Generalized Bateman's $G-$function and its bounds

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Abstract

In this paper, we presented some functional equations of the generalized Bateman's $G-$function $G_h(x)$ and its relation with the hypergeometric series ${}_3F_2$. We deduced an asymptotic expansion of the function $G_h(x)$ and studied the completely monotonic property of some functions involving it. Also, we presented some new bounds of the function $G_h(x)$. Our results generalize some recent results about the Bateman's $G-$function $G(x)$.

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1 Introduction.

The ordinary gamma function $\Gamma(x)$ is defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0$$

and the Psi or digamma function $\psi(x)$ is given by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x).$$

The gamma function and its logarithmic derivatives $\psi^{(n)}(x)$ are of the most widely used special functions encountered in advanced mathematics . For more details about the properties of these functions and their bounds, please refer to the papers [2], [3], [8], [9], [12]-[14], [16]-[20], [25]-[29] and plenty of references therein.

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The Bateman's $G-$function is defied by [7]

$$G(x)=\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right),\qquad x\neq 0,-1,-2,\dots\,. \tag{1}$$

The function $G(x)$ is very useful in estimating and summing certain numerical and algebraic series. For more details about the properties, bounds and applications of the $G(x)$, please refer to [7], [12], [14], [15], [17], [21], [30] and the references therein.

The function $G(x)$ satisfies the following relations [7]

$$G(x)=2\sum_{k=0}^{\infty}\frac{(-1)^k}{k+x}, \tag{2}$$

$$G(x+1)+G(x)=2x^{-1}, \tag{3}$$

$$G(nx)=2n^{-1}\sum_{k=0}^{n-1}(-1)^{k+1}\psi\left(x+\frac{k}{n}\right),\qquad n=2,4,6,... \tag{4}$$

$$G(nx)=n^{-1}\sum_{k=0}^{n-1}(-1)^{k}G\left(x+\frac{k}{n}\right),\qquad n=1,3,5,... \tag{5}$$

$$G(x)=2\int_0^{\infty}\frac{e^{-xt}}{1+e^{-t}}dt,\qquad x>0 \tag{6}$$

$$G(x)=2x^{-1}\ {}_2F_1(1,x;x+1;-1), \tag{7}$$

where

$${}_rF_s(\alpha_1,...,\alpha_r;\beta_1,...,\beta_s;x)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_r)_k}{(\beta_1)_k...(\beta_s)_k}\frac{x^k}{k!}$$

is the generalized hypergeometric series [3] defined for $r,s\in\mathbb{N}$, $\alpha_j,\beta_j\in\mathbb{C}$, $\beta_j\neq 0,-1,-2,...$ and the Pochhammer or shifted symbol $(\alpha)_n$ is given by

$$(\alpha)_0=1 \quad\text{and}\quad (\alpha)_m=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)},\qquad m\geq 1.$$

Qiu and Vuorinen [30] presented the double inequality

$$\frac{4(3/2-\ln 4)}{x^2}<G(x)-x^{-1}<\frac{1}{2x^2},\qquad x>0.5\,. \tag{8}$$

Mahmoud and Agarwal [12] deduced the following asymptotic formula for Bateman's G-function

$$G(x)\sim x^{-1}+\sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k},\qquad x\to\infty \tag{9}$$

and they improved the lower bound of the inequality (8) by

$$\frac{1}{2x^2+1.5}<G(x)-x^{-1}<\frac{1}{2x^2},\qquad x>0. \tag{10}$$

Also, Mahmoud and Almuashi [14] proved the following double inequality of the Bateman's $G-$function

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{n} x^{-2n} < G(x) - x^{-1} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{n} x^{-2n}, \qquad m \in \mathbb{N} \tag{11}$$

with the best possible bounds, where ${B_m}'s$ are the Bernoulli numbers [11]. Mortici [17] presented the double inequality

$$0 < \psi(x+\lambda) - \psi(x) \leq \psi(\lambda) + \gamma - \lambda + \lambda^{-1}, \qquad x \geq 1;\ 0 < \lambda < 1 \tag{12}$$

where γ is the Euler constant, which also improves the double inequality (8). Also, Alzer deduced the inequality [2]

$$x^{-1} - T_r(\lambda; x) - \omega_r(\lambda; x) < \psi(x+\lambda) - \psi(x) < x^{-1} - T_r(\lambda; x),$$

where $x > 0$, $r = 0, 1, 2, \ldots$, $0 < \lambda < 1$,

$$T_r(\lambda; x) = (1-\lambda)\left[\frac{1}{\lambda+r+1} + \sum_{i=0}^{r-1} \frac{1}{(x+i+1)(x+i+\lambda)}\right]$$

and

$$\omega_r(\lambda; x) = \frac{1}{x+r+\lambda} \log \frac{(x+r)^{(x+r)(1-\lambda)}(x+r+1)^{(x+r+1)\lambda}}{(x+r+\lambda)^{x+r+\lambda}}.$$

