Naimark-Sacker Bifurcation of Second Order Rational Difference Equation with Quadratic Terms

M. R. S. Kulenović*
Department of Mathematics,
University of Rhode Island,
Kingston, RI 02881,
USA

S. Moranjkić and Z. Nurkanović
Department of Mathematics,
University of Tuzla,
75350 Tuzla,
Bosnia and Herzegovina

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Abstract

We investigate the global asymptotic stability and Naimark-Sacker bifurcation of the difference equation

$$x_{n+1} = \frac{F}{bx_n x_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, \ldots$$

with non-negative parameters and nonnegative initial conditions x_{-1}, x_0 such that $bx_0 x_{-1} + cx_{-1}^2 + f > 0$. By using fixed point theorem for monotone maps we find the region of parameters where the unique equilibrium is globally asymptotically stable.

*Corresponding author, *e-mail: mkulenovic@mail.uri.edu*

1   Introduction and Preliminaries

In this paper we investigate the global dynamics of the following difference equation

$$x_{n+1} = \frac{F}{bx_n x_{n-1} + cx_{n-1}^2 + f} = H(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{1}$$

where

$$F, b, c, f \in (0, \infty),$$

and where the initial conditions x_{-1} and x_0 are arbitrary nonnegative real numbers. Equation (1) is the special case of a general second order quadratic fractional equation of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots \tag{2}$$

with non-negative parameters and initial conditions such that $A+B+C > 0$, $a+b+c+d+e+f > 0$ and $ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f > 0, n = 0, 1, \dots$. Several global asymptotic results for some special cases of Eq.(2) were obtained in [4, 5, 6, 15]. The systematic theory of the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots \tag{3}$$

with non-negative parameters and initial conditions such that $D + E + F > 0, d + e + f > 0$ and $dx_n + ex_{n-1} + f > 0, n = 0, 1, \dots$ was presented in [7] where it was shown that Eq.(3) does not exhibit Naimark-Sacker bifurcation and can only exhibit either conservative chaos or period doubling bifurcation, see [7, 10, 11]. In the case of quadratic fractional difference equation (2) we showed that Naimark-Sacker bifurcation is very common, see [12]. First systematic study of global dynamics of a special case of Eq.(2) where $A = C = D = a = c = d = 0$ was performed in [1, 2].

The global attractivity result in [7, 9], which is the fixed point theorem for monotone maps that will be used here is the following result.

Theorem 1 *Assume that the difference equation*

$$x_{n+1} = G(x_n, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{4}$$

where G is nondecreasing functions in all its arguments has the unique equilibrium $\overline{x} \in I$, where I is an invariant interval, that is $G : I^{k+1} \to I$. Then $\overline{x}$ is globally asymptotically stable.

In this paper we perform the local stability analysis of the unique equilibrium and give the necessary and sufficient conditions for the equilibrium to be locally asymptotically stable, a repeller or a non-hyperbolic equilibrium. The local stability analysis indicates that some possible dynamic scenarios for Eq.(1) include Naimark-Sacker bifurcation. We apply Theorem 1 in the part of the region of local asymptotic stability to obtain global asymptotic stability result. In the complement of the parametric region where the equilibrium is locally stable the equilibrium becomes repeller with two characteristic values to be complex conjugate numbers and on the boundary of this region two characteristic values are complex conjugate numbers on the unit circle. We show that in this case Eq.(1) exhibits Naimark-Sacker bifurcation resulting in the existence of locally stable periodic solution of unknown period.

2 Linearized Stability Analysis

In this section we present the local stability of the unique positive equilibrium of Eq.(1).

In view of the above restriction on the initial conditions of Eq.(1), the equilibrium points of Eq.(1) are the positive solutions of the equation

$$\overline{x} = \frac{F}{(b+c)\,\overline{x}^2 + f}, \tag{5}$$

or equivalently

$$(b+c)\overline{x}^3 + f\overline{x} - F = 0. \tag{6}$$

By Descartes rule of sign Eq.(6) has the unique positive solution $\overline{x}$ given as

$$\overline{x} = \sqrt[3]{\frac{1}{2(b+c)}\left(F + \sqrt{F^2 + \frac{4f^3}{27(b+c)}}\right)} - \frac{f}{3(b+c)\sqrt[3]{\frac{1}{2(b+c)}\left(F+\sqrt{F^2+\frac{4f^3}{27(b+c)}}\right)}}. \tag{7}$$

Now we investigate the stability of the positive equilibrium of Eq.(1). Set

$$H(u,v) = \frac{F}{buv + cv^2 + f}$$

and observe that

$$H'_u(u,v) = \frac{-bFv}{(buv+cv^2+f)^2}, \quad H'_v(u,v) = \frac{-F\,(bu+2cv)}{(buv+cv^2+f)^2}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(1), then the linearized equation associated with Eq.(1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = sz_n + tz_{n-1}$$

where

$$s = H'_u(\overline{x}, \overline{x}) \quad \text{and} \quad t = H'_v(\overline{x}, \overline{x}).$$

