y,a^{-1}z) - 2f(a^{-1}x,a^{-1}y,a^{-1}z), a^{6\beta}t/2\Big)\\
&= \mu\Big(\frac{f}{a^6}(x,y,z) - \frac{f}{a}(x,a^{-1}y,a^{-1}z), a^{6\beta}t/2^{\beta+1}\Big)\\
&\quad * \mu\Big(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\Big)\\
&= \mu\Big(\frac{1}{2}(\frac{2}{a^6}f(x,y,z) - \frac{2}{a^3}f(x,y,a^{-1}z))\\
&\quad + \frac{1}{2}(\frac{2}{a^3}f(x,y,a^{-1}z) - \frac{2}{a}f(x,a^{-1}y,a^{-1}z), a^{6\beta}t/2^{\beta+2} + a^{6\beta}t/2^{\beta+2}\Big)\\
&\quad * \mu\Big(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\Big)\\
&\geq \mu\Big(\frac{1}{a^6}(2f(x,y,z) - 2a^3f(x,y,a^{-1}z)\Big), a^{6\beta}t/2^{\beta+2}\Big)\\
&\quad * \mu(\frac{1}{a^3}(2f(x,y,a^{-1}z) - 2a^2f(x,a^{-1}y,a^{-1}z)\Big), a^{6\beta}t/2^{\beta+2}\Big)\\
&\quad * \mu\Big(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\Big)\\
&\geq \mu\Big(2f(x,y,z) - 2a^3f(x,y,a^{-1}z)\Big), a^{12\beta}t/2^{\beta+2}\Big)\\
&\quad * \mu(2f(x,y,a^{-1}z) - 2a^2f(x,a^{-1}y,a^{-1}z)\Big), a^{9\beta}t/2^{\beta+2}\Big)\\
&\quad * \mu\Big(2f(x,a^{-1}y,a^{-1}z) - 2af(a^{-1}x,a^{-1}y,a^{-1}z), a^{7\beta}t/2\Big)\\
&\geq \upsilon(a^{-1}x,y,a^{-1}z,0,a^{12\beta}t/2^{\beta+2}) * \varsigma(x,a^{-1}y,0,a^{-1}z,a^{9\beta}t/2^{\beta+2})\\
&\quad * \tau(a^{-1}x,0,a^{-1}y,a^{-1}z,a^{7\beta}t/2)
\end{aligned}
$$

but, we know that

$$\mu\Big(\frac{f(a^{-1}x,a^{-1}y,a^{-1}z)}{a^{-6}} - f(x,y,z),t\Big) \geq \mu\Big(\frac{f(x,y,z)}{a^6} - f(a^{-1}x,a^{-1}y,a^{-1}z),t\Big)$$

therefore

$$
\begin{aligned}
&\mu\Big(\frac{f(a^{-1}x,a^{-1}y,a^{-1}z)}{a^{-6}} - f(x,y,z),t\Big)\\
&\geq \upsilon(a^{-1}x,y,a^{-1}z,0,a^{12\beta}t/2^{\beta+2}) * \varsigma(x,a^{-1}y,0,a^{-1}z,a^{9\beta}t/2^{\beta+2})\\
&\quad * \tau(a^{-1}x,0,a^{-1}y,a^{-1}z,a^{7\beta}t/2)
\end{aligned}
$$

and so

$$\mu\Big(\frac{f(a^px,a^py,a^pz)}{a^{6p}} - f(x,y,z),t\Big) \geq \Phi(x,y,z,t). \tag{2.11}$$

Now, we consider the set

$$\mathcal{D} = \{h : E \times E \times E \to V : h(x,0,z) = 0 \text{ for all } x,z \in E\}$$

and introduce the modular ρ on $\mathcal{D}$ as follows:

$$\rho(h) = \inf\{c > 0 : \mu(h(x,y,z),ct) \geq \Phi(x,y,z,t)\}.$$

We know that ρ is even from $\rho(-h) = \rho(h)$ and $\rho(0) = 0$. If $\rho(h) = 0$, then, for each $c > 0$,

$$\mu(h(x,y,z),ct) \geq \Phi(x,y,z,t)$$

for all $t > 1$ and $x, y \in E$. Now, if $\epsilon = ct$ is fixed and $t \to +\infty$, then $\mu(h(x, y, z), \epsilon) = 1$, which implies that $h = 0$. It is sufficient to show that ρ satisfies the following condition:

$$\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h)$$

if $\alpha + \beta = 1$ for all $\alpha, \beta \geq 0$. Let $\epsilon > 0$ be given. Then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq \rho(g) + \epsilon,\ \ \mu(g(x, y, z), c_1 t) \geq \Phi(x, y, z, t)$$

and

$$c_2 \leq \rho(h) + \epsilon,\ \ \mu(h(x, y, z), c_2 t) \geq \Phi(x, y, z), t).$$

