ON NON-ARCHIMEDEAN MODULAR METRIC SPACES AND SOME NONLINEAR CONTRACTION MAPPINGS

M.PAKNAZAR, M.A.KUTBI, M.DEMMA, P.SALIMI

Abstract. Proving fixed point theorems in modular metric spaces are not possible for some nonlinear contraction mappings. For this we introduce the notion of non-Archimedean modular metric space. This new space allow us prove some fixed point theorems for such mapping. As some pattern we prove some fixed point results for such mappings in non-Archimedean modular metric spaces. Moreover, we present an example to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

Modular metric spaces are a natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. Modular metric spaces were introduced in [3, 4]. The introduction of this new concept is justified by the physical interpretation of the modular. Roughly, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But in this paper, we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [23] on vector spaces and modular function spaces introduced by Musielak [22] and Orlicz [24].

In recent years, many researchers studied the behavior of the electrorheological fluids, sometimes referred to as "smart fluids" (for instance lithium polymetachrylate). An interesting model for these fluids, is obtained by using Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, in the case that p is a function [6]. We remark that the usual approach in dealing with the Dirichlet energy problem [8,11] is to convert the energy functional, naturally defined by a modular, to a convoluted and complicated problem which involves the Luxemburg norm.

In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. Recently, there was a strong interest to study the existence of fixed points in the setting of modular function spaces after the first paper [15] was published in 1990. For more on metric fixed point theory, the reader may consult the book [13] and for modular function spaces the book [19].

Proving fixed point theorems in modular metric spaces is not possible for some nonlinear contraction mappings. For this we introduce the notion of non-Archimedean modular metric space. In the setting of this new space it is possible to give some fixed point theorems for such mappings. Moreover, we present an example to illustrate the validity of the obtained results.

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function, for semplicity, we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. [3,4] A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be modular metric on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$, for all $\lambda > 0$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$, and $x,y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$, for all $\lambda,\mu > 0$ and $x,y,z \in X$.

If in the Definition 1.1, we use the condition

(i') $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i), then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x,y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again, ω is called convex if for $\lambda,\mu > 0$ and $x,y,z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Remark 1.2. If ω is a pseudomodular metric on a set X, then the function $\lambda \to \omega_\lambda(x,y)$ is nonincreasing on $(0,+\infty)$ for all $x,y \in X$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

Definition 1.3. If in the Definition 1.1, we replace (iii) by

(iv) $\omega_{\max\{\lambda,\mu\}}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda,\mu > 0$ and $x,y,z \in X$;

then X_ω is called non-Archimedean modular metric space. Since (iv) implies (iii), then each non-Archimedean modular metric space is a modular metric space.

Definition 1.4 ([3,4]). Let ω be a pseudomodular on X and $x_0 \in X$ fixed. Consider the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \quad \text{such that} \quad \omega_\lambda(x,x_0) < +\infty\}.$$

X_ω and X_ω^* are called modular spaces (around x_0).

It is clear that $X_\omega \subset X_\omega^*$ but this inclusion may be proper in general. Let ω be a modular on X, from [3,4], we deduce that the modular space X_ω can be equipped with a (nontrivial) metric, induced by ω and defined by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\} \quad \text{for all} \quad x,y \in X_\omega.$$

If ω is a convex modular on X, according to [3,4] the two modular spaces coincide, that is, $X_\omega^* = X_\omega$, and this common set can be endowed with the metric d_ω^* defined by

$$d_\omega^*(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\} \quad \text{for all} \quad x,y \in X_\omega.$$

These distances will be called Luxemburg distances.

Example 2.1 presented by Abdou and Khamsi [1] is an important motivation for developing the theory of modular metric spaces. Other examples may be found in [3,4].

Definition 1.5. Let X_ω be a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Then

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n,x) \to 0$, as $n \to +\infty$. x will be called the ω-limit of (x_n).
(2) $(x_n)_{n\in N}$ is called ω-Cauchy if $\omega_1(x_m,x_n) \to 0$, as $m,n \to +\infty$.
(3) M is called ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.
(4) M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.

(5) M is called ω-bounded if we have $\delta_\omega(M) = \sup\{\omega_1(x,y); x,y \in M\} < +\infty$.

Definition 1.6 ([26]). Let T be a self-mapping on X and let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is an α-admissible mapping if

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \geq 1.$$

Definition 1.7 ([20]). Let T be an α-admissible self-mapping on X. We say that T is a triangular α-admissible mapping if, $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$ implies that $\alpha(x,z) \geq 1$.

Lemma 1.8 ([20]). *Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \quad \textit{for all} \quad m,n \in \mathbb{N} \quad \textit{with} \quad m < n.$$

By Ψ we will always denote the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ (which are called altering distance functions) such that the following conditions hold:

- φ is continuous and non-decreasing;
- $\varphi(t) = 0$ if and only if $t = 0$.

Motivated by the works of Kumam and Roldán [**?**] we introduce the following class of mappings which is suitable for our results.

Let Θ denote the set of all functions $\theta : [0,\infty)^4 \to [0,\infty)$ satisfying:

(Θ_1) θ is continuous and increasing in all its variables;
(Θ_2) $\theta(t_1,t_2,t_3,t_4) = 0$ iff, either $t_1 = 0$ or $t_4 = 0$.

2. Main Theorems

Now we are ready to prove our first theorem.