Theorem 2 *The unique equilibrium point $\overline{x}$ of Eq.(1) given by (7) is:*

i) locally asymptotically stable if $f^3 > \dfrac{c^3 F^2}{(b+2c)^2}$;

ii) a repeller if $f^3 < \dfrac{c^3 F^2}{(b+2c)^2}$;

iii) a non-hyperbolic point of elliptic type if $f^3 = \dfrac{c^3 F^2}{(b+2c)^2}.$

Proof. A straightforward calculation yields

$$s = H'_u(\overline{x}, \overline{x}) = \frac{-bF\overline{x}}{\left((b+c)\,\overline{x}^2 + f\right)^2} = \frac{-b\overline{x}^3}{F} < 0$$

and

$$t = H'_v(\overline{x}, \overline{x}) = \frac{-F\,(b+2c)\,\overline{x}}{\left((b+c)\,\overline{x}^2 + f\right)^2} = \frac{-\,(b+2c)\,\overline{x}^3}{F} < 0.$$

Thus $s - t > 0$ and

$$s^2 - (1-t)^2 = (s+t-1)\,(s-t+1) < 0,$$

i.e.

$$|s| < |1-t|\,.$$

The unique equilibrium point $\overline{x}$ is a non-hyperbolic point of elliptic type for $t = -1$. Eigenvalues are given by

$$\lambda_{1,2} = \frac{-b \pm i\sqrt{4\,(b+2c)^2 - b^2}}{2\,(b+2c)}, \tag{8}$$

and so $|\lambda_{1,2}| = 1$. Hence, for

$$t = -1 \Leftrightarrow \frac{-\,(b+2c)\,\overline{x}^3}{F} = -1 \Leftrightarrow \overline{x} = \sqrt[3]{\frac{F}{b+2c}}.$$

Thus

$$(b+c)\,\frac{F}{b+2c} + f\sqrt[3]{\frac{F}{b+2c}} - F = 0 \Leftrightarrow f^3 = \frac{c^3 F^2}{(b+2c)^2}$$

so

$$\overline{x} = \sqrt[3]{\frac{F}{b+2c}} = \sqrt{\frac{f}{c}} \quad \text{and} \quad f = c\left(\sqrt[3]{\frac{F}{b+2c}}\right)^2.$$

Then

$$s = \frac{-b\overline{x}^3}{F} = \frac{-b}{F}\frac{F}{b+2c} = -\frac{b}{b+2c}$$

and

$$\lambda^2 + \frac{b}{b+2c}\lambda + 1 = 0,$$

which solutions are given with (8). If $f^3 > \dfrac{c^3 F^2}{(b+2c)^2}$ the equilibrium point $\overline{x}$ is locally asymptotically stable $(t \in (-1,0))$, and if $f^3 < \dfrac{c^3 F^2}{(b+2c)^2}$ the equilibrium point $\overline{x}$ is a repeller $(t < -1)$. Finally, if $f^3 = \dfrac{c^3 F^2}{(b+2c)^2}$, then $|\lambda_{1,2}| = 1$. $\qquad\square$

3   Global Asymptotic Stability

In this section we give global asymptotic stability result for Eq.(1). We show that the unique equilibrium point is globally asymptotically stable in the subregion of the parametric region of local asymptotic stability.

Theorem 3 *The unique equilibrium point $\overline{x}$ of Eq.(1) is globally asymptotically stable if the following condition holds*

$$f^3 > \frac{1}{4}(b+c)F^2. \tag{9}$$

Proof. Every solution of Eq.(1) satisfies the fourth order difference equation

$$x_{n+1} = H(x_n, x_{n-1}) = H(H(x_{n-1}, x_{n-2}), H(x_{n-2}, x_{n-3})) = H_1(x_{n-1}, x_{n-2}, x_{n-3}), \quad n = 0, \ldots \tag{10}$$

where H_1 is increasing function in all its arguments. Simplifying the right hand side of Eq.(10) we obtain

$$x_{n+1} = \frac{FD(x_{n-1}, x_{n-2})D(x_{n-2}, x_{n-3})^2}{bF^2 D(x_{n-2}, x_{n-3}) + cF^2 D(x_{n-1}, x_{n-2}) + fD(x_{n-1}, x_{n-2})D(x_{n-2}, x_{n-3})^2}, \tag{11}$$

where

$$D(u, v) = buv + v^2 + f.$$

The equilibrium solution of Eq.(11) satisfies the equation

$$\left((b+c)F^2 + f((b+c)x^2 + f)^2\right)x - F((b+c)x^2 + f)^2 = 0. \tag{12}$$

Since the left hand side of Eq.(12) can be factored as

$$\left((b+c)x^3 + fx - F\right)\left((b+c)fx^2 - (b+c)Fx + f^2\right)$$

we conclude that the equilibrium solutions of Eq.(11) are either equilibrium solutions of Eq.(1) or the solutions of the quadratic equation

$$(b+c)fx^2 - (b+c)Fx + f^2 = 0. \tag{13}$$

Equation (13) has no real solutions under the condition (9). Now in view of the fact that $[0, \frac{F}{f}]$ is an invariant interval for H and so for H_1, an application of Theorem 1 completes the proof. $\square$

Remark 1 By Theorem 2 the equilibrium point $\overline{x}$ is locally asymptotically stable if

$$f^3 > \frac{c^3 F^2}{(b+2c)^2}, \tag{14}$$

and by Theorem 3 the equilibrium point $\overline{x}$ is globally asymptotically stable if the condition (9) holds. It can be shown that condition (9) implies (14), that is global asymptotic stability implies the local. We conjecture that the converse is true.

