Applications of Coupled fixed points for multivalued mappings in cone metric spaces

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Abstract: In this article we present a variant of the basic result of "*T. Gnana. Bhaskar and V. Lakshmikanthan, Fixed point theorem in partially ordered metric spaces and applications, Nonlinear Anal. TMA, 65 (2006), 1379–1393.*" to find the coupled fixed points of multivalued mappings without mixed monotone property in cone metric space involving non-normal cones. As applications we prove the existence of certain type of non-linear Fredholm integral equations in two variables and present a non-trivial example. In this way we provide a gateway to work in this applicable theory of coupled fixed and coincidence points.

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1 Introduction

In [1] Guo and Lakshamikantham proved the first important result for coupled fixed points of nonlinear operators and presented some applications in differential equations. After that in [2] Bhaskar and Lakshamikantham prove the following;

Thoerem[2] Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d\left(F\left(x,y\right),F\left(u,v\right)\right) \leq \frac{k}{2}[d(x,u)+d(y,v)],$$

for all $x \geq u$, $y \leq v$. If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y \geq F(y_0, x_0)$. Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

The coupled fixed point theory has many applications in the existence theory of many types of operators. Many authors divert their efforts in this direction and generalize and present many variants of the above result in many directions. A mutivalued result is presented in [3] by Samet and Vetro. Some researchers proved coupled fixed point results without monotone property (see [4-9]). In [10] Samet et al. show that most of the coupled fixed point theorems on ordered metric spaces are infact immediate consequences of well-known fixed point theorems in the literature. In this article we explain how our results nullify the discussion in [10]. We define mixed monotone property for multivalued mappings and generalized it. We provide a graphical presentation of sequences which are neither increasing nor decreasing but converge at one point, that is the sequence having compareable terms. We prove existence of solution of a certain type of integral equations and provide an example to validate the significance of our results.

2 Preliminaries

Let $\mathbb{E}$ be a real Banach space with its zero element θ. A nonempty subset K of $\mathbb{E}$ is called a cone if
(a). K is nonempty closed and $K \neq \{\theta\}$.
(b). $K \cap (-K) = \{\theta\}$;
(c). if α, β are nonnegative real numbers and $x, y \in K$, then $\alpha x + \beta y \in K$.

For a given cone $K \subseteq \mathbb{E}$, we define a partial ordering $\preccurlyeq$ with respect to K by $x \preccurlyeq y$ if and only if $y - x \in K$; $x \prec y$ stands for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in intK$, where $intK$ denotes the interior of K. The cone K is said to be solid if it has a nonempty interior.

The following definitions and lemmas will be used to prove our main results:

Definition 2.1. [11] Let X be a nonempty set. A vector-valued function $d : X \times X \to \mathbb{E}$ is said to be a *cone* metric if the following conditions hold:
$(M1)$ $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
$(M2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(M3)$ $d(x, z) \preccurlyeq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The pair (X, d) is then called a *cone* metric space.

The cone metric d in X generate a topology τ_d. The base of topology τ_d consist of the sets

$$B_c(y) = \{x \in X : d(x, y) \ll c\} \text{ for some } c \in \mathbb{E} \text{ with } \theta \ll c.$$

For $x_0 \in X$ and $\theta \ll r$, we define closed ball

$$\bar{B}(x_0, r) := \{x \in X : d(x_0, x) \preccurlyeq r\},$$

in cone metric space (X, d). A set $A \subset (X, d)$ is called closed if, for any sequence $\{x_n\} \subset A$ converges to x, we have $x \in A$.

Definition 2.2. [11] Let(X, d) be a *cone* metric space, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x if for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$;

(ii) $\{x_n\}$ is a Cauchy sequence if for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$;

(iii) (X, d) is complete if every Cauchy sequence in X is convergent.

Let (X, d) be a cone metric space. The following properties will be used very often (for more details, see [12, 13]).

$(P1)$ If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$.

$(P2)$ If $c \in intK$, $a_n \in \mathbb{E}$ and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$.

Definition 2.3. **[14]** A partially ordered set consists of a set X and a binary relation $\leq$ on X which satisfies the following conditions:
(i). $x \leq x$ (reflexivity),
(ii). if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric),
(iii). if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity),
for all x, y and z in X. A set with a partial order $\leq$ is called a partially ordered set. Let $(X, \leq)$ be a partially ordered set and $x, y \in X$. Elements x and y are said to be comparable elements of X if either $x \leq y$ or $y \leq x$.