Theorem 2.1. *Let X_ω be a complete non-Archimedean modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a $\omega-$continuous mapping. Assume that there exist a function $\alpha : X_\omega \times X_\omega \to [0,+\infty)$, two functions $\psi, \varphi \in \Psi$ and a function $\theta \in \Theta$ such that the following assertions hold:*

(i) *there exists $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(ii) *T is a triangular α-admissible mapping,*
(iii) *for all $x, y \in X_\omega$ with $\alpha(x,y) \geq 1$, we have*

$$\psi(\omega_1(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) + \theta\Big(\omega_1(x,Tx), \omega_1(y,Ty), \omega_1(x,Ty), \omega_1(y,Tx)\Big) \tag{2.1}$$

where

$$M(x,y) = \max\left\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{1}{2}[\omega_1(x,Ty) + \omega_1(y,Tx)]\right\}.$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ and let $\{x_n\}$ be a Picard sequence starting at x_0, that is, $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Let there exists n_0 such that $\omega_1(x_{n_0}, x_{n_0+1}) = 0$, since ω is regular, we get $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. So x_{n_0} is a fixed point of T. Hence we assume that $\omega_1(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Now, since T is a triangular α-admissible mapping, by Lemma 1.8, we have

$$\alpha(x_m, x_n) \geq 1 \quad \text{for all} \quad m,n \in \mathbb{N} \quad \text{with} \quad m < n.$$

Then by (iii), we have

$$\begin{aligned}\psi(\omega_1(x_n, x_{n+1})) &= \psi(\omega_1(Tx_{n-1}, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)) \\ &\quad + \theta\big(\omega_1(x_{n-1}, Tx_{n-1}), \omega_1(x_n, Tx_n), \omega_1(x_{n-1}, Tx_n), \omega_1(x_n, Tx_{n-1})\big),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\Big\{\omega_1(x_{n-1},x_n),\omega_1(x_{n-1},Tx_{n-1}),\omega_1(x_n,Tx_n),\tfrac{\omega_1(x_{n-1},Tx_n)+\omega_1(x_n,Tx_{n-1})}{2}\Big\} \\ &= \max\Big\{\omega_1(x_{n-1},x_n),\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1}),\tfrac{\omega_1(x_{n-1},x_{n+1})+\omega_1(x_n,x_n)}{2}\Big\} \\ &= \max\Big\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1}),\tfrac{\omega_{\max\{1,1\}}(x_{n-1},x_{n+1})}{2}\Big\} \\ &\le \max\Big\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1}),\tfrac{\omega_1(x_{n-1},x_n)+\omega_1(x_n,x_{n+1})}{2}\Big\} \\ &= \max\big\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1})\big\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\theta\big(\omega_1(x_{n-1},Tx_{n-1}),\omega_1(x_n,Tx_n),\omega_1(x_{n-1},Tx_n),\omega_1(x_n,Tx_{n-1})\big) \\ &= \theta\big(\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1}),\omega_1(x_{n-1},x_{n+1}),\omega_1(x_n,x_n)\big) \\ &= \theta\big(\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1}),\omega_1(x_{n-1},x_{n+1}),0\big) = 0. \end{aligned} \tag{2.4}$$

By (2.2)-(2.4) and the properties of ψ and φ, we obtain,

$$\begin{aligned} \psi(\omega_1(x_n,x_{n+1})) &\le \psi\big(\max\big\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1})\big\}\big) - \varphi\big(M(x_{n-1},x_n)\big) \\ &< \psi\big(\max\big\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1})\big\}\big). \end{aligned} \tag{2.5}$$

If there exists n_0 such that

$$\max\big\{\omega_1(x_{n_0-1},x_{n_0}),\omega_1(x_{n_0},x_{n_0+1})\big\} = \omega_1(x_{n_0},x_{n_0+1}),$$

then by (2.5), we get

$$\begin{aligned} \psi(\omega_1(x_{n_0},x_{n_0+1})) &\le \psi(\omega_1(x_{n_0},x_{n_0+1})) - \varphi(M(x_{n_0-1},x_{n_0})) \\ &< \psi(\omega_1(x_{n_0},x_{n_0+1})), \end{aligned}$$

which is a contradiction. Hence

$$\max\big\{\omega_1(x_{n-1},x_n),\omega_1(x_n,x_{n+1})\big\} = \omega_1(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. Therefore,

$$\psi(\omega_1(x_n,x_{n+1})) \le \psi(\omega_1(x_n,x_{n-1})) - \varphi(\omega_1(x_{n-1},x_n)) < \psi(\omega_1(x_n,x_{n-1})). \tag{2.6}$$

Since ψ is a non-decreasing mapping, then $\{\omega_1(x_n,x_{n+1})\}$ is a non-increasing sequence of positive numbers. Thus, there exists $r_1 \ge 0$ such that

$$\lim_{n\to\infty} \omega_1(x_n,x_{n+1}) = r_1.$$

Letting $n \to \infty$ in (2.6), we have,

$$\psi(r_1) \le \psi(r_1) - \varphi(r_1) \le \psi(r_1).$$

Therefore, $\varphi(r_1) = 0$, and hence $r_1 = 0$, *i.e.*,

$$\lim_{n\to\infty} \omega_1(x_n,x_{n+1}) = 0. \tag{2.7}$$

Now, we show that $\{x_n\}$ is a $\omega-$Cauchy sequence in X_ω. Assume to the contrary that $\{x_n\}$ is not a $\omega-$Cauchy sequence, that is, $\lim_{m,n\to\infty} \omega_1(x_m,x_n) > 0$. Then there exists $\varepsilon > 0$ and two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \omega_1(x_{m_i},x_{n_i}) \ge \varepsilon \text{ and} \tag{2.8}$$

$$\omega_1(x_{m_i},x_{n_i-1}) < \varepsilon. \tag{2.9}$$

By using (2.8), (2.9) and the triangular inequality (iv), we have

$$\begin{aligned}\varepsilon &\le \omega_1(x_{m_i}, x_{n_i}) = \omega_{\max\{1,1\}}(x_{m_i}, x_{n_i})\\ &\le \omega_1(x_{m_i}, x_{n_i-1}) + \omega_1(x_{n_i-1}, x_{n_i})\\ &\le \varepsilon + \omega_1(x_{n_i-1}, x_{n_i})\end{aligned}$$