Conjecture 1 *The equilibrium point $\overline{x}$ of Eq.(1) is globally asymptotically stable if it is locally asymptotically stable.*

4 Naimark-Sacker Bifurcation for Eq.(1)

In this section we consider bifurcation of a fixed point of a map associated to Eq.(1) in the case where the eigenvalues are complex conjugate numbers on the unit circle.

The Naimark-Sacker bifurcation occurs for a discrete system depending on a parameter, λ, with a fixed point whose Jacobian matrix has a pair of complex conjugate eigenvalues $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which cross the unit circle transversally at $\lambda = \lambda_0$

For the sake of completeness we include the Naimark-Sacker bifurcation theorem of the interior fixed point. See [3, 8, 13, 14, 16] for detailed description and especially [13] for the detailed proof of the result.

Theorem 4 *(Poincare-Andronov-Hopf bifurcation or Naimark-Sacker bifurcation for Maps) Let*

$$F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x)$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, 0) = 0$ for λ near some fixed λ_0;

(ii) $DF(\lambda, 0)$ has two non-real eigenvalues $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) \neq 0$ at $\lambda = \lambda_0$;

(iv) $\mu^k(\lambda_0) \neq 1$ for $k = 1, 2, 3, 4$.

Then there is a smooth λ-dependent change of coordinate bringing f into the form

$$F(\lambda, x) = \mathcal{F}(\lambda, x) + O(||x||^5)$$

and there are smooth function $a(\lambda), b(\lambda)$ and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r - a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{15}$$

If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.
If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $F(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\overline{\mu(\lambda)} = \alpha(\lambda) - i\beta(\lambda)$ satisfying $(\alpha(\lambda))^2 + (\beta(\lambda))^2 = 1$ and $\beta(\lambda) \neq 0$. By transforming the linear part of such a map into Jordan normal form, we may assume F to have the following form near the origin

$$F(\lambda, x) = \begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(\lambda, x_1, x_2) \\ g_2(\lambda, x_1, x_2) \end{pmatrix}. \tag{16}$$

Then the coefficient $a(\lambda_0)$ of the cubic term in Eq.(15) in polar coordinates is equal to

$$a\left(\lambda_0\right) = Re\left(\frac{\left(1 - 2\mu\left(\lambda_0\right)\right)\overline{\mu\left(\lambda_0\right)}^2}{1 - \mu\left(\lambda_0\right)}\xi_{11}\xi_{20}\right) + \frac{1}{2}\left|\xi_{11}\right|^2 + \left|\xi_{02}\right|^2 - Re\left(\overline{\mu\left(\lambda_0\right)}\xi_{21}\right), \qquad (17)$$

where

$$\xi_{20} = \frac{1}{8}\left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} - \frac{\partial^2 g_1(0,0)}{\partial x_2^2} + 2\frac{\partial^2 g_2(0,0)}{\partial x_1 \partial x_2} + i\left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} - \frac{\partial^2 g_2(0,0)}{\partial x_2^2} - 2\frac{\partial^2 g_1(0,0)}{\partial x_1 \partial x_2}\right)\right), \qquad (18)$$

$$\xi_{11} = \frac{1}{4}\left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} + \frac{\partial^2 g_1(0,0)}{\partial x_2^2} + i\left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} + \frac{\partial^2 g_2(0,0)}{\partial x_2^2}\right)\right), \qquad (19)$$

$$\xi_{02} = \frac{1}{8}\left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} - \frac{\partial^2 g_1(0,0)}{\partial x_2^2} - 2\frac{\partial^2 g_2(0,0)}{\partial x_1 \partial x_2} + i\left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} - \frac{\partial^2 g_2(0,0)}{\partial x_2^2} + 2\frac{\partial^2 g_1(0,0)}{\partial x_1 \partial x_2}\right)\right), \qquad (20)$$

and

$$\xi_{21} = \frac{1}{16}\left(\frac{\partial^3 g_1}{\partial x_1^3} + \frac{\partial^3 g_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 g_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 g_2}{\partial x_2^3} + i\left(\frac{\partial^3 g_2}{\partial x_1^3} + \frac{\partial^3 g_2}{\partial x_1 \partial x_2^2} - \frac{\partial^3 g_1}{\partial x_1^2 \partial x_2} - \frac{\partial^3 g_1}{\partial x_2^3}\right)\right). \qquad (21)$$
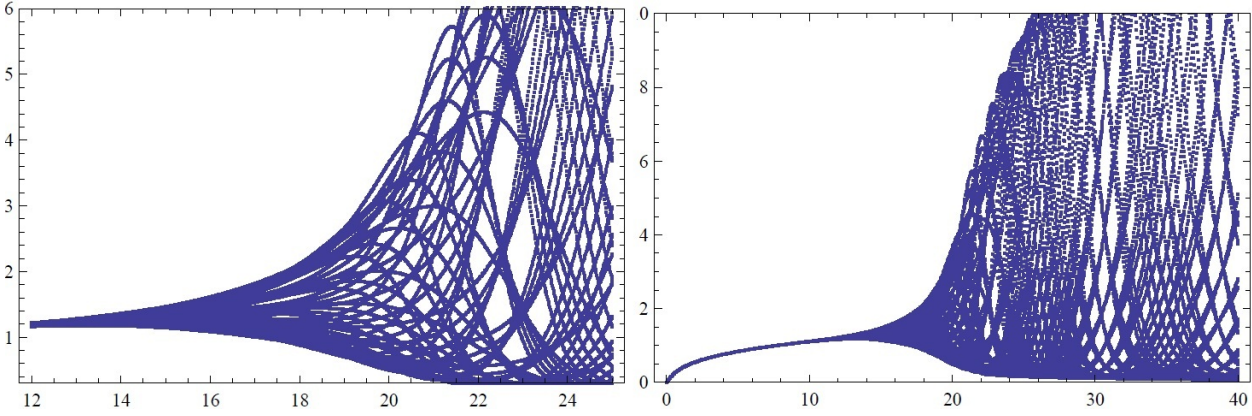