Definition 2.4. [15] Let A and B be two non-empty subsets of $(X, \leq)$, the relations between A and B are denoted and defined as follows:
(i). $A \leq_1 B$: if for every $a \in A$ there exists $b \in B$ such that $a \leq b$,
(ii). $A \leq_2 B$: if for every $b \in B$ there exists $a \in A$ such that $a \leq b$,
(iii). $A \leq_3 B$: if $A \leq_1 B$ and $A \leq_2 B$.
In addition we define the following relations
(iv). $A \leq_4 B$: if for every $a \in A$ there exists $b \in B$ such that $a \asymp b$, (read as; "a is compareable with b”).
(v). $A \leq_5 B$: if for every $b \in B$ there exists $a \in A$ such that $a \asymp b$.

Definition 2.5. An ordered cone metric space is said to have a subsequential limit comparison property if for every non decreasing sequence $\{x_n\}$ in X with $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \asymp x$ for all n.

Definition 2.6. An ordered cone metric space is said to have a sequential limit comparison property if for every non decreasing sequence $\{x_n\}$ in X with $x_n \to x$, we have $x_n \asymp x$ for all n.

Let $C(X)$ denotes the family of nonempty closed subsets of X. According to [16], let us denote for $p \in \mathbb{E}$

$$s(p) = \{q \in \mathbb{E} : p \preccurlyeq q\} \text{ for } q \in \mathbb{E}$$

For $A, B \in C(X)$, we denote and define

$$\sigma(A,B) = \underset{a\in A, b\in B}{\cap} s\left(d\left(a,b\right)\right).$$

Lemma 2.1 Let (X,d) be a *cone* metric space with a cone K. If $q \in \sigma(A,B)$, then $d(a,b) \preccurlyeq q$ for all $a \in A$, $b \in B$.

Proof: Since $q \in \sigma(A,B) = \underset{a\in A, b\in B}{\cap} s\left(d\left(a,b\right)\right)$, which means $q \in s\left(d\left(a,b\right)\right)$ for all $a \in A$ and $b \in B$. This further implies that

$$d\left(a,b\right) \preccurlyeq q$$

for all $a \in A$ and $b \in B$.

Remark 2.1 [12] The vector cone metric is not continuous in the general case, *i.e.* from $x_n \to x$, $y_n \to y$ it need not follow that $d(x_n, y_n) \to d(x,y)$.

Remark: 2.2 Let (X,d) be a tvs-cone metric space. If $E = R$ and $P = [0, +\infty)$, then (X,d) is a metric space. Moreover, for $A, B \in CB(X)$, $H(A,B) = \inf s(A,B)$ is the Hausdorff distance induced by d. Also note that $\inf \sigma(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$.

3 Set valued results

Definition 3.1 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to 2^X$ be a set valued mapping. We say that F has comparable combined monotone (CCM) property if for any $x, y \in X$,

$$x_1, x_2, y_1, y_2 \in X,\ x_1 \asymp x_2 \ \ \textit{and}\ y_1 \asymp y_2 \Rightarrow\ F(x_1, y_1) \leq_4 F(x_2, y_2).$$

Definition 3.2 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to 2^X$ be a set valued mapping. We say that F has combined monotone (CM) property if for any $x, y \in X$,

$$x_1, x_2, y_1, y_2 \in X,\ x_1 \leq x_2 \ \ \textit{and}\ y_1 \geq y_2 \Rightarrow\ F(x_1, y_1) \leq_1 F(x_2, y_2).$$

Note that (CM) property implies (CCM) property.

Remark: 3.1 Above definition of combined monotone property is equivalent to the mixed monotone property in multivalued mappings. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to 2^X$ be a set valued mapping. From litrature we say that F has mixed monotone (MM) property if for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \leq x_2 \ \Rightarrow\ F(x_1, y) \leq_1 F(x_2, y) \tag{a}$$

and

$$y_1, y_2 \in X,\ y_1 \leq y_2 \ \Rightarrow\ F(x, y_2) \leq_1 F(x, y_1). \tag{b}$$

From (a) we have for $x_1 \leq x_2 \;\Rightarrow\; F(x_1, y_1) \leq_1 F(x_2, y_1)$ and (b) implies $y_2 \leq y_1 \;\Rightarrow\; F(x_2, y_1) \leq_1 F(x_2, y_2)$, thus we have $F(x_1, y_1) \leq_1 F(x_2, y_2)$. Hence (MM) implies (CM).