Now, using (2.7) and taking the limit as $i \to +\infty$, we get

$$\lim_{i\to+\infty} \omega_1(x_{m_i}, x_{n_i}) = \varepsilon. \tag{2.10}$$

Again, by using (iv), deduce

$$\begin{aligned}\varepsilon &\le \omega_1(x_{m_i}, x_{n_i})\\ &= \omega_{\max\{1,1\}}(x_{m_i}, x_{n_i})\\ &\le \omega_1(x_{m_i}, x_{n_i-1}) + \omega_1(x_{n_i-1}, x_{n_i})\\ &= \omega_{\max\{1,1\}}(x_{m_i}, x_{n_i-1}) + \omega_1(x_{n_i-1}, x_{n_i})\\ &\le \omega_1(x_{m_i}, x_{m_i-1}) + \omega_1(x_{m_i-1}, x_{n_i-1}) + \omega_1(x_{n_i-1}, x_{n_i})\end{aligned}$$

and

$$\omega_1(x_{m_i-1}, x_{n_i-1}) \le \omega_1(x_{m_i-1}, x_{n_i}) + \omega_1(x_{m_i}, x_{n_i}) + \omega_1(x_{m_i}, x_{n_i-1}).$$

Taking limit in the above inequalities, we get

$$\lim_{i\to+\infty} \omega_1(x_{m_i-1}, x_{n_i-1}) = \varepsilon. \tag{2.11}$$

Again, using the triangular inequality (iv), we have

$$\begin{aligned}\varepsilon &\le \omega_1(x_{m_i}, x_{n_i}) \le \omega_1(x_{m_i}, x_{m_i-1}) + \omega_1(x_{m_i-1}, x_{n_i})\\ &\le \omega_1(x_{m_i}, x_{m_i-1}) + \omega_1(x_{m_i-1}, x_{n_i-1}) + \omega_1(x_{n_i-1}, x_{n_i})\end{aligned} \tag{2.12}$$

and

$$\varepsilon \le \omega_1(x_{m_i}, x_{n_i}) \le \omega_1(x_{m_i}, x_{n_i-1}) + \omega_1(x_{n_i-1}, x_{n_i}). \tag{2.13}$$

Using (2.7) and (2.11) and taking the limit as $i \to +\infty$ in (2.12), we get

$$\lim_{i\to+\infty} \omega_1(x_{m_i-1}, x_{n_i}) = \varepsilon. \tag{2.14}$$

Similar, we deduce

$$\lim_{i\to+\infty} \omega_1(x_{m_i}, x_{n_i-1}) = \varepsilon. \tag{2.15}$$

Since, $\alpha(x_{m_i-1}, x_{n_i-1}) \ge 1$, then from (iii) we have,

$$\begin{aligned}&\psi(\omega_1(x_{m_i}, x_{n_i})) = \psi(\omega_1(Tx_{m_i-1}, Tx_{n_i-1}))\\ &\le \psi(M(x_{m_i-1}, x_{n_i-1})) - \varphi(M(x_{m_i-1}, x_{n_i-1}))\\ &+\theta\Big(\omega_1(x_{m_i-1}, Tx_{m_i-1}), \omega_1(x_{n_i-1}, Tx_{n_i-1}), \omega_1(x_{m_i-1}, Tx_{n_i-1}), \omega_1(x_{n_i-1}, Tx_{m_i-1})\Big),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}M(x_{m_i-1}, x_{n_i-1}) &= \max\Big\{\omega_1(x_{m_i-1}, x_{n_i-1}), \omega_1(x_{m_i-1}, Tx_{m_i-1}), \omega_1(x_{n_i-1}, Tx_{n_i-1}),\\ &\qquad \tfrac{\omega_1(x_{m_i-1}, Tx_{n_i-1}) + \omega_1(Tx_{m_i-1}, x_{n_i-1})}{2}\Big\}\\ &= \max\Big\{\omega_1(x_{m_i-1}, x_{n_i-1}), \omega_1(x_{m_i-1}, x_{m_i}), \omega_1(x_{n_i-1}, x_{n_i}),\\ &\qquad \tfrac{\omega_1(x_{m_i-1}, x_{n_i}) + \omega_1(x_{m_i}, x_{n_i-1})}{2}\Big\},\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&\theta\Big(\omega_1(x_{m_i-1},Tx_{m_i-1}),\omega_1(x_{n_i-1},Tx_{n_i-1}),\omega_1(x_{m_i-1},Tx_{n_i-1}),\omega_1(x_{n_i-1},Tx_{m_i-1})\Big)\\&=\theta\Big(\omega_1(x_{m_i-1},x_{m_i}),\omega_1(x_{n_i-1},x_{n_i}),\omega_1(x_{m_i-1},x_{n_i}),\omega_1(x_{n_i-1},x_{m_i})\Big).\end{aligned}\tag{2.18}$$

Taking the limit as $i\to+\infty$ in (2.17) and (2.18) and using (2.7), (2.11), (2.14) and (2.15), we get

$$\lim_{i\to+\infty}M(x_{m_i-1},x_{n_i-1})=\varepsilon,\tag{2.19}$$

and

$$\begin{aligned}&\lim_{i\to+\infty}\theta\Big(\omega_1(x_{m_i-1},Tx_{m_i-1}),\omega_1(x_{n_i-1},Tx_{n_i-1}),\omega_1(x_{m_i-1},Tx_{n_i-1}),\omega_1(x_{n_i-1},Tx_{m_i-1})\Big)\\&=\lim_{i\to+\infty}\theta\Big(\omega_1(x_{m_i-1},x_{m_i}),\omega_1(x_{n_i-1},x_{n_i}),\omega_1(x_{m_i-1},x_{n_i}),\omega_1(x_{n_i-1},x_{m_i})\Big)=0.\end{aligned}\tag{2.20}$$

Now, taking the limit as $i\to+\infty$ in (2.16) and using (2.19) and (2.20) we have

$$\psi(\varepsilon)\le\psi(\varepsilon)-\varphi(\varepsilon)$$

which is a contradiction. So $\{x_n\}$ is a $\omega-$Cauchy sequence in X_ω.