Figure 1: Bifurcation diagrams in (F, x) plane for $b = 4.8, c = 1, f = 1.9$, generated by *Dynamica 3* [8].

Theorem 5 *Assume that $b, c, f > 0$ and*

$$F_0 = \frac{f\sqrt{f}\,(b + 2c)}{c\sqrt{c}} \quad and \quad \overline{x} = \frac{\sqrt{f}}{\sqrt{c}}.$$

i) *If $0 < b < \left(\frac{2}{3} + 4\cos\frac{1}{3}\left(\arctan\frac{1}{53}\sqrt{107}\right)\right)c$, then there is a neighborhood U of the equilibrium point $\overline{x}$ and a $\rho > 0$ such that for $|F - F_0| < \rho$ and $x_0, x_{-1} \in U$, then ω-limit set of solution of Eq(1), with initial condition x_0, x_{-1} is the equilibrium point $\overline{x}$ if $F < F_0$ and belongs to a closed invariant C^1 curve $\Gamma(F_0)$ encircling the equilibrium point $\overline{x}$ if $F > F_0$. Furthermore, $\Gamma(F_0) = 0$.*

ii) *If $b > \left(\frac{2}{3} + 4\cos\frac{1}{3}\left(\arctan\frac{1}{53}\sqrt{107}\right)\right)c$, then there is a neighborhood U of the equilibrium point $\overline{x}$ and a $\rho > 0$ such that for $|F - F_0| < \rho$ and $x_0, x_{-1} \in U$, then α-limit set of x_0, x_{-1} is the equilibrium point $\overline{x}$ if $F > F_0$ and belongs to a closed invariant C^1 curve $\Gamma(F_0)$ encircling the equilibrium point $\overline{x}$ if $F < F_0$. Furthermore, $\Gamma(F_0) = 0$.*
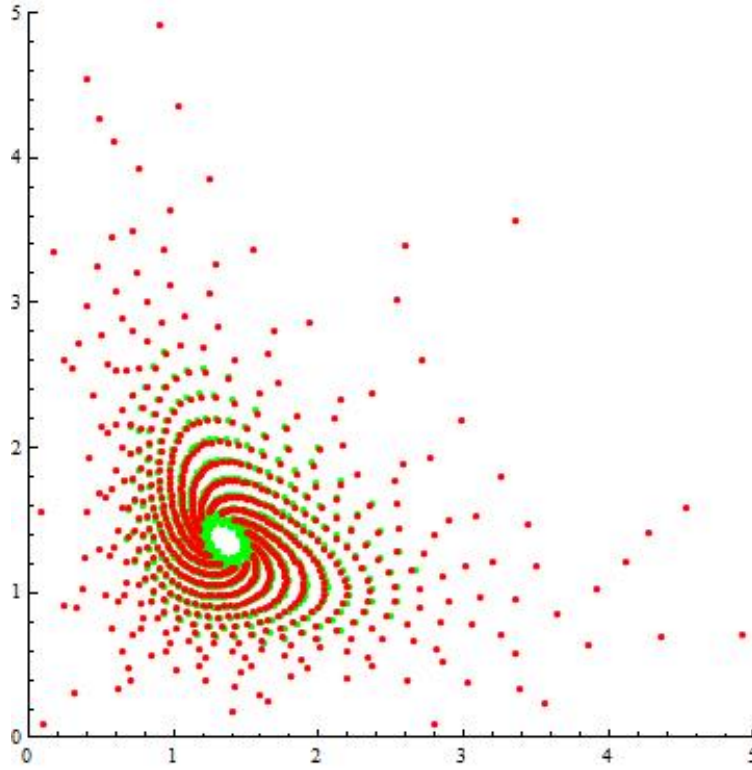
Figure 2: Phase portraits when $F = 17 < F_0(c = 1, f = 1.9, b = 4.8 > 2/3 + \cos((1/3)arctan\sqrt{107}/53)$, $x_{-1} = x_0 = 1.4$ (green), $x_{-1} = x_0 = 3.1$ (red), generated by *Dynamica* 3 [8].