If $\alpha + \beta = 1$ for all $\alpha, \beta \geq 0$, then we get

$$\begin{aligned}\mu(\alpha g(x, y, z) + \beta h(x, y, z), c_1 t + c_2 t) &\geq \mu(g(x, y, z), c_1 t) * \mu(h(x, y, z), c_2 t)\\ &\geq \Phi(x, y, z, t)\end{aligned}$$

and

$$\rho(\alpha g + \beta h) \leq c_1 + c_2 \leq \rho(g) + \rho(h) + 2\epsilon$$

thus

$$\rho(\alpha g + \beta h) \leq \rho(g) + \rho(h).$$

Now, we show that ρ has the $\triangle_2$-condition, where $\kappa = 2^\beta$. For all $\epsilon > 0$, there exists $c > 0$ such that

$$c \leq \rho(h) + \epsilon,\ \ \mu(h(x, y, z), ct) \geq \Phi(x, y, z, t).$$

Since $(V, \mu, *)$ is a β-homogeneous $\mathcal{FM}$-space, we get

$$\mu(2h(x, y, z), 2^\beta ct) = \mu(h(x, y, z), ct) \geq \Phi(x, y, z, t),$$

where $\rho(2h) \leq 2^\beta c \leq 2^\beta \rho(h) + 2^\beta \epsilon$ and so $\rho(2h) \leq 2^\beta \rho(h)$. Thus ρ satisfies the $\triangle_2$-condition with $\kappa = 2^\beta$.

Moreover, ρ satisfies the l.s.c. (in the modular sense). Indeed, if the sequence $\{h_n\}$ in $\mathcal{D}$ is ρ-convergent to h, then we can easily see that $h_n(x, y, z)$ is μ-convergent to $h(x, y, z)$ for all $x, y, z \in E$.

Let $\rho := \liminf_{n\to\infty} \rho(h_n) < \infty$ and $\rho(h) > \rho$. Then, we have

$$\mu(h(x, y, z), \rho t) < \Phi(x, y, z, t)$$

for all $t > 0$. Since μ satisfies the l.s.c. (in the fuzzy modular sense), we have

$$\limsup_{n\to\infty} \mu(h_n(x, y, z), \rho t) \leq \mu(h(x, y, z), \rho t) < \Phi(x, y, z, t).$$

From the last inequality, we know that there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\mu(h_n(x, y, z), \rho t) < \Phi(x, y, z, t)$$

and so $\rho(h_n) > \rho$ for all $n \geq n_0$. Thus lim inf$\rho(h_n) > \rho$ where $n \to \infty$, which is a contradiction. Therefore, ρ satisfies the l.s.c..

If $\delta > 0$ and $\lambda \in (0, 1)$ are given, it follows from $\Phi(x, y, z) \in \triangle$ that there exists $t_0 > 0$ such that $\Phi(x, y, z, t_0) > 1 - \lambda$. Let $\{h_n\}$ be a ρ-Cauchy sequence in $\mathcal{D}_\rho$ and let $\epsilon < \frac{\delta}{t_0}$ be given. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $\rho(h_n - h_m) \leq \epsilon$ for all $n, m \geq n_0$.

Now, by considering the definition of the modular ρ, we see that

$$\begin{aligned}\mu(h_n(x, y, z) - h_m(x, y, z), \delta) &\geq \mu(h_n(x, y, z) - h_m(x, y, z), \epsilon t_0)\\ &\geq \Phi(x, y, z, t_0)\\ &> 1 - \lambda\end{aligned} \tag{2.12}$$

for all $x, y, z \in E$ and $n, m \geq n_0$.