Now, since X_ω is a $\omega-$complete modular metric space, there exists $x^*\in X$ such that, $\omega_1(x_n,x^*)\to 0$ as $n\to+\infty$. Thus $\omega_1(Tx_n,Tx^*)\to 0$ as $n\to+\infty$, since T is an $\omega-$continuous mapping. Then by the triangular inequality (iv), we obtain

$$\omega_1(x^*,Tx^*)\le\omega_1(x^*,Tx_n)+\omega_1(Tx_n,Tx^*)=\omega_1(x^*,x_{n+1})+\omega_1(Tx_n,Tx^*).$$

Letting $n\to+\infty$ in the above inequality, we get $\omega_1(x^*,Tx^*)=0$. Since ω is regular, we deduce $Tx^*=x^*$. □

For self-mappings that are not $\omega-$continuous we have the following result.

Theorem 2.2. *Let X_ω be a complete non-Archimedean modular metric space with ω regular and let $T:X_\omega\to X_\omega$ be a self-mapping. Assume that there exist a function $\alpha:X_\omega\times X_\omega\to[0,+\infty)$, two functions $\psi,\varphi\in\Psi$ and a function $\theta\in\Theta$ such that the following assertions hold:*

(i) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;*
(ii) *T is triangular α-admissible mapping;*
(iii) *for all $x,y\in X$ with $\alpha(x,y)\ge 1$, we have*

$$\psi(\omega_1(Tx,Ty))\le\psi(M(x,y))-\varphi(M(x,y))+\theta\Big(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\Big),\tag{2.21}$$

where

$$M(x,y)=\max\left\{\omega_1(x,y),\omega_1(x,Tx),\omega_1(y,Ty),\frac{\omega_1(x,Ty)+\omega_1(y,Tx)}{2}\right\},$$

(iv) *if $\{x_n\}$ be a sequence such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$ as $n\to+\infty$, then $\alpha(x_n,x)\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

Proof. Let $x_0\in X$ be such that $\alpha(x_0,Tx_0)\ge 1$ and let $\{x_n\}$ be a Picard sequence starting at x_0. Following the proof of the Theorem 2.1, we obtain that $\{x_n\}$ is a $\omega-$Cauchy sequence such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$. Since X is $\omega-$complete, then there is $x^*\in X$ such that the sequence $\{x_n\}$ ω-converges to x^*. Using the assumption (iv), we have $\alpha(x_n,x^*)\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$. By (iii), we have

$$\begin{aligned}\psi(\omega_1(x_{n+1},Tx^*))&=\psi(\omega_1(Tx_n,Tx^*))\\&\le\psi(M(x_n,x^*))-\varphi(M(x_n,x^*))\\&+\theta\Big(\omega_1(x_n,Tx_n),\omega_1(x^*,Tx^*),\omega_1(x_n,Tx^*),\omega_1(x^*,Tx_n)\Big)\end{aligned}\tag{2.22}$$

where

$$\begin{aligned} M(x_n,x^*) &= \max\left\{\omega_1(x_n,x^*),\omega_1(x_n,Tx_n),\omega_1(x^*,Tx^*),\tfrac{\omega_1(x_n,Tx^*)+\omega_1(Tx_n,x^*)}{2}\right\} \\ &= \max\left\{\omega_1(x_n,x^*),\omega_1(x_n,x_{n+1}),\omega_1(x^*,Tx^*),\tfrac{\omega_1(x_n,Tx^*)+\omega_1(x_{n+1},x^*)}{2}\right\} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &\theta\Big(\omega_1(x_n,Tx_n),\omega_1(x^*,Tx^*),\omega_1(x_n,Tx^*),\omega_1(x^*,Tx_n)\Big) \\ &= \theta\Big(\omega_1(x_n,x_{n+1}),\omega_1(x^*,Tx^*),\omega_1(x_n,Tx^*),\omega_1(x^*,x_{n+1})\Big). \end{aligned} \tag{2.24}$$

Letting $n\to+\infty$ in (2.23) and (2.24), considering that $\limsup\omega_1(x_n,Tx^*)\le\omega_1(x^*,Tx^*)$, we get

$$M(x_n,x^*)\to\omega_1(x^*,Tx^*) \text{ as } n\to+\infty, \tag{2.25}$$

and

$$\theta\Big(\omega_1(x_n,Tx_n),\omega_1(x^*,Tx^*),d(x_n,Tx^*),\omega_1(x^*,Tx_n)\Big)\to 0 \text{ as } n\to+\infty.$$

Now, taking the sup limit as $i\to+\infty$ in (2.22), we get

$$\psi(\omega_1(x^*,Tx^*))\le\psi(\limsup_{n\to+\infty}\omega_1(x_{n+1},Tx^*))\le\psi(\omega_1(x^*,Tx^*))-\varphi(\omega_1(x^*,Tx^*)).$$

Hence, $\varphi(\omega_1(x^*,Tx^*))=0$ and this implies $x^*=Tx^*$. □

Example 2.3. Let $X_\omega=\mathbb{R}$ be endowed with the non-Archimedean modular metric

$$\omega_\lambda(x,y)=\begin{cases}\frac{1}{\lambda}(|x|+|y|), & \text{if } x\neq y\\ 0 & \text{if } x=y\end{cases}$$

for all $x,y\in X_\omega$ and $\lambda>0$. Define $T:X_\omega\to X_\omega$ and $\alpha:X_\omega\times X_\omega\to[0,+\infty)$ by

$$Tx=\begin{cases}2x^{10}, & \text{if } x\in(-\infty,0)\\ \frac{1}{8}x^2, & \text{if } x\in[0,1)\\ \frac{1}{8}x, & \text{if } x\in[1,2)\\ \frac{1}{4} & \text{if } x\in[2,+\infty)\end{cases},\qquad \alpha(x,y)=\begin{cases}2, & \text{if } x,y\in[0,+\infty)\\ 0, & \text{otherwise}\end{cases}$$