Proof. In order to apply Theorem (4) we make a change of variable $y_n = x_n - \overline{x}$. Then, the new equation is given by

$$y_{n+1} = \frac{F}{b\left(y_n + \overline{x}\right)\left(y_{n-1} + \overline{x}\right) + c\left(y_{n-1} + \overline{x}\right)^2 + f} - \overline{x}.$$

Set

$$u_n = y_{n-1} \quad \text{and} \quad v_n = y_n \quad \text{for } n = 0, 1, \dots$$

and write

$$\left.\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = \frac{F}{b(v_n+\overline{x})(u_n+\overline{x})+c(u_n+\overline{x})^2+f} - \overline{x}, \end{array}\right\} \tag{22}$$

where

$$F\left(u, v\right) = \left(\begin{array}{c} v \\ \frac{F}{b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2+f} - \overline{x} \end{array} \right).$$

Then $F\left(u, v\right)$ has the unique fixed point $(0, 0)$. The Jacobian matrix of $F\left(u, v\right)$ is given by

$$J_F\left(u, v\right) = \left(\begin{array}{cc} 0 & 1 \\ \frac{-F(b(v+\overline{x})+2c(u+\overline{x}))}{\left(b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2+f\right)^2} & \frac{-Fb(u+\overline{x})}{\left(b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2+f\right)^2} \end{array} \right),$$
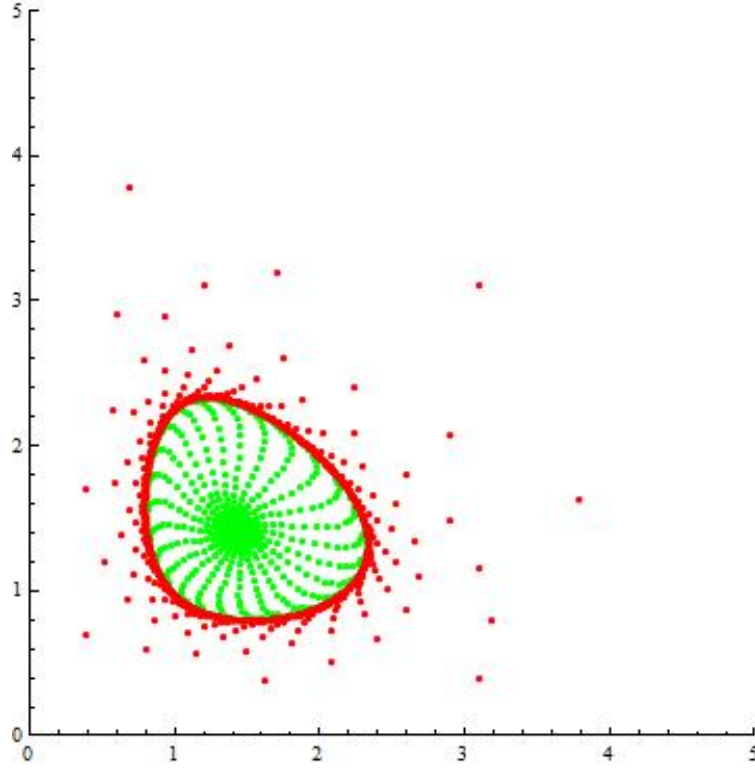
Figure 3: Phase portraits when $F = 19.4 > F_0(c = 1, f = 1.9, b = 4.8 > 2/3 + \cos((1/3)arctan\sqrt{107}/53)$, $x_{-1} = x_0 = 1.4$ (green) $x_{-1} = x_0 = 3.1$ (red), generated by *Dynamica* 3 [8].

and its value at the zero equilibrium is

$$J_0 = J_F(0,0) = \begin{pmatrix} 0 & 1 \\ \frac{-F(b+2c)\overline{x}}{\left(b\overline{x}^2+c\overline{x}^2+f\right)^2} & \frac{-Fb\overline{x}}{\left(b\overline{x}^2+c\overline{x}^2+f\right)^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-(b+2c)}{F}\overline{x}^3 & \frac{-b}{F}\overline{x}^3 \end{pmatrix}.$$

The eigenvalues are $\mu(F)$ and $\overline{\mu(F)}$ where

$$\mu(F) = \frac{-b\overline{x}^3 \pm i\sqrt{\overline{x}^3\left(4F(b+2c) - b^2\overline{x}^3\right)}}{2F},$$

and

$$4F(b+2c) - b^2\overline{x}^3 = 4\left(((b+c)\overline{x}^3 + f\overline{x})(b+2c) - b^2\overline{x}^3 > 0.$$

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-(b+2c)}{F}\overline{x}^3 & \frac{-b}{F}\overline{x}^3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(F,u,v) \\ f_2(F,u,v) \end{pmatrix},$$

and

$$f_1\left(F,u,v\right) = 0,$$

$$f_2\left(F,u,v\right) = \frac{F}{b\left(v+\overline{x}\right)\left(u+\overline{x}\right)+c\left(u+\overline{x}\right)^2+f}+\frac{\left(b+2c\right)u}{F}\overline{x}^3+\frac{bv}{F}\overline{x}^3-\overline{x}.$$

Let

$$F_0 = \frac{f\sqrt{f}\left(b+2c\right)}{c\sqrt{c}}.$$

For $F = F_0$ we obtain

$$\overline{x}=\frac{\sqrt{f}}{\sqrt{c}},\quad \overline{x}^3=\frac{F_0}{b+2c}.$$