If x, y and z are arbitrary given points of E, then (2.12) implies that $\{h_n(x, y, z)\}$ is a μ-Cauchy sequence in $(V, \mu, *)$. Since it is μ-complete, it follows that $\{h_n(x, y, z)\}$ is μ-convergent in $(V, \mu, *)$ for all $x, y, z \in E$. Thus, we can define

$$h(x, y, z) = \lim_{n\to\infty} h_n(x, y, z),$$

where a function $h : E \times E \times E \to (V, \mu, *)$ for all $x, y, z \in E$. Moreover, μ has the l.s.c.. Then, we have

$$\rho(h_n - h) \leq \epsilon$$

for all $n \geq n_0$. Thus $\{h_n\}$ is a ρ-convergent sequence in $\mathcal{D}_\rho$. Therefore, $\mathcal{D}_\rho$ is ρ-complete. Now, we consider the function $\mathcal{T} : \mathcal{D}_\rho \to \mathcal{D}_\rho$ defined by

$$\mathcal{T}h(x, y, z) := a^{-6p} h(a^p x, a^p y, a^p z)$$

for all $h \in \mathcal{D}_\rho$. Let $g, h \in \mathcal{D}_\rho$ and $c \in [0, \infty]$ be an arbitrary constant with $\rho(g - h) \leq c$. From the definition of ρ, we have

$$\mu(g(x, y, z) - h(x, y, z), ct) \geq \Phi(x, y, z, t)$$

for all $x, y, z \in E$. By the assumption and the last inequality, we get

$$\begin{aligned}
&\mu(\mathcal{T}g(x, y, z) - \mathcal{T}h(x, y, z), Lct)\\
&= \mu(a^{-6p} g(a^p x, a^p y, a^p z) - a^{-6p} h(a^p x, a^p y, a^p z), Lct)\\
&= \mu(g(a^p x, a^p y, a^p z) - h(a^p x, a^p y, a^p z), a^{6\beta p} Lct)\\
&\geq \Phi(a^p x, a^p y, a^p z, a^{6\beta p} Lt)\\
&\geq \Phi(x, y, z, t)
\end{aligned}$$

for all $x, y, z \in E$ and so $\rho(\mathcal{T}g - \mathcal{T}h) \leq L\rho(g - h)$ for all $g, h \in \mathcal{D}_\rho$, that is, $\mathcal{T}$ is a ρ-contraction.

Now, we show that the ρ-strict mapping $\mathcal{T}$ satisfies the conditions of Theorem (2.14). Observe that

$$\mu(a^{-6p} f(a^{2p}x, a^{2p}y, a^{2p}z) - f(a^p x, a^p y, a^p z), t) \geq \Phi(a^p x, a^p y, a^p z, t)$$

and so

$$\begin{aligned}
&\mu(a^{-2(6)p} f(a^{2p}x, a^{2p}y, a^{2p}z) - a^{-6p} f(a^p x, a^p y, a^p z), Lt)\\
&= \mu(a^{-6p} f(a^{2p}x, a^{2p}y, a^{2p}z) - f(a^p x, a^p y, a^p z), a^{6\beta p} Lt)\\
&\geq \Phi(a^p x, a^p y, a^p z), a^{6\beta p} Lt)\\
&\geq \Phi(x, y, z, t).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\mu\Big(\frac{f(a^{2p}x, a^{2p}y, a^{2p}z)}{a^{2(6)p}} - f(x, y, z), 2^\beta(Lt + t)\Big)\\
&\geq \mu\Big(\frac{f(a^{2p}x, a^{2p}y, a^{2p}z)}{a^{2(6)p}} - \frac{f(a^p x, a^p y, a^p z)}{a^{6p}}, Lt\Big)\\
&\quad * \mu\Big(\frac{f(a^p x, a^p y, a^p z)}{a^{6p}} - f(x, y, z), t\Big)\\
&\geq \Phi(x, y, z)(t)
\end{aligned} \tag{2.13}$$

for all $x, y, z \in E$. By replacing x, y and z by $a^p x, a^p y$ and $a^p z$ in (2.13), respectively, we get

$$\mu\big(a^{-2(6)p} f(a^{3p}x, a^{3p}y, a^{3p}z) - f(a^p x, a^p y, a^p z), a^{6\beta p} 2^\beta (L^2 t + Lt)\big)$$

$$\geq \Phi(a^p x, a^p y, a^p z), a^{6\beta p} L t)$$
$$\geq \Phi(x, y, z, t)$$

and so

$$\mu\big(a^{-3(6)p} f(a^{3p}x, a^{3p}y, a^{3p}z) - a^{-6p} f(a^p x, a^p y, a^p z), 2^\beta(L^2 t + Lt)\big) \geq \Phi(x, y, z, t).$$