Also, define, $\psi,\varphi:[0,+\infty)\to[0,+\infty)$ and $\theta:[0,+\infty)^4\to[0,+\infty)$ by $\psi(t)=t$, $\varphi(t)=\frac{3}{4}t$ and $\theta(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3,t_4\}$. Clearly, X_ω is a $\omega-$complete modular metric space, $\psi,\varphi\in\Psi$ and $\theta\in\Theta$. Let, $\alpha(x,y)\ge 1$, then $x,y\in[0,+\infty)$. On the other hand, $Tw\in[0,+\infty)$ for all $w\in[0,+\infty)$. Then $\alpha(Tx,Ty)\ge 1$. That is, T is an α-admissible mapping. Let $\alpha(x,y)\ge 1$ and $\alpha(y,z)\ge 1$. So, $x,y,z\in[0,+\infty)$, *i.e.*, $\alpha(x,z)\ge 1$. Hence, T is a triangular α-admissible mapping. Let $\{x_n\}$ be a sequence in X_ω such that $\alpha(x_n,x_{n+1})\ge 1$ with $x_n\to x$ as $n\to+\infty$. Then $x_n\in[0,+\infty)$ for all $n\in\mathbb{N}$. Also, $[0,+\infty)$ is a closed set. Then $x\in[0,+\infty)$, that is, $\alpha(x_n,x)\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$. Clearly, $\alpha(0,T0)\ge 1$.

Let, $\alpha(x,y)\ge 1$. So $x,y\in[0,+\infty)$.

Now we consider the following cases:

- Let $x,y\in[0,1)$, then,

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) &= \omega_1(Tx,Ty) = \tfrac{1}{8}x^2 + \tfrac{1}{8}y^2\\ &= \tfrac{1}{8}(x^2+y^2)\\ &\le \tfrac{1}{4}(x+y)\\ &= \tfrac{1}{4}\omega_1(x,y)\\ &\le \tfrac{1}{4}M(x,y)\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &\le \psi(M(x,y)) - \varphi(M(x,y))\Big] + \theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).\end{aligned}$$

- Let $x, y \in [1,2)$, then,

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) = \omega_1(Tx,Ty) &= \tfrac{1}{8}x + \tfrac{1}{8}y\\ &= \tfrac{1}{8}(x+y)\\ &\le \tfrac{1}{4}(x+y)\\ &= \tfrac{1}{4}\omega_1(x,y)\\ &\le \tfrac{1}{4}M(x,y)\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &\psi(M(x,y)) - \varphi(M(x,y))\\ &+\theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).\end{aligned}$$

- Let $x, y \in [2,\infty)$, then,

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) &= \omega_1(Tx,Ty) = \tfrac{1}{4} + \tfrac{1}{4}\\ &= \tfrac{1}{2}\\ &= \tfrac{1}{4}(1+1)\\ &\le \tfrac{1}{4}(x+y)\\ &= \tfrac{1}{4}\omega_1(x,y)\\ &\le \tfrac{1}{4}M(x,y)\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &\le \psi(M(x,y)) - \varphi(M(x,y))\\ &+\theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).\end{aligned}$$

- Let $x \in [0,1)$ and $y \in [1,2)$, then,

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) &= \omega_1(Tx,Ty) = \tfrac{1}{8}x^2 + \tfrac{1}{8}y\\ &\le \tfrac{1}{8}x + \tfrac{1}{8}y\\ &\le \tfrac{1}{4}(x+y)\\ &= \tfrac{1}{4}\omega_1(x,y) \le \tfrac{1}{4}M(x,y)\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &+\theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).\end{aligned}$$

- Let $x \in [0,1)$ and $y \in [2,\infty)$, then,

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) &= \omega_1(Tx,Ty) = \tfrac{1}{8}x^2 + \tfrac{1}{4}\\ &\le \tfrac{1}{8}x + \tfrac{1}{8}y\\ &= \tfrac{1}{8}(x+y)\\ &\le \tfrac{1}{4}(x+y)\\ &= \tfrac{1}{4}\omega_1(x,y)\\ &\le \tfrac{1}{4}M(x,y)\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &+\theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).\end{aligned}$$

- Let $x \in [1,2)$ and $y \in [2,\infty)$, then,

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) &= \omega_1(Tx,Ty) = (\tfrac{1}{8}x + \tfrac{1}{4})\\ &\le \tfrac{1}{8}x + \tfrac{1}{8}y\\ &= \tfrac{1}{8}(x+y)\\ &\le \tfrac{1}{4}(x+y)\\ &= \tfrac{1}{4}\omega_1(x,y)\\ &\le \tfrac{1}{4}M(x,y)\\ &= \psi(M(x,y)) - \varphi(M(x,y))\\ &\le \psi(M(x,y)) - \varphi(M(x,y))\Big]\\ &+\theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).\end{aligned}$$

Therefore, $\alpha(x,y) \ge 1$ implies

$$\psi(\omega_1(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) + \theta(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)).$$

Hence, all conditions of Theorem 2.2 hold and T has a fixed point. Here, $x = 0$ is the fixed point of T.

Theorem 2.4. *Let X_ω be a complete non-Archimedean modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist a function $\alpha : X_\omega \times X_\omega \to [0,+\infty)$, two functions $\psi, \varphi \in \Psi$ and a function $\theta \in \Theta$ such that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(ii) *T is triangular α-admissible mapping;*
(iii) *for all $x, y \in X$, we have*

$$\psi(\alpha(x,y)\omega_1(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) + \theta\Big(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\Big),$$

where

$$M(x,y) = \max\left\{\omega_1(x,y),\omega_1(x,Tx),\omega_1(y,Ty),\frac{\omega_1(x,Ty)+\omega_1(y,Tx)}{2}\right\},$$

(iv) *if $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

Proof. Let $\alpha(x,y) \ge 1$. Since ψ is increasing, by(iii) we have

$$\begin{aligned}\psi(\omega_1(Tx,Ty)) &\le \psi(\alpha(x,y)\omega_1(Tx,Ty))\\ &\le \psi(M(x,y)) - \varphi(M(x,y))\\ &+\theta\Big(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\Big).\end{aligned}$$

Therefore, all conditions of Theorem 2.2 holds and T has a fixed point. □

If in Theorem 2.4 we take $\alpha(x,y)=1$ for all $x,y\in X$, then we have the following Corollary.

