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Approximation of solutoins of quasi-variational inclusion and fixed points of nonexpansive mappings

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Abstract

The purpose of this article is to study common solution problems of quasi-variational inclusion problems and nonlinear operator equations involving nonexpansive mappings. Strong convergence theorems are obtained without any compactness assumptions imposed on the operators and the spaces.

Keywords: Hilbert space, convergent theorem, fixed point, contractive mapping, monotone operator
2010 MSC: 47H05, 65J15.

1. Introduction

Convex feasibility problems have recently attracted much attention due to their applications in signal processing and image reconstruction [11] with particular progress in intensity modulated therapy [6]. Recently, the convex feasibility problems have been studied extensively by many authors; see, for instance, [2], [3], [16], [33] and the references therein. The quasi-variational inclusion problem has the reformulations which require finding solutions of evolution equations, complementarity problems, mini-max problems, variational inequalities; see [7]-[10], [31]-[35] and the references therein. It is well known that minimizing a convex function g can be reduced to finding zero points of the subdifferential mapping ∂g.

In this paper, we study a convex feasibility problem based on quasi-variational inclusion and fixed points of nonexpansvie mappings. Strong convergence of solutions are established in the framework of Hilbert spaces. The organization of this paper is as follows. In Section 2, we give the necessary definitions and lemmas. In Section 3, strong convergence theorems are established based on a new iterative algorithm.

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2. Preliminaries

From now on, we assume that H is a real Hilbert and C a nonempty convex closed subset of H. Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be inverse strongly monotone iff there exists a constant $\kappa > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \kappa\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\kappa > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \kappa\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be κ-strongly monotone. A is said to be strongly monotone mapping iff A^{-1} is inverse strongly monotone.

Recall that a mapping $B : H \rightrightarrows H$ is said to be monotone iff, $f \in Bx$ and $g \in By$ imply $\langle x-y, f-g\rangle \geq 0$, for all $x, y \in H$. From now on, we denote the zero point set of B by $B^{-1}(0)$. A monotone mapping $B : H \rightrightarrows H$ is maximal iff its graph $Graph(B)$ is not properly contained in the graph of any other monotone operators. In this paper, we use $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain, to denote the resolvent operator.

The so called quasi-variational inclusion problem is to a point $\bar{x}$ such that

$$0 \in (A + B)\bar{x}. \tag{2.1}$$

A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see, for instance, [23], [7], [12] and the references therein. The problem includes many important problems as special cases.

(1) If $B = \partial\phi$, where $\phi \to R \cup \infty$ is a proper convex lower semi-continuous function and and $\partial\phi$ is the subdifferential of ϕ, then the variational inclusion problem is reduced to the following: find $\bar{x} \in H$, such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \forall y \in H. \tag{2.2}$$

This is called the mixed quasi-variational inequality; see, [24] and the references therein.

(2) If $B = \partial\delta_C$, where δ_C is the indicator function of C, i.e.

$$\begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the variational inclusion problem is reduced finding a point $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \forall y \in C. \tag{2.3}$$

This is called the classical variational inequality; see, [14] and the references therein.

It is known that variational inequality 2.3 is equivalent to a fixed point problem. $\bar{x}$ is a solution to variational inequality 2.3 iff it is a fixed point of some nonlinear operators. Recently, iterative methods have extensively studied for solving solutions of problem 2.1, 2.2 and 2.3; see, [2], [15]-[21], [25]-[30] and the references therein.

Let $T : C \to C$ be a mapping. From now on, we use $Fix(T)$ to denote the fixed point set of T, that is, $Fix(T) = \{x \in C | Tx = x\}$.

Recall that T is said to be contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

We also say T is α-contractive. T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.1 ([1])**.** *Let H be a Hilbert space, and A an maximal monotone operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 2.2 ([3])**.** *Let C be a convex closed and nonempty subset of a real Hilbert space H. Let $A : C \to H$ be a mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $F(J_r(I - rA)) = (A + B)^{-1}(0)$.*

Lemma 2.3 ([22])**.** *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \le (1 - t_n)a_n + t_n b_n + c_n$, $\forall n \ge 0$, where $\{t_n\}$ is a number sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \le 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4 ([5])**.** *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, $\{x_n\}$ converges weakly to some point $\bar{x}$ and $x_n - Tx_n$ converges in norm to 0, $\bar{x} = T\bar{x}$.*

3. Main results

We are now in a position to give the main results in this article.