Therefore, we get

$$\begin{aligned}
&\mu\Big(\frac{f(a^{3p}x, a^{3p}y, a^{3p}z)}{a^{3(6)p}} - f(x, y, z), 2^\beta\{2^\beta(L^2 t + Lt) + t\}\Big) \\
&\geq \mu\Big(\frac{f(a^{3p}x, a^{3p}y, a^{3p}z)}{a^{3(6)p}} - \frac{f(a^p x, a^p y, a^p z)}{a^{6p}}, 2^\beta(L^2 t + Lt)\Big) \\
&\quad * \mu\Big(\frac{f(a^p x, a^p y, a^p z)}{a^{6p}} - f(x, y, z), t\Big) \\
&\geq \Phi(x, y, z, t)
\end{aligned}$$

for all $x, y, z \in E$. By induction, we can easily see that

$$\mu\Big(\frac{f(a^{np}x, a^{np}y, a^{np}z)}{a^{6np}} - f(x, y, z), \Big\{(2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta L)^{i-1}\Big\} t\Big) \geq \Phi(x, y, z, t)$$

for all $x, y, z \in E$ and so

$$\rho(\mathcal{T}^n f - f) \leq (2^\beta L)^{n-1} + 2^\beta \sum_{i=1}^{n-1} (2^\beta L)^{i-1} \leq 2^\beta \sum_{i=1}^{n} (2^\beta L)^{i-1} \leq \frac{2^\beta}{1 - 2^\beta L}. \tag{2.14}$$

Next, we confirm that $\delta_\rho(f) = \sup\{\rho(\mathcal{T}^n(f) - \mathcal{T}^m(f)) : n, m \in \mathbb{N}\} < \infty$. Since ρ satisfies the $\triangle_2$-condition with $\kappa = 2^\beta$, it follows from the inequality (2.14) that

$$\begin{aligned}
\rho(\mathcal{T}^n f - \mathcal{T}^m f) &\leq \frac{1}{2}\rho(2\mathcal{T}^n f - 2f) + \frac{1}{2}\rho(2\mathcal{T}^m f - 2f) \\
&\leq \frac{\kappa}{2}\rho(\mathcal{T}^n f - f) + \frac{\kappa}{2}\rho(\mathcal{T}^m f - f) \\
&\leq \frac{2^{2\beta}}{1 - 2^\beta L}
\end{aligned} \tag{2.15}$$

for all $n, m \in \mathbb{N}$. By the definition of $\delta_\rho(f)$, we have $\delta_\rho(f) < \infty$. Thus Theorem (2.14) shows that $\{\mathcal{T}^n(f)\}$ is ρ-convergent to a point $s \in \mathcal{D}_\rho$. Since ρ has the l.s.c., the inequality (2.14) gives $\rho(\mathcal{T}(s) - f) < \infty$.

If we replace m by $n + 1$ in the inequality (2.15), then we obtain

$$\rho(\mathcal{T}^{n+1} f - \mathcal{T}^n f) \leq \frac{2^{2\beta}}{1 - 2^\beta L}.$$

Therefore, we get $\rho(\mathcal{T}(s) - s) \leq \frac{2^{2\beta}}{1-2^\beta L} < \infty$. Therefore, it follows from Theorem (2.14) that ρ-limit of $\{\mathcal{T}^n(f)\}, s \in \mathcal{D}_\rho$, is a fixed point of the mapping $\mathcal{T}$.