Mahmoud, Talat and Moustafa [15] presented the following family of approximations of the function $G(x)$

$$M(\mu, x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)}, \qquad x > 0;\ 1 \leq \mu \leq 2$$

which is of an order of convergence of $O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$ for $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ and is asymptotically equivalent to $G(x)$ as $x \to \infty$. Also, they presented the new double inequality

$$\ln\left(1 + \frac{1}{x + \frac{4}{e^2-4}}\right) + \frac{2}{x(x+1)} < G(x) < \ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)},$$

where the constants 1 and $\frac{4}{e^2-4}$ are the best possible.

In this paper, we presented some functional equations of the generalized Bateman's $G-$function

$$G_h(x) = \psi\left(\frac{x+h}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad 0 < h < 2;\ x \neq -2m, -2m-h \text{ for } m = 0, 1, 2, \ldots \tag{13}$$

and its relation with the hypergeometric function ${}_3F_2$. We deduced an asymptotic expansion of the function $G_h(x)$ and studied the completely monotonic property of the function $G_h(x) - \frac{s}{x^r}$ for different values of the parameter s. Also, some bounds of the generalized Bateman's $G-$function are given.

2 Some relations of the function $G_h(x)$.

Lemma 2.1. *The function $G_h(x)$ satisfies the functional equation*

$$G_h(x+1) + G_h(x) = 2\left(\psi(x+h) - \psi(x)\right), \qquad x > 0. \tag{14}$$

Proof. Using the integral representation [3]

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-tz}}{1 - e^{-t}} dt, \qquad R(z) > 0$$

we get

$$G_h(x) = 2\int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}}\, e^{-xt} dt, \qquad x > 0. \tag{15}$$

Also,

$$\begin{aligned}
\psi(x+h) - \psi(x) &= \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-t}}\, e^{-xt} dt \\
&= \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}}\, e^{-(x+1)t} dt + \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}}\, e^{-xt} dt \\
&= \frac{1}{2}\left[G_h(x+1) + G_h(x)\right].
\end{aligned}$$

□

In case of $h = 1$ and using the functional equation $\psi(x+1) = \frac{1}{x} + \psi(x)$, we get the relation (3).

Lemma 2.2. *The function $G_h(x)$ satisfies the functional equation*

$$G_h(mx) = \frac{1}{m}\sum_{r=0}^{m-1} G_{\frac{h}{m}}\left(x + \frac{2r}{m}\right), \qquad x > 0;\ m \in \mathbb{N}. \tag{16}$$

Proof.

$$\begin{aligned}
\sum_{r=0}^{m-1} G_{\frac{h}{m}}\left(x + \frac{2r}{m}\right) &= \int_0^\infty \left(\sum_{r=0}^{m-1} e^{\frac{-2rt}{m}}\right) \frac{1 - e^{\frac{-ht}{m}}}{1 - e^{-2t}}\, e^{-xt} dt \\
&= \int_0^\infty \left(\frac{1 - e^{-2t}}{1 - e^{\frac{-2t}{m}}}\right) \frac{1 - e^{\frac{-ht}{m}}}{1 - e^{-2t}}\, e^{-xt} dt \\
&= \int_0^\infty \frac{1 - e^{\frac{-ht}{m}}}{1 - e^{\frac{-2t}{m}}}\, e^{-xt} dt \\
&= m\, G_h(mx).
\end{aligned}$$

□

As a special case, when $h = 1$, we get the following new functional equation of the ordinary function $G(x)$ in terms of the generalized function $G_h(x)$.