Theorem 3.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let B be a maximal monotone operator on H such that $Dom(B) \subset C$ and let $A : C \to H$ be an inverse κ-strongly monotone mapping. Let $S : C \to C$ be a fixed α-contraction and let $T : C \to C$ be a nonexpansive mapping. Assume that $(A+B)^{-1}(0) \cap Fix(T)$ is not empty. Let $\{\alpha_n\}$ is a real number sequence in $(0, 1)$, $\{\beta_n\}$ is a real number sequence in $(0, 2\kappa)$. x_0 is an initial in C.. $\{x_n\}$ is a sequence such that $x_{n+1} = T(I + \beta_n B)^{-1}(y_n - \beta_n Ay_n)$, $n \ge 0$, where $y_n = (1 - \alpha_n)x_n + \alpha_n Sx_n$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and $\{\beta_n\}$ is a sequence such that $0 < a \le \beta_n \le b < 2\kappa$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, where a and b are two real numbers. Then $\{x_n\}$ converges strongly $\bar{x} \in Fix(T) \cap (A + B)^{-1}(0)$, where $\bar{x} = Proj_{Fix(T)\cap(A+B)^{-1}(0)} S\bar{x}$.*

Proof. Fix $p \in Fix(T) \cap (A + B)^{-1}(0)$. Using Lemma 2.2, one has

$$\begin{aligned}
\|y_n - p\| &\le \alpha_n\|Sx_n - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\le \alpha_n\|Sx_n - Sp\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Sp - p\| \\
&\le \alpha_n\|Sp - p\| + \big(1 - \alpha_n(1 - \alpha)\big)\|x_n - p\| \\
&\le \max\{\frac{\|Sp - p\|}{1 - \alpha}, \|x_n - p\|\}.
\end{aligned} \tag{3.1}$$

Since A is inverse κ-strongly monotone, one has

$$\begin{aligned}
&\|(I - \beta_n A)x - (I - \beta_n A)y\|^2 \\
&= \|x - y\|^2 - 2\beta_n\langle x - y, Ax - Ay\rangle + \beta_n{}^2\|Ax - Ay\|^2 \\
&\le \|x - y\|^2 - \beta_n(2\kappa - \beta_n)\|Ax - Ay\|^2.
\end{aligned}$$

From the restriction imposed on $\{\beta_n\}$, one has $\|(I - \beta_n A)x - (I - \beta_n A)y\| \le \|x - y\|$. This shows that $I - \beta_n A$ is nonexpansive. It follows from 3.1 that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \|(I + \beta_n B)^{-1}(y_n - \beta_n Ay_n) - p\| \\
&\le \|y_n - \beta_n Ay_n - p + \beta_n Ap\| \\
&\le \|y_n - p\| \\
&\le \max\{\frac{\|Sp - p\|}{1 - \alpha}, \|x_n - p\|\}.
\end{aligned}$$

This implies that sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Since $Proj_{(A+B)^{-1}(0)}S$ is contractive, one has it has a unique fixed point. Next, we denote the unique fixed point by $\bar{x}$. Now, we are in a position to show

$$\limsup_{n\to\infty}\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle \le 0.$$

To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle = \lim_{i\to\infty}\langle S\bar{x}-\bar{x}, y_{n_i}-\bar{x}\rangle \le 0,$$

Since $\{y_{n_i}\}$ is bounded, we find that there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \hat{x}$.