Conversely: For $x_1 \leq x_2$ *and* $y_1 \geq y_2 \Rightarrow\; F(x_1, y_1) \leq_1 F(x_2, y_2) \Rightarrow F(x_1, y) \leq_1 F(x_2, y)$ for $y_1 = y_2 = y$. Also $F(x, y_1) \leq_1 F(x, y_2)$ for $x_1 = x_2 = x$. Thus (CM) implies (MM).

So we have

$$(MM) \Leftrightarrow (CM) \Rightarrow (CCM).$$

Example 3.1. Let $X = [-1, 1]$ and $F : X \times X \to 2^X$ be a mapping defined by

$$F(x, y) = [-1, x \cdot \sin(\frac{1}{y})]$$

We need to show that F does not satisfy (a) and (b).

For any $y \in X$ and for $x_1, x_2 \in X,\; x_1 \leq x_2 \;\Rightarrow\; F(x_1, y) \leq_1 F(x_2, y)$ *i.e.*, $[-1, x_1 \cdot \sin(\frac{1}{y})] \subseteq [-1, x_2 \cdot \sin(\frac{1}{y})]$ but for any $x \in X$, take $y_1 = 0.211, y_2 = 0.573 \in X$ the clearly $y_1 \leq y_2 \;\not\Rightarrow\; F(x, y_2) = [-1, 0.988x] \leq_1 F(x, y_1) = [-1, -0.999]$. Thus F does not satisfy (MM) property, but satisfy (CCM) property.

Theorem 3.1 Let (X, d) be a complete cone metric space endowed with a partial order $\leq$ on X. Let $F : X \times X \to C(X)$ be a multivalued mapping having CCM- property on X. Assume that there exists a $k \in [0, 1)$ such that

$$\frac{k}{2}[d(x, u) + d(y, v)] \in \sigma(F(x, y), F(u, v)]$$

for all $x \asymp u$, $y \asymp v$. If there exist $x_0, y_0 \in X$, such that

$$\{x_0\} \leq_4 F(x_0, y_0) \text{ and } F(y_0, x_0) \leq_5 \{y_0\}.$$

If X has limit comparison property then there exist $\bar{x}, \bar{y} \in X$, such that

$$\bar{x} \in F(\bar{x}, \bar{y}) \text{ and } \bar{y} \in F(\bar{y}, \bar{x}).$$

Proof: Since $\{x_0\} \leq_4 F(x_0, y_0)$ and $F(y_0, x_0) \leq_5 \{y_0\}$, then there exist some $x_1 \in F(x_0, y_0)$ and $y_1 \in F(y_0, x_0)$ such that

$$x_0 \asymp x_1 \text{ and } y_1 \asymp y_0, \tag{1}$$

so by given condition we have

$$\frac{k}{2}[d(x_0, x_1) + d(y_0, y_1)] \in \sigma(F(x_0, y_0), F(x_1, y_1)],$$

and

$$\frac{k}{2}[d(y_0, y_1) + d(x_0, x_1)] \in \sigma(F(y_0, x_0), F(y_1, x_1)].$$

As

$x_0 \asymp x_1$ and $y_1 \asymp y_0 \Rightarrow F(x_0, y_0) \leq_4 F(x_1, y_1)$ and $F(y_0, x_0) \leq_4 F(y_1, x_1)$

then there exist $x_2 \in F(x_1, y_1)$ and $y_2 \in F(y_1, x_1)$ such that $x_1 \asymp x_2$, and $y_1 \asymp y_2$, so using lemma 2.1 we have

$$\frac{k}{2}[d(x_0, x_1) + d(y_0, y_1)] \in \sigma(d(x_1, x_2))$$

which gives

$$d(x_1, x_2) \preccurlyeq \frac{k}{2}[d(x_0, x_1) + d(y_0, y_1)],$$

also

$$\frac{k}{2}[d(y_0, y_1) + d(x_0, x_1)] \in s(d(y_1, y_2))$$

this gives

$$d(y_1, y_2) \preccurlyeq \frac{k}{2}[d(y_0, y_1) + d(x_0, x_1)].$$

Continuing in this way we will get, $x_{n+2} \in F(x_{n+1}, y_{n+1})$ and $y_{n+2} \in F(y_{n+1}, x_{n+1})$ such that $x_{n+1} \asymp x_{n+2}$, and $y_{n+1} \asymp y_{n+2}$, so we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\preccurlyeq \frac{k}{2}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] \\ &\preccurlyeq \frac{k}{2}d(x_{n+1}, x_n) + \frac{k}{2}d(y_{n+1}, y_n). \end{aligned}$$

and

$$\begin{aligned} d(y_{n+2}, y_{n+1}) &\preccurlyeq \frac{k}{2}[d(y_{n+1}, y_n) + d(x_{n+1}, x_n)] \\ &\preccurlyeq \frac{k}{2}d(y_{n+1}, y_n) + \frac{k}{2}d(x_{n+1}, x_n). \end{aligned}$$