Corollary 2.5. *Let X_ω be a complete non-Archimedean modular metric space with ω regular and let $T: X_\omega \to X_\omega$ be a self-mapping. Assume that there exist two functions $\psi,\varphi\in\Psi$ and a function $\theta\in\Theta$ such that for all $x,y\in X$, we have*

$$\psi(\omega_1(Tx,Ty))\le\psi(M(x,y))-\varphi(M(x,y))+\theta\Big(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\Big).$$

Then T has a fixed point.

3. Some results in $b-$metric spaces endowed with a graph

As in [?], let (X_ω,ω) be a modular metric space and Δ denotes the diagonal of the Cartesian product of $X\times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G)\supseteq\Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph (see [7], p. 309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N\in\mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x_0=x,\ x_N=y$ and $(x_{i-1},x_i)\in E(G)$ for $i=1,\dots,N$.

Definition 3.1. [?] Let (X,d) be a metric space endowed with a graph G. We say that a self-mapping $T:X\to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is,

$$\text{for all } x,\ y\in X,\quad (x,y)\in E(G)\Longrightarrow (Tx,Ty)\in E(G)$$

and T decreases the weights of the edges of G in the following way:

$$\exists\,\alpha\in(0,1)\text{ such that for all } x,y\in X,\quad (x,y)\in E(G)\Longrightarrow d(Tx,Ty)\le\alpha d(x,y).$$

Definition 3.2. [?] A mapping $T:X\to X$ is called G-continuous, if given $x\in X$ and sequence $\{x_n\}$

$$x_n\to x \text{ as } n\to\infty \text{ and } (x_n,x_{n+1})\in E(G) \text{ for all } n\in\mathbb{N} \text{ imply } Tx_n\to Tx.$$

In this section, we will show that many fixed point results in non-Archimedean modular metric spaces endowed with a graph G can be deduced easily from our presented theorems.

Theorem 3.3. *Let X_ω be a complete modular metric space endowed with a graph G satisfies the Δ_2-condition. Let, $T:X_\omega\to X_\omega$ be a self-mapping satisfying the following assertions:*

(i) *there exists $x_0\in X_\omega$ such that, $(x_0,Tx_0)\in E(G)$,*
(ii) *T is ω-continuous,*
(iii) $\forall x,y\in X_\omega[(x,y)\in E(G)\Rightarrow(T(x),T(y))\in E(G)]$
(iv) $\forall x,y,z\in X_\omega[(x,y)\in E(G)\ and\ (y,z)\in E(G)\Rightarrow(x,z)\in E(G)]$
(v) *for all $x,y\in X_\omega$ with $(x,y)\in E(G)$ we have,*

$$\psi(\omega_1(Tx,Ty))\le\psi(M(x,y))-\varphi(M(x,y))\Big]+\theta\Big(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\Big)$$

where, $\psi,\varphi\in\Psi$, $\theta\in\Theta$ and

$$M(x,y)=\max\left\{\omega_1(x,y),\omega_1(x,Tx),\omega_1(y,Ty),\frac{\omega_1(x,Ty)+\omega_1(y,Tx)}{2s}\right\}.$$

Then T has a fixed point.

Proof. Define $\alpha: X_\omega\times X_\omega\to[0,+\infty)$ by $\alpha(x,y)=\begin{cases}2, & \text{if } (x,y)\in E(G)\\ \frac{1}{2}, & \text{otherwise}\end{cases}$. At first we show that T is a triangular α-admissible mapping. Let, $\alpha(x,y)\ge1$, then $(x,y)\in E(G)$. From (iii), we have, $(Tx,Ty)\in E(G)$. That is, $\alpha(Tx,Ty)\ge1$. Also, let, $\alpha(x,y)\ge1$ and $\alpha(y,z)\ge1$. So,

$(x,y) \in E(G)$ and $(y,z) \in E(G)$. From (iv) we get, $(x,z) \in E(G)$. i.e., $\alpha(x,z) \geq 1$. Thus T is a triangular α-admissible mapping. From (i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$. Let $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. Now, from (v) we have

$$\psi(\omega_1(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y))\Big] + \theta\Big(\omega_1(x,Tx), \omega_1(y,Ty), \omega_1(x,Ty), \omega_1(y,Tx)\Big)$$

Hence, all conditions of Theorem 2.1 are satisfied and T has a fixed point. □

If in Theorem 3.3 we take $\theta(t_1,t_2,t_3,t_4) = \min\{t_1,t_2,t_3,t_4\}$, then we have the following Corollary.

Corollary 3.4. *Let X_ω be a complete modular metric space endowed with a graph G satisfies the Δ_2-condition. Let, $T : X_\omega \to X_\omega$ be a self-mapping satisfying the following assertions:*

(i) *there exists $x_0 \in X$ such that, $(x_0, Tx_0) \in E(G)$,*
(ii) *T is ω-continuous,*
(iii) $\forall x,y \in X_\omega[(x,y) \in E(G) \Rightarrow (T(x),T(y)) \in E(G)]$
(iv) $\forall x,y,z \in X_\omega[(x,y) \in E(G)\, and\, (y,z) \in E(G) \Rightarrow (x,z) \in E(G)]$
(v) *for all $x,y \in X_\omega$ with $(x,y) \in E(G)$ we have,*

$$\psi(\omega_1(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) + L\min\{\omega_1(x,Tx), \omega_1(y,Ty), \omega_1(x,Ty), \omega_1(y,Tx)\}$$

where, $\psi, \varphi \in \Psi$, $L \geq 0$ and

$$M(x,y) = \max\left\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_1(x,Ty) + \omega_1(y,Tx)}{2}\right\}.$$

Then T has a fixed point.