Note that

$$\|y_n-y_{n-1}\| \le |\alpha_n-\alpha_{n-1}|\|x_{n-1}-Sx_{n-1}\| + \big(1-\alpha_n(1-\alpha)\big)\|x_{n-1}-x_n\|. \tag{3.2}$$

Putting $z_n = y_n - \beta_n Ay_n$, we find from 3.2 that

$$\begin{aligned}\|z_n-z_{n-1}\| &\le \|y_n-y_{n-1}\| + \|\beta_n-\beta_{n-1}\|\|Ay_{n-1}\|\\ &\le |\alpha_n-\alpha_{n-1}|\|x_{n-1}-Sx_{n-1}\| + \big(1-\alpha_n(1-\alpha)\big)\|x_{n-1}-x_n\|\\ &\quad + |\beta_n-\beta_{n-1}|\|Ay_{n-1}\|.\end{aligned} \tag{3.3}$$

Set $J^B_{\beta_n} = (I+\beta_n B)^{-1}$. Using Lemma 2.1 and 3.3, one has

$$\begin{aligned}\|x_n-x_{n+1}\| &\le \|J^B_{\beta_{n-1}}z_{n-1} - J^B_{\beta_n}z_n\|\\ &= \|J^B_{\beta_{n-1}}z_{n-1} - J^B_{\beta_{n-1}}\big(\frac{\beta_{n-1}}{\beta_n}z_n + (1-\frac{\beta_{n-1}}{\beta_n})J^B_{\beta_n}z_n\big)\|\\ &\le \|(1-\frac{\beta_{n-1}}{\beta_n})(J^B_{\beta_n}z_n - z_{n-1}) + \frac{\beta_{n-1}}{\beta_n}(z_n-z_{n-1})\|\\ &\le \|(1-\frac{\beta_{n-1}}{\beta_n})(J^B_{\beta_n}z_n - z_n) + (z_n-z_{n-1})\|\\ &\le \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\|z_n-J^B_{\beta_n}z_n\| + \|z_{n-1}-z_n\|\\ &\le \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\|z_n-J^B_{\beta_n}z_n\| + |\alpha_n-\alpha_{n-1}|\|x_{n-1}-Sx_{n-1}\|\\ &\quad + \big(1-\alpha_n(1-\alpha)\big)\|x_{n-1}-x_n\| + |\beta_n-\beta_{n-1}|\|Ay_{n-1}\|.\end{aligned}$$

Using Lemma 2.3, we find $\lim_{n\to\infty}\|x_n-x_{n+1}\| = 0$. Since $\alpha_n \to 0$ as $n\to\infty$, we find $\lim_{n\to\infty}\|x_n-y_n\| = 0$. Note that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|J^B_{\beta_n}(p-\beta_n Ap) - J^B_{\beta_n}(y_n-\beta_n Ay_n)\|^2\\ &\le \|(p-\beta_n Ap) - (y_n-\beta_n Ay_n)\|^2\\ &\le \|y_n-p\|^2 - \beta_n(2\kappa-\beta_n)\|Ay_n-Ap\|^2\\ &\le \alpha_n\|Sx_n-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 - \beta_n(2\kappa-\beta_n)\|Ay_n-Ap\|^2\\ &\le \alpha_n\|Sx_n-p\|^2 + \|x_n-p\|^2 - \beta_n(2\kappa-\beta_n)\|Ay_n-Ap\|^2\end{aligned}$$

It follows that

$$\begin{aligned}\beta_n(2\kappa-\beta_n)\|Ap-Ay_n\|^2 &\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n\|p-Sx_n\|^2\\ &\le (\|x_n-p\| + \|x_{n+1}-p\|)\|x_n-x_{n+1}\| + \alpha_n\|p-Sx_n\|^2.\end{aligned}$$

This yields that

$$\lim_{n\to\infty} \|Ap - Ay_n\| = 0. \tag{3.4}$$

Since $J^B_{\beta_n}$ is firmly nonexpansive, one has

$$\begin{aligned}
&\|J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\|^2 \\
&\le \langle (y_n - \beta_n Ay_n) - (p - \beta_n Ap), J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\rangle \\
&\le \frac{1}{2}\Big(\|(y_n - \beta_n Ay_n) - (p - \beta_n Ap)\|^2 + \|J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\|^2 \\
&\quad - \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n) - \beta_n(Ay_n - Ap)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|y_n - p\|^2 + \|J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\|^2 \\
&\quad - \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|^2 - \beta_n^2\|Ay_n - Ap\|^2 \\
&\quad + 2\beta_n\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\|\Big) \\
&\le \frac{1}{2}\Big(\|y_n - p\|^2 + \|J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\|^2 \\
&\quad - \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|^2 + 2\beta_n\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\|\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\|^2 \le\ & \|y_n - p\|^2 - \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|^2 \\
& + 2\beta_n\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|J^B_{\beta_n}(y_n - \beta_n Ay_n) - p\|^2 \\
&\le \|y_n - p\|^2 - \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|^2 \\
&\quad + 2\beta_n\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\| \\
&\le \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|^2 \\
&\quad + 2\beta_n\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|^2 &\le \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\beta_n\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\| \\
&\le \alpha_n\|Sx_n - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2\|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\|\|Ay_n - Ap\|.
\end{aligned}$$