Consider,

$$\begin{aligned} d(y_{n+2}, y_{n+1}) &\preccurlyeq \frac{k}{2}[d(y_{n+1}, y_n) + d(x_{n+1}, x_n)] \\ &\preccurlyeq \frac{k}{2}d(y_{n+1}, y_n) + \frac{k}{2}d(x_{n+1}, x_n) \\ &\preccurlyeq \frac{k^2}{2^2}[d(y_n, y_{n-1}) + d(x_n, x_{n-1})] + \frac{k^2}{2^2}[d(y_n, y_{n-1}) + d(x_n, x_{n-1}))] \\ &\preccurlyeq \frac{k^2}{2}[d(y_n, y_{n-1}) + d(x_n, x_{n-1})] \\ &. \\ &. \\ &. \\ &\preccurlyeq \frac{k^{n+1}}{2}[d(y_1, y_0) + d(x_1, x_0)] \end{aligned}$$

Similarly

$$d(x_{n+2}, x_{n+1}) \preccurlyeq \frac{k^{n+1}}{2}[d(x_1, x_0) + d(y_1, y_0)]$$

Now for $m > n$, consider

$$\begin{aligned} d(x_n, x_m) &\preccurlyeq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preccurlyeq \frac{1}{2}[k^n + k^{n+1} + \cdots + k^{m-1}][d(x_1, x_0) + d(y_1, y_0)] \\ &\prec \frac{k^n}{2(1-k)}[d(x_1, x_0) + d(y_1, y_0)]. \end{aligned}$$

Since $k^n \to 0 \; as \; n \to \infty$, this gives us $\frac{k^n}{2(1-k)}[d(x_1, x_0) + d(y_1, y_0)] \to \theta$ as $n \to \infty$. Now, using properties $(P1)$ and $(P2)$ of cone metric space, for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number n_1 such that $d(x_n, x_m) \ll c$ for all $m, n \geq n_1$, so $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, $\{x_n\}$ is convergent in X and $\lim\limits_{n\to\infty} x_n = \bar{x}$. Hence, for every $c \in \mathbb{E}$ with $\theta \ll c$, there is a natural number k_1 such that

$$d(\bar{x}, x_{n+1}) \ll \frac{c}{3}, \quad for\ all\ n \geq k_1.$$

Similarly we can prove that $\{y_n\}$ is cauchy sequence in X, by completeness of (X, d) we have $\lim\limits_{n\to\infty} y_n = \bar{y}$. Hence, for every $c \in \mathbb{E}$ with $\theta \ll c$, there is a natural number k_2 such that

$$d(y, y_{n+1}) \ll \frac{c}{3}, \quad for\ all\ n \geq k_2.$$

Now to prove $\bar{x} \in F(\bar{x}, \bar{y})$ and $\bar{y} \in F(\bar{y}, \bar{x})$. By limit comparison property of X, we have $x_n \asymp \bar{x}$ and $y_n \asymp \bar{y}$ for all n, we have

$$\frac{k}{2}[d(x_n, \bar{x}) + d(y_n, \bar{y})] \in \sigma(F(x_n, y_n), F(\bar{x}, \bar{y})]$$

and

$$\frac{k}{2}[d(y_n, \bar{y}) + d(x_n, \bar{x})] \in \sigma(F(y_n, x_n), F(\bar{y}, \bar{x})]$$

there exists a sequence v_n in $F(\bar{x}, \bar{y})$ such that

$$d(x_{n+1}, v_n) \preccurlyeq \frac{k}{2}[d(x_n, \bar{x}) + d(y_n, \bar{y})]$$

and a sequence u_n in $F(\bar{y}, \bar{x})$ such that

$$d(y_{n+1}, u_n) \preccurlyeq \frac{k}{2}[d(y_n, \bar{y}) + d(x_n, \bar{x})].$$

Now consider,

$$\begin{aligned} d(\bar{x}, v_n) &\preccurlyeq d(x_{n+1}, \bar{x}) + d(x_{n+1}, v_n) \\ &\preccurlyeq d(x_{n+1}, \bar{x}) + \frac{k}{2}[d(x_n, \bar{x}) + d(y_n, \bar{y})] \\ &\ll c \text{ for all } v \geq k_3(c), \text{ where } k_3 = \max\{k_1, k_2\} \end{aligned}$$

Which iplies $v_n \to \bar{x}$, since $F(\bar{x}, \bar{y})$ is closed so $\bar{x} \in F(\bar{x}, \bar{y})$. Similarly $u_n \to \bar{y}$, and $F(\bar{y}, \bar{x})$ is closed so $\bar{y} \in F(\bar{y}, \bar{x})$.