The eigenvalues of J_0 are $\mu\left(F_0\right)$ and $\overline{\mu\left(F_0\right)}$ where

$$\mu\left(F_0\right)=\frac{-b+i\sqrt{\left(3b+4c\right)\left(b+4c\right)}}{2\left(b+2c\right)},\quad \left|\mu\left(F_0\right)\right|=1.$$

The eigenvectors coresponding to $\mu\left(F_0\right)$ and $\overline{\mu\left(F_0\right)}$ are $v\left(F_0\right)$ and $\overline{v\left(F_0\right)}$ where

$$v\left(F_0\right)=\left(\frac{-b-i\sqrt{\left(3b+4c\right)\left(b+4c\right)}}{2\left(b+2c\right)},1\right).$$

Then

$$\left|\mu\left(F_0\right)\right| = 1,$$

$$\mu^2\left(F_0\right) = -\frac{8bc+b^2+8c^2}{2\left(b+2c\right)^2}-ib\frac{\sqrt{\left(3b+4c\right)\left(b+4c\right)}}{2\left(b+2c\right)^2},$$

$$\mu^3\left(F_0\right) = b\frac{6bc+b^2+6c^2}{\left(b+2c\right)^3}-i\frac{2c\left(b+c\right)\sqrt{\left(3b+4c\right)\left(b+4c\right)}}{\left(b+2c\right)^3},$$

$$\mu^4\left(F_0\right) = \frac{64bc^3+32b^2c^2-b^4+32c^4}{2\left(b+2c\right)^4}+i\frac{\left(8bc+b^2+8c^2\right)b\sqrt{\left(3b+4c\right)\left(b+4c\right)}}{2\left(b+2c\right)^4}$$

and $\mu^k\left(F_0\right)\neq 1$ for $k=1,2,3,4$.
For $F=F_0$ and $\overline{x}=\frac{\sqrt{f}}{\sqrt{c}}$ we get

$$F\left(\begin{array}{c}u\\v\end{array}\right)=\left(\begin{array}{cc}0&1\\-1&\frac{-b}{b+2c}\end{array}\right)\left(\begin{array}{c}u\\v\end{array}\right)+\left(\begin{array}{c}h_1\left(u,v\right)\\h_2\left(u,v\right)\end{array}\right)$$

and

$$h_1\left(u,v\right) = f_1\left(F_0,u,v\right)=0,$$

$$h_2\left(u,v\right) = f_2\left(F_0,u,v\right)$$

$$= \frac{\sqrt{c}\left(2c^{\frac{5}{2}}u^3+b^2\sqrt{f}u^2+b^2\sqrt{f}v^2+2c^2\sqrt{f}u^2+bc^{\frac{3}{2}}u^3+b^2\sqrt{c}uv^2+b^2\sqrt{c}u^2v+3bc\sqrt{f}u^2+3bc^{\frac{3}{2}}u^2v+b^2\sqrt{f}uv+2bc\sqrt{f}uv\right)}{\left(b+2c\right)\left(c^2u^2+bf+2cf+2c^{\frac{3}{2}}\sqrt{f}u+b\sqrt{c}\sqrt{f}u+b\sqrt{c}\sqrt{f}v+bcuv\right)}.$$

Hence, for $F = F_0$ system (22) takes the form

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{-b}{b+2c} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_1(u_n, v_n) \\ h_2(u_n, v_n) \end{pmatrix}. \tag{23}$$

For

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$P = \begin{pmatrix} \frac{-b}{2(b+2c)} & \frac{\sqrt{(3b+4c)(b+4c)}}{2(b+2c)} \\ 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{2(b+2c)}{\sqrt{(3b+4c)(b+4c)}} & \frac{b}{\sqrt{(3b+4c)(b+4c)}} \end{pmatrix},$$

system (22) is equivalent to its normal form

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{b}{2(b+2c)} & -\frac{\sqrt{(3b+4c)(b+4c)}}{2(b+2c)} \\ \frac{\sqrt{(3b+4c)(b+4c)}}{2(b+2c)} & -\frac{b}{2(b+2c)} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + P^{-1}H\left(P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}\right)$$

where

$$H\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} h_1(u, v) \\ h_2(u, v) \end{pmatrix}.$$

Let

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix} = P^{-1}H\left(P \begin{pmatrix} u \\ v \end{pmatrix}\right).$$