In view of 3.4, we find that

$$\lim_{n\to\infty} \|y_n - J^B_{\beta_n}(y_n - \beta_n Ay_n)\| = 0. \tag{3.5}$$

Note that

$$\begin{aligned}
\|y_n - Ty_n\| &\le \|y_n - x_n\| + \|x_n - TJ^B_{\beta_n}(y_n - \beta_n Ay_n)\| + \|TJ^B_{\beta_n}(y_n - \beta_n Ay_n) - Ty_n\| \\
&\le \|y_n - x_n\| + \|x_n - TJ^B_{\beta_n}(y_n - \beta_n Ay_n)\| + \|J^B_{\beta_n}(y_n - \beta_n Ay_n) - y_n\|.
\end{aligned}$$

By 3.5, one gets that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. Using Lemma 2.4, one has $\hat{x} \in Fix(T)$. Putting $r_n = J^B_{\beta_n}(y_n - \beta_n Ay_n)$, We have $r_{n_i} \rightharpoonup \hat{x}$.

Next, we prove $\hat{x} \in (A+B)^{-1}(0)$.

Notice that $y_n - \beta_n A y_n \in r_n + \beta_n B r_n$; that is, $\frac{y_n-\beta_n A y_n - r_n}{\beta_n} \in Br_n$. Let $\eta \in B\tau$. Since B is maximal monotone, we find

$$\left\langle \eta - \frac{y_n - r_n}{\beta_n} + Ay_n, \tau - r_n \right\rangle \geq 0.$$

This implies that $\langle \eta + A\hat{x}, \tau - \hat{x}\rangle \geq 0$. This implies that $-A\hat{x} \in B\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}(0)$. Hence, one has

$$\limsup_{n\to\infty} \langle \bar{x} - S\bar{x}, \bar{x} - y_n\rangle \leq 0. \tag{3.6}$$

Since

$$\begin{aligned}\|y_n - \bar{x}\|^2 &= (1-\alpha_n)\langle x_n - \bar{x}, y_n - \bar{x}\rangle + \alpha_n \langle Sx_n - \bar{x}, y_n - \bar{x}\rangle \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\|\|y_n - \bar{x}\| + \alpha_n\langle Sx_n - S\bar{x}, y_n - \bar{x}\rangle + \alpha_n \langle S\bar{x} - \bar{x}, y_n - \bar{x}\rangle \\ &\leq \big(1 - \alpha_n(1-\alpha)\big)\|y_n - \bar{x}\|\|x_n - \bar{x}\| + \alpha_n\langle S\bar{x} - \bar{x}, y_n - \bar{x}\rangle,\end{aligned}$$

we find

$$\|y_n - \bar{x}\|^2 \leq 2\alpha_n \langle S\bar{x} - \bar{x}, y_n - \bar{x}\rangle + \big(1 - \alpha_n(1-\alpha)\big)\|x_n - \bar{x}\|^2. \tag{3.7}$$