Corollary 3.1. Let (X, d) be a complete cone metric space endowed with a partial order $\leq$ on X. Let $F : X \times X \to C(X)$ be a multivalued mapping having CCM- property on X. Assume that there exists a $k \in [0, 1)$ such that

$$\frac{k}{2}[d(x, u) + d(y, v)] \in \sigma(F(x, y), F(u, v)]$$

for all $x \asymp u$, $y \asymp v$. If there exists $x_0, y_0 \in X$, such that

$$\{x_0\} \leq_4 F(x_0, y_0) \text{ and } F(y_0, x_0) \leq_5 \{y_0\}.$$

If X has limit comparison property then there exist $\bar{x}, \bar{y} \in X$, such that

$$\bar{x} \in F(\bar{x}, \bar{y}) \text{ and } \bar{y} \in F(\bar{y}, \bar{x}).$$

Corollary 3.2. Let (X, d) be a complete metric space endowed with a partial order $\leq$ on X. Let $F : X \times X \to CB(X)$ be a multivalued mapping having CCM- property on X. Assume that there exists a $k \in [0, 1)$ such that

$$\delta(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$$

for all $x \asymp u$, $y \asymp v$. If there exists $x_0, y_0 \in X$, such that

$$\{x_0\} \leq_1 F(x_0, y_0) \text{ and } F(y_0, x_0) \leq_2 \{y_0\}.$$

If X has limit comparison property then there exist $\bar{x}, \bar{y} \in X$, such that

$$\bar{x} \in F(\bar{x}, \bar{y}) \text{ and } \bar{y} \in F(\bar{y}, \bar{x}).$$

Remark: 3.2. Samet et al [10] ,with the help of their lemma 2.1, showed that most of the coupled fixed point theorems for single valued mappings (on ordered metric spaces) are in fact immediate consequences of well-known fixed point theorems. However in cone metric spaces lemma 2.1(a) of [10] is not valid. Moreover in the case of multivalued mappings validity of lemma 2.1(b) of [10] is also suspicious. Therefore, the extensions of coupled fixed point results to multivalued mappings in cone metric spaces are reasonable.

4 Single valued results

Definition 4.1 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. We say that F has comparable combined monotone (CCM) property if for any $x, y \in X$,

$$x_1, x_2, y_1, y_2 \in X,\ x_1 \asymp x_2 \ \ and\ y_1 \asymp y_2 \Rightarrow \ F(x_1, y_1) \asymp F(x_2, y_2).$$

Definition 4.2 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. We say that F has combined monotone (CM) property if for any $x, y \in X$,

$$x_1, x_2, y_1, y_2 \in X,\ x_1 \leq x_2 \ \ and\ y_1 \geq y_2 \Rightarrow \ F(x_1, y_1) \leq F(x_2, y_2).$$

Clearly also for single valued mappings

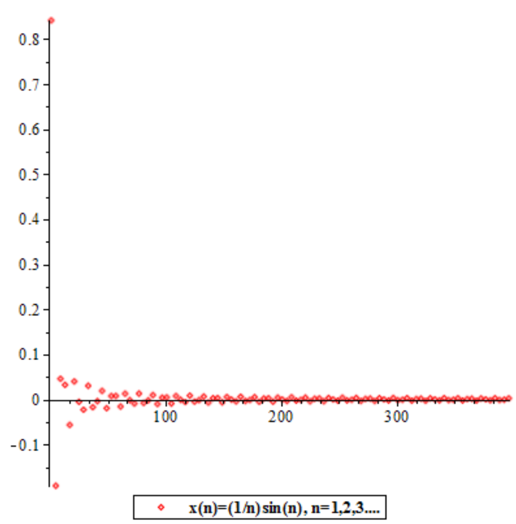
$$(MM) \Leftrightarrow (CM) \Rightarrow (CCM).$$

Remark 4.1 In the litrature many authors discuss the convergent sequences having compareable terms, we provide an example for a convergent sequence which is niether nonincreasing nor nondecreasing but have comparable terms. The sequence

$$x_n = \frac{1}{n}\sin(n),\ \ n = 1, 2, 3 \cdots$$

has comparable terms and converges to 0.