By straightforward calculation we obtain that

$$g_1(u, v) = \frac{\sqrt{c}}{b+2c}\Lambda(u, v),$$

with

$$\Lambda(u, v) = \frac{(b+2c)c\sqrt{c}(h(u,v))^3 + \left((b+c)(b+2c)\sqrt{f} + (b+3c)bu\sqrt{c}\right)(h(u,v))^2 + bu\left(bu\sqrt{c} + (b+2c)\sqrt{f}\right)h(u,v) + b^2u^2\sqrt{f}}{c^2(h(u,v))^2 + \left(bcu + (b+2c)\sqrt{cf}\right)h(u,v) + bu\sqrt{cf} + (b+2c)f}$$

where

$$h(u, v) = \frac{v\sqrt{(3b+4c)(b+4c)} - bu}{2(b+2c)}$$

and

$$g_2(u, v) = \frac{b}{\sqrt{(3b+4c)(b+4c)}}g_1(u, v).$$

Another straightforward calculation gives

$$\frac{\partial^2 g_1(0,0)}{\partial u^2} = \frac{b^2\sqrt{c}(3b+5c)}{2\sqrt{f}(b+2c)^3},$$

$$\frac{\partial^2 g_1(0,0)}{\partial u \partial v} = \frac{bc^{\frac{3}{2}}\sqrt{(3b+4c)(b+4c)}}{2\sqrt{f}(b+2c)^3},$$

$$\frac{\partial^2 g_1(0,0)}{\partial v^2} = \frac{\sqrt{c}\,(b+4c)\,(3b+4c)\,(b+c)}{2\sqrt{f}\,(b+2c)^3},$$

$$\frac{\partial^3 g_1(0,0)}{\partial u^3} = -\frac{3b^3 c\,(5b+8c)}{4f\,(b+2c)^4},$$

$$\frac{\partial^3 g_1(0,0)}{\partial u \partial v^2} = \frac{b^2 c\,(3b+4c)\,(b+4c)}{4f\,(b+2c)^4},$$

$$\frac{\partial^3 g_1(0,0)}{\partial u^2 \partial v} = -\frac{b^2 c\,(3b+8c)\,\sqrt{16bc+3b^2+16c^2}}{4f\,(b+2c)^4},$$

$$\frac{\partial^3 g_1(0,0)}{\partial v^3} = -\frac{3bc\left(\sqrt{16bc+3b^2+16c^2}\right)^3}{4f\,(b+2c)^4}.$$

By using (17)-(21) for $\lambda_0 = F_0, x_1 = u$ and $x_2 = v$ we obtain

$$\xi_{11} = \frac{\sqrt{c}\,(3b^2+6bc+4c^2)}{4\sqrt{f}\,(b+2c)^2}\left(1 + i\frac{b}{\sqrt{(3b+4c)(b+4c)}}\right),$$

$$\xi_{20} = \frac{1}{8}\left(\frac{-2\left(\sqrt{c}\right)^3(3b+2c)}{(b+2c)^2\sqrt{f}} + i\left(\frac{-b\left(\sqrt{c}\right)^3(32bc+10b^2+24c^2)}{\sqrt{16bc+3b^2+16c^2}\sqrt{f}\,(b+2c)^3}\right)\right),$$

$$\xi_{20} = -\frac{c^{\frac{3}{2}}\left((3b+2c)\sqrt{3b^2+16bc+16c^2} + ib(6c+5b)\right)}{4\sqrt{f}\,(b+2c)^2\sqrt{3b^2+16bc+16c^2}},$$

$$\xi_{02} = \frac{-c^{\frac{3}{2}}}{\sqrt{f}\,(b+2c)^3}\left((b+c)^2 + i\left(\frac{b(b^2-2c^2)}{2\sqrt{3b^2+16bc+16c^2}}\right)\right),$$

$$\xi_{21} = \frac{bc}{8f\,(b+2c)^4}\left(-b(8bc+3b^2+4c^2) + i\left(\frac{(b+2c)(80bc^2+38b^2c+3b^3+48c^3)}{\sqrt{3b^2+16bc+16c^2}}\right)\right),$$

$$\frac{(1-2\mu(F_0))\,\overline{\mu(F_0)}^2}{1-\mu(F_0)} = \frac{-(3b+4c)(10bc+b^2+12c^2) + i\left(10bc+5b^2+4c^2\right)\sqrt{3b^2+16bc+16c^2}}{2(b+2c)^2(3b+4c)},$$

$$\xi_{11}\xi_{20} = \frac{-\left(6bc+3b^2+4c^2\right)}{2f\,(b+2c)^4}\left(\frac{c^2\left(20bc^2+12b^2c+b^3+8c^3\right)}{2(3b+4c)(b+4c)} + i\frac{bc^2(b+c)}{\sqrt{16bc+3b^2+16c^2}}\right),$$

$$Re\left(\frac{(1-2\mu(F_0))\,\overline{\mu(F_0)}^2}{1-\mu(F_0)}\xi_{11}\xi_{20}\right) = \frac{c^2\left(6bc+3b^2+4c^2\right)\left(26bc+11b^2+12c^2\right)}{8f\,(b+2c)^3(b+4c)(3b+4c)},$$

$$\frac{1}{2}|\xi_{11}|^2 = \frac{c\left(3b^2+6bc+4c^2\right)^2}{8f\,(b+2c)^2(3b+4c)(b+4c)},$$

$$\xi_{02} = \frac{-c^{\frac{3}{2}}}{\sqrt{f}\,(b+2c)^3}\left((b+c)^2 + i\left(\frac{b(b^2-2c^2)}{2\sqrt{3b^2+16bc+16c^2}}\right)\right),$$

$$|\xi_{02}|^2 = \frac{c^3\left(28bc^2 + 34b^2c + 13b^3 + 8c^3\right)}{4f\left(b+2c\right)^3\left(3b+4c\right)\left(b+4c\right)},$$