This in turn implies from 3.7 that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \|\bar{x} - J^B_{\beta_n}(y_n - \beta_n A y_n)\|^2 \\ &\leq \|(y_n - \beta_n A y_n) - (\bar{x} - \beta_n A\bar{x})\|^2 \\ &\leq \|y_n - \bar{x}\|^2 \\ &\leq \big(1 - \alpha_n(1-\alpha)\big)\|x_n - \bar{x}\|^2 + 2\alpha_n \langle S\bar{x} - \bar{x}, y_n - \bar{x}\rangle.\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, we find from 3.6 that $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an inverse κ-strongly monotone mapping. Let $S : C \to C$ be a fixed α-contraction and let $T : C \to C$ be a nonexpansive mapping. Assume that $VI(C, A) \cap Fix(T)$ is not empty. Let $\{\alpha_n\}$ is a real number sequence in $(0, 1)$, $\{\beta_n\}$ is a real number sequence in $(0, 2\kappa)$. x_0 is an initial in C.. $\{x_n\}$ is a sequence such that $x_{n+1} = TProj_C(y_n - \beta_n A y_n)$, $n \geq 0$, where $y_n = (1-\alpha_n)x_n + \alpha_n S x_n$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and $\{\beta_n\}$ is a sequence such that $0 < a \leq \beta_n \leq b < 2\kappa$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, where a and b are two real numbers. Then $\{x_n\}$ converges strongly $\bar{x} \in Fix(T) \cap VI(C, A)$, where $\bar{x} = Proj_{Fix(T)\cap VI(C,A)} S\bar{x}$.*

Proof. Letting $x = J^B_\beta y$, we find that

$$\begin{aligned} y \in x + r\partial i_C x &\Longleftrightarrow y \in x + rN_C x \\ &\Longleftrightarrow \langle y - x, v - x\rangle \leq 0, \forall v \in C \\ &\Longleftrightarrow x = Proj_C y,\end{aligned}$$

where $Proj_C$ is the metric projection from H onto C and $N_C x := \{e \in H : \langle e, v - x\rangle, \forall v \in C\}$. This find the desired conclusion immediately. □

Finally, we consider a problem of finding a solution of a Ky Fan inequality, which is known as an equilibrium problem in the terminology of Blum and Oettli; see [4] and [13] and the references therein.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{3.8}$$

The following standard assumptions are also essential in this paper.

(1) $0 = F(x, x) \geq F(x, y) + F(y, x)$ for all $x \in C$;

(2) $F(x, y) \geq \limsup_{t\downarrow 0} F(tz + (1-t)x, y)$ for all $x, y, z\ inC$:

(3) $y \mapsto F(x, y)$ is convex and lower semi-continuous, for all $x \in C$.

Lemma 3.3 ([4])**.** *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (1)-(3)*. Then, for any $\beta > 0$ and $x \in H$, there exists $z \in C$ such that $\beta F(z, y) + \langle y - z, z - x\rangle \geq 0$, $\forall y \in C$. Further, define*

$$T_\beta x = \Big\{z \in C : \beta F(z, y) + \langle y - z, z - x\rangle \geq 0,\ \forall y \in C\Big\} \tag{3.9}$$

for all $\beta > 0$ and $x \in H$. Then (1) T_β is single-valued and firmly nonexpansive; (2) $F(T_\beta) = EP(F)$ is closed and convex.

Lemma 3.4 ([27])**.** *Let C be a nonempty convex closed subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (1)-(3)*, and let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.10}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$, where $FP(F)$ stands for the solution set of 3.8, and $T_\beta x = (I + \beta A_F)^{-1}x$, $\forall x \in H$, $\beta > 0$, where T_β is defined as in 3.9

Theorem 3.5. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (1)-(3)*. Let $S : C \to C$ be a fixed α-contraction and let $T : C \to C$ be a nonexpansive mapping. Assume that $EP(F) \cap Fix(T)$ is not empty. Let $\{\alpha_n\}$ is a real number sequence in $(0, 1)$, $\{\beta_n\}$ is a real number sequence in $(0, 2\kappa)$. x_0 is an initial in C.. $\{x_n\}$ is a sequence such that $x_{n+1} = T(I+\beta_n A_F)^{-1}(y_n - \beta_n Ay_n)$, $n \geq 0$, where $y_n = (1-\alpha_n)x_n + \alpha_n Sx_n$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and $\{\beta_n\}$ is a sequence such that $0 < a \leq \beta_n \leq b < 2\kappa$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, where a and b are two real numbers. Then $\{x_n\}$ converges strongly $\bar{x} \in Fix(T) \cap (A + B)^{-1}(0)$, where $\bar{x} = Proj_{Fix(T)\cap(A+B)^{-1}(0)} S\bar{x}$.*

Proof. Putting $A = 0$ in Theorem 3.1, we find that $J^B_{\beta_n} = (I + \beta_n A_F)^{-1}$. From Theorem 3.1, we can draw the desired conclusion immediately.

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