Theorem 3.5. *Let X_ω be a complete modular metric space endowed with a graph G satisfies the Δ_2-condition. Let, $T : X_\omega \to X_\omega$ be a self-mapping satisfying the following assertions:*

(i) *there exists $x_0 \in X_\omega$ such that, $(x_0, Tx_0) \in E(G)$,*
(ii) $\forall x,y \in X_\omega[(x,y) \in E(G) \Rightarrow (T(x),T(y)) \in E(G)]$
(iv) $\forall x,y,z \in X_\omega[(x,y) \in E(G)\, and\, (y,z) \in E(G) \Rightarrow (x,z) \in E(G)]$
(v) *for all $x,y \in X_\omega$ with $(x,y) \in E(G)$ we have,*

$$\psi(\omega_1(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) + \theta\Big(\omega_1(x,Tx), \omega_1(y,Ty), \omega_1(x,Ty), \omega_1(y,Tx)\Big) \quad (3.1)$$

where, $(\psi, \varphi \in \Psi)$, $\theta \in \Theta$ and

$$M(x,y) = \max\left\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_1(x,Ty) + \omega_1(y,Tx)}{2}\right\}.$$

(vi) *if $\{x_n\}$ be a sequence in X_ω such that, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then we have, $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Proof. Define the mapping $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ as in the proof of Theorem 3.3. Similar to the proof of Theorem 3.3 we can prove that the conditions (i)-(iii) of Theorem 2.2 are satisfied. Let $\{x_n\}$ be a sequence in X such that, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. From (vi) we get, $(x_n, x) \in E(G)$. That is, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, all conditions of Theorem 2.2 holds and T has a fixed point. □

Corollary 3.6. *Let X_ω be a complete modular metric space endowed with a graph G satisfies the Δ_2-condition. Let, $T : X_\omega \to X_\omega$ be a self-mapping satisfying the following assertions:*

(i) *there exists $x_0 \in X$ such that, $(x_0, Tx_0) \in E(G)$,*
(ii) *T is ω-continuous,*
(iii) $\forall x,y \in X_\omega[(x,y) \in E(G) \Rightarrow (T(x),T(y)) \in E(G)]$
(iv) $\forall x,y,z \in X_\omega[(x,y) \in E(G)\, and\, (y,z) \in E(G) \Rightarrow (x,z) \in E(G)]$

(v) *for all* $x, y \in X_\omega$ *with* $(x, y) \in E(G)$ *we have,*

$$\psi(\omega_1(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + L \min\{\omega_1(x, Tx), \omega_1(y, Ty), \omega_1(x, Ty), \omega_1(y, Tx)\}$$

where, $\psi, \varphi \in \Psi$, $L \geq 0$ *and*

$$M(x, y) = \max\left\{\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty), \frac{\omega_1(x, Ty) + \omega_1(y, Tx)}{2}\right\}.$$

(vi) *if* $\{x_n\}$ *be a sequence in* X_ω *such that,* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to \infty$, *then we have,* $(x_n, x) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

4. Some results in modular metric spaces endowed with a partial ordered

The existence of fixed points in partially ordered sets has been considered in [21]. Let X_ω be a modular metric space and let $\preceq$ be a partially ordered on X_ω. Then $(X_\omega, \preceq)$ is called a partially ordered modular metric space. Two elements $x, y \in X_\omega$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A mapping $T : X_\omega \to X_\omega$ is said to be non-decreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X_\omega$.

In this section, we will show that many fixed point results in partially ordered modular metric spaces can be deduced easily from our obtained results.

Theorem 4.1. *Let* $(X_\omega, \preceq)$ *be a complete partially ordered modular metric space with* ω *regular and let* $T : X_\omega \to X_\omega$ *be a self-mapping. Assume that there exist two functions* $\psi, \varphi \in \Psi$ *and a function* $\theta \in \Theta$ *such that the following assertions hold:*

(i) *there exists* $x_0 \in X_\omega$ *such that* $x_0 \preceq Tx_0$*;*
(ii) T *is an* $\omega-$*continuous mapping;*
(iii) T *is an increasing mapping;*
(iv) *for all* $x, y \in X$ *with* $x \preceq y$, *we have*

$$\psi(\omega_1(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + \theta\big(\omega_1(x, Tx), \omega_1(y, Ty), \omega_1(x, Ty), \omega_1(y, Tx)\big),$$

where

$$M(x, y) = \max\left\{\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty), \frac{\omega_1(x, Ty) + \omega_1(y, Tx)}{2}\right\}.$$

Then T *has a fixed point.*

Proof. Define $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x \preceq y \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

First, we prove that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geq 1$, then $x \preceq y$. Since T is increasing, then we have $Tx \preceq Ty$. That is, $\alpha(Tx, Ty) \geq 1$. Also, we suppose that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. Then, $x \preceq y$ and $y \preceq z$. Hence, $x \preceq z$, that is, $\alpha(x, z) \geq 1$. Therefore, T is a triangular α-admissible mapping. From (i) there exists $x_0 \in X_\omega$ such that $x_0 \preceq Tx_0$, that is, $\alpha(x_0, Tx_0) \geq 1$. Let $\alpha(x, y) \geq 1$, then $x \preceq y$. Now, from (iv) we have

$$\psi(\omega_1(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + \theta\Big(\omega_1(x, Tx), \omega_1(y, Ty), \omega_1(x, Ty), \omega_1(y, Tx)\Big).$$

Hence, all conditions of Theorem 2.1 are satisfied and T has a fixed point. □

If in Theorem 4.1, we take $\theta(t_1, t_2, t_3, t_4) = L\psi(\min\{t_1, t_4\})$ where $L \geq 0$, then we have the following Corollary.

Corollary 4.2. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist two functions $\psi, \varphi \in \Psi$ and a non negative real number L such that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *T is an $\omega-$continuous mapping;*
(iii) *T is an increasing mapping;*
(v) *for all $x, y \in X$ with $x \preceq y$, we have*

$$\psi(\omega_1(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) + L\psi(\min\{\omega_1(x,Tx), \omega_1(y,Tx)\}),$$

where

$$M(x,y) = \max\left\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_1(x,Ty)+\omega_1(y,Tx)}{2s}\right\}.$$

Then T has a fixed point.