$$Re\left(\overline{\mu\left(F_0\right)}\xi_{21}\right) = \frac{bc}{8f(b+2c)^4}\left(\frac{b^2\left(8bc+3b^2+4c^2\right)}{2(b+2c)} + \frac{\left(80bc^2+38b^2c+3b^3+48c^3\right)}{2}\right) = \frac{bc\left(14bc+3b^2+12c^2\right)}{8f(b+2c)^3},$$

$$\xi_{21} = \frac{bc}{8f(b+2c)^4}\left(-b\left(8bc+3b^2+4c^2\right) + i\left(\frac{(b+2c)\left(80bc^2+38b^2c+3b^3+48c^3\right)}{\sqrt{3b^2+16bc+16c^2}}\right)\right),$$

$$a\left(F_0\right) = \frac{c^2\left(-3b^3 + 6b^2c + 32bc^2 + 24c^3\right)}{8f\left(b+2c\right)^3\left(3b+4c\right)}.$$

So,

$$a\left(F_0\right) = 0 \Leftrightarrow -3b^3 + 6b^2c + 32bc^2 + 24c^3 = 0.$$

By substituting $b = kc$ we obtain

$$-3\left(kc\right)^3 + 6\left(kc\right)^2 c + 32\left(kc\right)c^2 + 24c^3 = 0$$

i.e.

$$k^3 - 2k^2 - \frac{32}{3}k - 8 = 0.$$

By using Cardano's substitution $k = y + \frac{2}{3}$, we obtain

$$y^3 - 12y - \frac{424}{27} = 0,$$

with the corresponding positive solution

$$y = \frac{1}{3}\sqrt[3]{212 + 4i\sqrt{107}} + \frac{1}{3}\sqrt[3]{212 - 4i\sqrt{107}}.$$

The corresponding angle and modulus are

$$\tan\alpha = \frac{\sqrt{107}}{53},$$

$$\alpha = \arctan\frac{1}{53}\sqrt{107},$$

$$r = \sqrt{\left(212\right)^2 + \left(16\right)\left(107\right)} = 216,$$

for which we obtain

$$y = \frac{1}{3}\left(6\left(\cos\frac{\alpha}{3} + i\sin\frac{\alpha}{3}\right)\right) + \frac{1}{3}\left(6\left(\cos\frac{\alpha}{3} - i\sin\frac{\alpha}{3}\right)\right) = 4\cos\frac{1}{3}\alpha$$

and

$$k = y + \frac{2}{3} = \frac{2}{3} + 4\cos\frac{1}{3}\alpha.$$

Hence,

$$a(F_0) = 0 \quad \text{for} \quad b = \left(\frac{2}{3} + 4\cos\frac{1}{3}\arctan\frac{\sqrt{107}}{53}\right)c.$$

Now

$$\mu\left(F\right) = \frac{-b\overline{x}^3 \pm i\sqrt{\overline{x}^3\left(4F\left(b+2c\right) - b^2\overline{x}^3\right)}}{2F},$$

and so

$$\mu\left(F\right)\overline{\mu\left(F\right)} = \frac{\left(b+2c\right)x^3}{F}.$$

Thus

$$\left|\mu\left(F\right)\right| = \sqrt{\frac{\left(b+2c\right)x^3}{F}}.$$

Differentiating the equilibrium equation

$$\left(b+c\right)x^3 + fx - F = 0,$$

with respect to F and solving for $x'(F)$ we obtain

$$\begin{aligned}
x'\left(F\right) &= \frac{1}{3\left(b+c\right)\left(x\left(F\right)\right)^2 + f}, \quad x\left(F_0\right) = \sqrt{\frac{f}{c}}, \\
x'\left(F_0\right) &= \frac{1}{3\left(b+c\right)\frac{f}{c} + f} = \frac{c}{f\left(3b+4c\right)}.
\end{aligned}$$

By substituting $x'(F)$ in the expression

$$\frac{d\left|\mu\left(F\right)\right|}{dF} = \frac{1}{2\sqrt{\frac{\left(b+2c\right)x^3}{F}}}\left(\frac{3\left(b+2c\right)x^2 x'}{F}\right), x = x\left(F\right)$$

we obtain that

$$\frac{d\left|\mu\left(F\right)\right|}{dF}\left(F_0\right) = \frac{1}{2\sqrt{\frac{\left(b+2c\right)}{F_0} \cdot \frac{F_0}{\left(b+2c\right)}}}\left(\frac{3\left(b+2c\right)\left(\sqrt{\frac{f}{c}}\right)^2 \frac{c}{f\left(3b+4c\right)}}{F_0}\right)$$

and

$$\frac{d\left|\mu\left(F\right)\right|}{dF}\left(F_0\right) = \frac{3\left(b+2c\right)}{2F_0\left(3b+4c\right)} = \frac{3c\sqrt{c}}{2f\sqrt{f}\left(3b+4c\right)} > 0,$$

which completes the proof of theorem. □

The visual illustration of Theorem 4 is given in Figures 1-3. Figure 1 shows the bifurcation diagram for a parameter range where the Naimark-Sacker bifurcation takes the place. Figures 2 and 3 show the transition from the global asymptotic stability of the equilibrium to the existence of a periodic solution.

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