If we replace x_1, x_2, y and z by $a^{np}x_1, a^{np}x_2, a^{np}y$ and $a^{np}z$ in the inequality (2.1), respectively, then we obtain

$$\begin{aligned}
&\mu\Big(\frac{f(a^{np}(ax_1 + bx_2), a^{np}y, a^{np}z)}{a^{6np}} + \frac{f(a^{np}(ax_1 - bx_2), a^{np}y, a^{np}z)}{a^{6np}} \\
&\quad - 2a\frac{f(a^{np}x_1, a^{np}y, a^{np}z)}{a^{6np}}, t\Big) \\
&= \mu(f(a^{np}(ax_1 + bx_2), a^{np}y, a^{np}z) + f(a^{np}(ax_1 - bx_2), a^{np}y, a^{np}z) \\
&\quad - 2af(a^{np}x_1, a^{np}y, a^{np}z), a^{6\beta np}t) \\
&\geq \tau(a^{np}x_1, a^{np}x_2, a^{np}y, a^{np}z), a^{6\beta np}t).
\end{aligned} \tag{2.16}$$

Similarly, by replacing x, y_1, y_2 and z by $a^{np}x, a^{np}y_1, a^{np}y_2$ and $a^{np}z$ in the inequality (2.2), respectively, we get

$$\begin{aligned}&\mu\Big(\frac{f(a^{np}x, a^{np}(ay_1+by_2), a^{np}z)}{a^{6np}} + \frac{f(a^{np}x, a^{np}(ay_1-by_2), a^{np}z)}{a^{6np}}\\&- 2a^2\frac{f(a^{np}x, a^{np}y_1, a^{np}z)}{a^{6np}} - 2b^2\frac{f(a^{np}x, a^{np}y_2, a^{np}z)}{a^{6np}}, t\Big)\\ \geq\ & \varsigma(a^{np}x, a^{np}y_1, a^{np}y_2, a^{np}z), a^{6\beta np}t)\end{aligned} \tag{2.17}$$

and, also by replacing x, y, z_1 and z_2 by $a^{np}x, a^{np}y, a^{np}z_1$ and $a^{np}z_2$ in the inequality (2.3), respectively, we get

$$\begin{aligned}&\mu\Big(\frac{f(a^{np}x, a^{np}y, a^{np}(az_1+bz_2))}{a^{6np}} + \frac{f(a^{np}x, a^{np}y, a^{np}(az_1-bz_2))}{a^{6np}}\\&- ab^2\frac{f(a^{np}x, a^{np}y, a^{np}(z_1+z_2))}{a^{6np}} + \frac{f(a^{np}x, a^{np}y, a^{np}(z_1-z_2))}{a^{6np}}\\&- 2a(a^2-b^2)\frac{f(a^{np}x, a^{np}y, a^{np}z_1)}{a^{6np}}, t\Big)\\ \geq\ & \upsilon(a^{np}x, a^{np}y, a^{np}z_1, a^{np}z_2), a^{6\beta np}t)\end{aligned} \tag{2.18}$$

for all $x, x_i, y, y_i, z, z_i \in E, i = 1, 2$. Taking $n \to \infty$ in the inequalities (2.16), (2.17) and (2.18), we deduce that s satisfies the system (1.1), that is, s is sextic. It follows from the inequality (2.14) that

$$\rho(s-f) \leq \frac{2^\beta}{1-2^\beta L}.$$

Hence (2.5) holds. If s^* is another fixed point of $\mathcal{T}$, then we get

$$\begin{aligned}\rho(s-s^*) &\leq \frac{1}{2}\rho(2\mathcal{T}(s)-2f) + \frac{1}{2}\rho(2\mathcal{T}(s^*)-2f)\\&\leq \frac{\kappa}{2}\rho(\mathcal{T}(s)-f) + \frac{\kappa}{2}\rho(\mathcal{T}(s^*)-f)\\&\leq \frac{2^{2\beta}}{1-2^\beta L}\\&< \infty.\end{aligned}$$

Since $\mathcal{T}$ is ρ-contraction, we get

$$\begin{aligned}\rho(s-s^*) &= \rho(\mathcal{T}(s)-\mathcal{T}(s^*))\\&\leq L\rho(s-s^*),\end{aligned}$$

which implies that $\rho(s-s^*) = 0$ or $s = s^*$. Since $\rho(s-s^*) < \infty$, which proves the uniqueness of s. This completes the proof. □

Concluding remarks

Our results guarantee the generalized UHR stability of sextic mappings, whose codomain is equipped with a β-homogeneous and l.s.c. modular.

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