Theorem 4.3. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist two functions $\psi, \varphi \in \Psi$ and a function $\theta \in \Theta$ such that the following assertions hold:*

(i) *there exists $x_0 \in X_\omega$ such that $x_0 \preceq Tx_0$;*
(ii) *T is an increasing mapping;*
(iii) *for all $x, y \in X$ with $x \preceq y$, we have*

$$\psi(\omega_1(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) + \theta\big(\omega_1(x,Tx), \omega_1(y,Ty), \omega_1(x,Ty), \omega_1(y,Tx)\big),$$

where

$$M(x,y) = \max\left\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_1(x,Ty)+\omega_1(y,Tx)}{2}\right\}.$$

(iv) *if $\{x_n\}$ be a sequence in X_ω such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$, then we have $x_n \preceq x$ for all $n \in \mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

Proof. Define the mapping $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ as in the proof of Theorem 4.1. Analogous to the proof of Theorem 4.1 we can prove that the conditions (i)-(iii) of Theorem 2.2 are satisfied. Let $\{x_n\}$ be a sequence in X such that, $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. From (iv), we get $x_n \preceq x$. That is, $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$. Therefore, all conditions of Theorem 2.2 holds and T has a fixed point. □

If in Theorem 4.3, we take $\theta(t_1, t_2, t_3, t_4) = L\psi(\min\{t_1, t_4\})$ where $L \ge 0$, then we have the following Corollary.

Corollary 4.4. *Let $(X_\omega, \preceq)$ be a complete partially ordered modular metric space with ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist two functions $\psi, \varphi \in \Psi$ and a non negative real number L such that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *T is an increasing mapping;*
(iii) *for all $x, y \in X$ with $x \preceq y$, we have*

$$\psi(\omega_1(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) + L\psi(\min\{\omega_1(x,Tx), \omega_1(y,Tx)\}),$$

where

$$M(x,y) = \max\left\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_1(x,Ty)+\omega_1(y,Tx)}{2s}\right\}.$$

(iv) *if* $\{x_n\}$ *be an increasing sequence in* X_ω *such* $x_n \to x$ *as* $n \to +\infty$, *then we have* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

5. Some Integral type contractions

Let Φ denotes the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

- every $\phi \in \Phi$ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$;
- for any $\phi \in \Phi$ and any $\epsilon > 0$, $\int_0^\epsilon \phi(\tau)d\tau > 0$.

Note that if we take $\psi(t) = \int_0^t \phi(\tau)d\tau$ where $\phi \in \Phi$, then $\psi \in \Psi$. Also, note that, if $\psi \in \Psi$ and $\theta \in \Theta$, then $\psi\theta \in \Theta$.

Now, if in the Theorem 2.1 and Theorem 2.2, we take $\psi(t) = \int_0^t \phi(\tau)d\tau$, $\varphi(t) = (1-r)\int_0^t \phi(\tau)d\tau$ for all $t \in [0, +\infty)$, where $0 \leq r < 1$ and replace θ by $\psi\theta$, then we have the following theorems.

Theorem 5.1. *Let* X_ω *be a complete non-Archimedean modular metric space with* ω *regular and let* $T : X_\omega \to X_\omega$ *be a* $\omega-$*continuous mapping. Assume that there exist a function* $\alpha : X_\omega \times X_\omega \to [0, +\infty)$, *two functions* $\phi \in \Phi$ *and* $\theta \in \Theta$ *and a real number* $r \in [0, 1)$ *such that the following assertions hold:*

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(ii) T *is a triangular* α*-admissible mapping;*

(iii) *for all* $x, y \in X$ *with* $\alpha(x, y) \geq 1$, *we have*

$$\int_0^{\omega_1(Tx,Ty)} \phi(\tau)d\tau \leq r\int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\right)} \phi(\tau)d\tau.$$

Then T *has a fixed point.*

Theorem 5.2. *Let* X_ω *be a complete non-Archimedean modular metric space with* ω *regular and let* $T : X_\omega \to X_\omega$ *be a* $\omega-$*continuous mapping. Assume that there exist a function* $\alpha : X_\omega \times X_\omega \to [0, +\infty)$, *two functions* $\phi \in \Phi$ *and* $\theta \in \Theta$ *and a real number* $r \in [0, 1)$ *such that the following assertions hold:*

(i) *there exists* $x_0 \in X$ *such that,* $\alpha(x_0, Tx_0) \geq 1$;

(ii) T *is a triangular* α*-admissible mapping;*

(iii) *for all* $x, y \in X$ *with* $\alpha(x, y) \geq 1$, *we have*

$$\int_0^{\omega_1(Tx,Ty)} \phi(\tau)d\tau \leq r\int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\right)} \phi(\tau)d\tau;$$

(iv) *if* $\{x_n\}$ *be a sequence such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

In the setting of partially ordered modular metric space, for example from Theorem 4.3, we deduce the following result.

Theorem 5.3. *Let* $(X_\omega, \preceq)$ *be a complete partially ordered modular metric space with* ω *regular and let* $T : X_\omega \to X_\omega$ *be a self-mapping. Assume that there exist two functions* $\phi \in \Phi$ *and* $\theta \in \Theta$ *and a real number* $r \in [0, 1)$ *such that the following assertions hold:*

(i) *there exists* $x_0 \in X_\omega$ *such that* $x_0 \preceq Tx_0$;

(ii) T *is an increasing mapping;*

(iii) *for all* $x, y \in X$ *with* $x \preceq y$, *we have*

$$\int_0^{\omega_1(Tx,Ty)} \phi(\tau)d\tau \leq r \int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(\omega_1(x,Tx),\omega_1(y,Ty),\omega_1(x,Ty),\omega_1(y,Tx)\right)} \phi(\tau)d\tau$$

(iv) *if* $\{x_n\}$ *be an increasing sequence in* X_ω *such that* $x_n \to x$ *as* $n \to +\infty$, *then we have* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

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(M. Paknazar) Department of Mathematics, Farhangian University, Iran
E-mail address: m.paknazar@cfu.ac.ir

(M.A.Kutbi) Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: E-mail address: mkutbi@yahoo.com

(M. Demma) Università degli Studi di Palermo, Italy
E-mail address: martanoir91@hotmail.it

(P. Salimi) Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran
E-mail address: salimipeyman@gmail.com