The following graph shows the comparable terms of the sequence for some values of n;



Corollary 4.1 Let (X,d) be a complete cone metric space endowed with a partial order $\leq$ on X. Let $F : X \times X \to X$ be a multivalued mapping having CCM- property on X. Assume that there exists a $k \in [0,1)$ such that

$$d(F(x,y),F(u,v)) \preccurlyeq \frac{k}{2}[d(x,u)+d(y,v)]$$

for all $x \asymp u$, $y \asymp v$. If there exists $x_0, y_0 \in X$, such that

$$x_0 \asymp F(x_0,y_0) \text{ and } F(y_0,x_0) \asymp y_0,$$

If X has limit comparison property then there exist $\bar{x}, \bar{y} \in X$, such that

$$\bar{x} = F(\bar{x},\bar{y}) \text{ and } \bar{y} = F(\bar{y},\bar{x}).$$

Proof: In theorem 3.1 take F as a single valued mapping as:

$$\frac{k}{2}[d(x,u)+d(y,v)] \in \sigma(F(x,y),F(u,v)] = \bigcap_{a=F(x,y),b=F(u,v)} s(d(a,b))$$

$$\frac{k}{2}[d(x,u)+d(y,v)] \in s(d(F(x,y),F(u,v))$$

which implies

$$d(F(x,y),F(u,v)) \preccurlyeq \frac{k}{2}[d(x,u)+d(y,v)].$$

Thus following the proof of theorem 3.1 we will get the result.

Corollary 4.2 Let (X,d) be a complete metric space endowed with a partial order $\leq$ on X. Let $F : X \times X \to X$ be a mapping having CCM- property on X. Assume that there exists a $k \in [0,1)$ such that

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)]$$

for all $x \asymp u$, $y \asymp v$. If there exists $x_0, y_0 \in X$, such that

$$x_0 \asymp F(x_0,y_0) \text{ and } F(y_0,x_0) \asymp y_0,$$

If X has limit comparison property then there exist $\bar{x}, \bar{y} \in X$, such that

$$\bar{x} = F(\bar{x},\bar{y}) \text{ and } \bar{y} = F(\bar{y},\bar{x}).$$

Corollary 4.3 [2] Let (X,d) be a complete metric space endowed with a partial order $\leq$ on X. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$$d\left(F\left(x,y\right),F\left(u,v\right)\right) \leq \frac{k}{2}[d(x,u)+d(y,v)],$$

for all $x \geq u$, $y \leq v$. If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0,y_0)$ and $y \geq F(y_0,x_0)$. Then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

5 Applications:

In the next theorem we provide some conditions for existence of solution of a certain type of a nonlinear integral equation.

Theorem: Consider the nonlinear integral equation of fredholm type:

$$u(x,y)=h(x,y)+\int_0^a\int_0^b K_1(x,y,\tau,s)+K_2(x,y,\tau,s)]\left(f(\tau,s,u(\tau,s))+g(\tau,s,v(\tau,s))\right)d\tau ds \qquad (*)$$

where $K_1,K_2\in C(I_a\times I_b\times I_a\times I_b,\mathbb{R})$ and $f,g\in C(I_a\times I_b\times\mathbb{R},\mathbb{R})$, assume that there exist $\lambda,\mu>0$ and $\theta\in[0,\frac{1}{2})$ such that $\sup\limits_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b K_1(x,y,\tau,s)d\tau ds\right|\leq\frac{\theta}{2(\lambda+\mu)}$ and $\sup\limits_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b K_2(x,y,\tau,s)d\tau ds\right|\leq\frac{\theta}{2(\lambda+\mu)}$. For $u_1\geq u_2$, for all $(x,y)\in I_a\times I_b$, f and g satisfy

$$0\leq f(x,y,u)-f(x,y,v)\leq\lambda(u-v)$$

and

$$-\mu(u-v)\leq g(x,y,u)-g(x,y,v)\leq 0;$$

If the coupled lower solution of $(*)$ exists, then there exists a unique solution of the integral equation $(*)$.

Proof: Let $X=C(I_a\times I_b,\mathbb{R})$, where $I_a=[0,a]$ and $I_b=[0,b]$, then X is a complete metric space with metric defined by;
for $w_1,w_2\in X$, $d(w_1,w_2)(x,y)=\sup\limits_{(x,y)\in I_a\times I_b}|w_1(x,y)-w_2(x,y)|$.

Define $F:X\times X\to X$ by $(F(u,v))(x,y)=\int_0^a\int_0^b K_1(x,y,\tau,s)\left(f(\tau,s,u(\tau,s))+g(\tau,s,v(\tau,s))\right)d\tau ds$

$+\int_0^a\int_0^b K_2(x,y,\tau,s)\left(f(\tau,s,v(\tau,s))+g(\tau,s,u(\tau,s))\right)d\tau ds+h(x,y)$ for all $(x,y)\in I_a\times I_b$ and $u,v\in X$. It is obvious that F satisfies CCM property. For $u,v\in X$ define a partial oeder $u\geq v$ iff $u(x,y)\geq v(x,y)$ for all $(x,y)\in I_a\times I_b$. Now for $u_1,u_2,v_1,v_2\in X$ with $u_1\geq u_2$ and $v_1\geq v_2$, consider

$$d\left(F(u_1,v_1),F(u_2,v_2)\right)=\sup_{(x,y)\in I_a\times I_b}|F(u_1,v_1)(x,y)-F(u_2,v_2)(x,y)|$$

$$
\begin{aligned}
= \sup_{(x,y)\in I_a\times I_b} \Bigg| & \int_0^a\int_0^b K_1(x,y,\tau,s)\left(f(\tau,s,u_1(\tau,s))+g(\tau,s,v_1(\tau,s))\right)d\tau ds+ \\
& \int_0^a\int_0^b K_2(x,y,\tau,s)\left(f(\tau,s,v_1(\tau,s))+g(\tau,s,u_1(\tau,s))\right)d\tau ds \\
& -\int_0^a\int_0^b K_1(x,y,\tau,s)\left(f(\tau,s,u_2(\tau,s))+g(\tau,s,v_2(\tau,s))\right)d\tau ds- \\
& \int_0^a\int_0^b K_2(x,y,\tau,s)\left(f(\tau,s,v_2(\tau,s))+g(\tau,s,u_2(\tau,s))\right)d\tau ds \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
= \sup_{(x,y)\in I_a\times I_b} \Bigg| & \int_0^a\int_0^b K_1(x,y,\tau,s)[f(\tau,s,u_1(\tau,s))-f(\tau,s,u_2(\tau,s))+ \\
& g(\tau,s,v_1(\tau,s))-g(\tau,s,v_2(\tau,s))]d\tau ds \\
& +\int_0^a\int_0^b K_2(x,y,\tau,s)[f(\tau,s,v_1(\tau,s))-f(\tau,s,v_2(\tau,s))+ \\
& g(\tau,s,u_1(\tau,s))-g(\tau,s,u_2(\tau,s))]d\tau ds \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq \sup_{(x,y)\in I_a\times I_b} \Bigg| & \int_0^a\int_0^b K_1(x,y,\tau,s)[f(\tau,s,u_1(\tau,s))-f(\tau,s,u_2(\tau,s)) \\
& +g(\tau,s,v_1(\tau,s))-g(\tau,s,v_2(\tau,s))]d\tau ds \Bigg| \\
+ \sup_{(x,y)\in I_a\times I_b} \Bigg| & \int_0^a\int_0^b K_2(x,y,\tau,s)[f(\tau,s,v_1(\tau,s))-f(\tau,s,v_2(\tau,s)) \\
& +g(\tau,s,u_1(\tau,s))-g(\tau,s,u_2(\tau,s))]d\tau ds \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \sup_{(x,y)\in I_a\times I_b} \left| \int_0^a\int_0^b K_1(x,y,\tau,s)[\lambda\{u_1(\tau,s)-u_2(\tau,s)\}+\mu\{v_1(\tau,s))-v_2(\tau,s)\}]d\tau ds \right| \\
+ & \sup_{(x,y)\in I_a\times I_b} \left| \int_0^a\int_0^b K_2(x,y,\tau,s)[\lambda\{v_1(\tau,s)-v_2(\tau,s)\}+\mu\{u_1(\tau,s))-u_2(\tau,s)\}]d\tau ds \right|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \sup_{(x,y)\in I_a\times I_b} \left| \int_0^a\int_0^b K_1(x,y,\tau,s)[\lambda d(u_1,u_2)+\mu d(v_1,v_2)]d\tau ds \right| + \\
& \sup_{(x,y)\in I_a\times I_b} \left| \int_0^a\int_0^b K_2(x,y,\tau,s)[\lambda d(v_1,v_2)+\mu(u_1,u_2)]d\tau ds \right|
\end{aligned}
$$

$$\leq \sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b K_1(x,y,\tau,s)d\tau ds\right|\cdot[\lambda d(u_1,u_2)+\mu d(v_1,v_2)]+$$

$$\sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b K_2(x,y,\tau,s)d\tau ds\right|\cdot[\lambda d(v_1,v_2)+\mu(u_1,u_2)]$$

$$\leq \frac{\theta}{2(\lambda+\mu)}\cdot[\lambda d(u_1,u_2)+\mu d(v_1,v_2)]+\frac{\theta}{2(\lambda+\mu)}\cdot[\lambda d(v_1,v_2)+\mu(u_1,u_2)]$$
$$=\frac{\theta}{2(\lambda+\mu)}[(\lambda+\mu)d(u_1,u_2)+(\lambda+\mu)d(v_1,v_2)]$$
$$=\frac{\theta}{2}[d(u_1,u_2)+d(v_1,v_2)]$$

Now let $\alpha(x,y)$ and $\beta(x,y)$ be the coupled upper-lower solutions of $(*)$. We have

$$\alpha(x,y)\leq F(\alpha(x,y),\beta(x,y))$$

and

$$\beta(x,y)\leq F(\beta(x,y),\alpha(x,y))$$

for all $(x,y)\in I_a\times I_b$. Thus all the hypothesis of the corollary 4.1 are satisfied, thus there exists a unique coupled solution (u,v) of $(*)$.

Example 4.1. Consider the nonlinear Fredholm integral equation

$$u(x,y)=\frac{1}{(1+x+y)^2}-\frac{x}{6(8+y)}+\int_0^1\int_0^1\frac{x}{(8+y)(1+t+s)}\cdot u^2(t,s)dtds$$

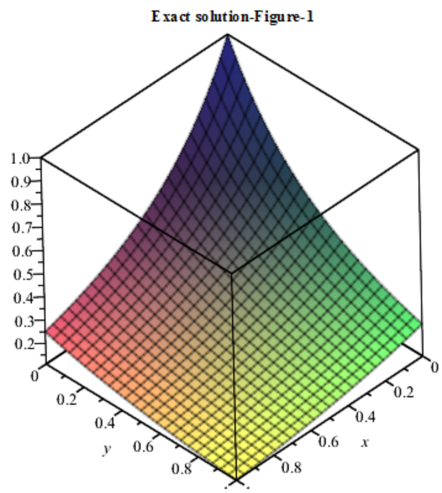
Here we have $h(x,y)=\frac{1}{(1+x+y)^2}-\frac{x}{6(8+y)}$, $K_1(x,y,\tau,s)=\frac{x}{(8+y)(1+\tau+s)}$ and $f(\tau,s,u(\tau,s)=u^2(\tau,s)$.

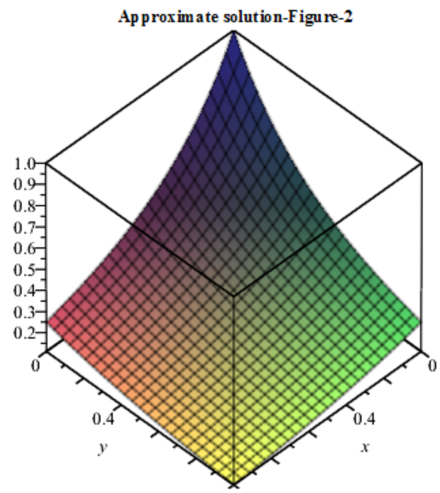
Taking intial $u_0(t,s)=\frac{1}{1+t+s}$, and using the iterative scheme;

$$u_{n+1}=\frac{1}{(1+x+y)^2}-\frac{x}{6(8+y)}+\int_0^1\int_0^1\frac{x}{(8+y)(1+t+s)}\cdot u_n^2\ dtds.$$

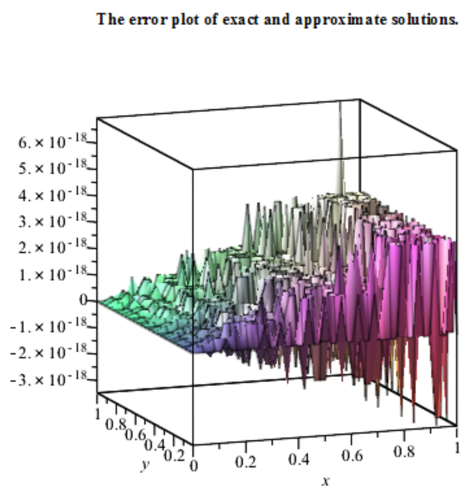
The exact solution is $u(x,y)=\frac{1}{(1+x+y)^2}$.

The figure-1 is the exact solution while the figure-2 shows the approximate solution given after 5th iteration.





The error plot of exact and approximate solution is given below.



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