Corollary 2.3. *The function $G(x)$ satisfies the functional equation*

$$G(mx) = \frac{1}{m} \sum_{r=0}^{m-1} G_{\frac{1}{m}} \left(x + \frac{2r}{m} \right), \qquad x > 0;\ m \in \mathbb{N}. \tag{17}$$

The following result relates the function $G_h(x)$ and the hypergeometric function ${}_3F_2$.

Lemma 2.4. *The function $G_h(x)$ satisfies*

$$G_h(x) = \frac{h}{x+h}\, {}_3F_2 \left(1, 1, \frac{h+2}{2}; 2, \frac{x+h+2}{2}; 1 \right), \qquad x > 0. \tag{18}$$

Proof. Using the integral representation [3]

$$\psi(z) = -\gamma + \int_0^1 \frac{1-t^{z-1}}{1-t} dt, \qquad R(z) > 0$$

we get

$$G_h(x) = \int_0^1 \frac{t^{\frac{x-2}{2}} - t^{\frac{x+h-2}{2}}}{1-t} dt = \int_0^1 \left(t^{\frac{x-2}{2}} - t^{\frac{x+h-2}{2}} \right) \left(\sum_{n=0}^{\infty} t^n \right) dt, \qquad x > 0$$

and then

$$G_h(x) = \sum_{n=0}^{\infty} \frac{2h}{(x+2n)(x+h+2n)}, \qquad x > 0. \tag{19}$$

Using the relation

$$x + n = \frac{x(x+1)_n}{(x)_n},$$

we obtain

$$\begin{aligned} G_h(x) &= \frac{2h}{x(x+h)} \sum_{n=0}^{\infty} \frac{\left(\frac{x+h}{2}\right)_n \left(\frac{x}{2}\right)_n}{\left(\frac{x+h+2}{2}\right)_n \left(\frac{x+2}{2}\right)_n} \\ &= \frac{2h}{x(x+h)}\, {}_3F_2 \left(1, \frac{x}{2}, \frac{x+h}{2}; \frac{x+2}{2}, \frac{x+h+2}{2}; 1 \right), \qquad x > 0. \end{aligned}$$

Now using the two-term Thomae transformation formula [32], [23]

$${}_3F_2 (\alpha, \beta, \sigma; \delta, \eta; 1) = \frac{\Gamma(\delta)\Gamma(\theta-\sigma)}{\Gamma(\theta)\Gamma(\delta-\sigma)}\, {}_3F_2 (\eta-\alpha, \eta-\beta, \sigma; \theta, \eta; 1), \quad \theta = \delta + \eta - \alpha - \beta$$

with

$$\alpha = \frac{x}{2}, \quad \beta = \frac{x+h}{2}, \quad \sigma = 1, \quad \eta = \frac{x+h+2}{2}, \quad \delta = \frac{x+2}{2}$$

we have

$${}_3F_2 \left(1, \frac{x}{2}, \frac{x+h}{2}; \frac{x+2}{2}, \frac{x+h+2}{2}; 1 \right) = \frac{x}{2}\, {}_3F_2 \left(1, 1, \frac{h+2}{2}; 2, \frac{x+h+2}{2}; 1 \right),$$

which complete the proof. □

Remark 1. From the formulas (7) and (18) for $h = 1$, we can conclude that

$${}_3F_2 \left(1, 1, 3/2; 2, \frac{x+3}{2}; 1 \right) = \frac{2(x+1)}{x}\, {}_2F_1 (1, x; x+1; -1), \qquad x > 0. \tag{20}$$

3 An asymptotic expansion of the function $G_h(x)$.

Ii is well known that the Psi function has the asymptotic expansion [6]

$$\psi(z) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} \frac{1}{z^k}$$

and its generalization is given by

$$\psi(z+l) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k(l)}{k} \frac{1}{z^k},$$

where $B_k(l)$ are the Bernoulli polynomials defied by the generating function [11]

$$\frac{ze^{zt}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(l)}{k!} z^k$$

and the Bernoulli constants $B_k = B_k(0)$. Using the operations of the asymptotic expansions [5]; [22], we obtain

$$\psi(z+l) - \psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[B_k(l) - B_k\right] \frac{1}{z^k}.$$

For more details about the general theory of the asymptotic expansion of the function $f(z+t)$ by the asymptotic expansion of the function $f(z)$ using Appell polynomials, we refer to [4]. Now, using the identity [11]

$$B_k(l) = \sum_{r=0}^{k} \binom{k}{r} B_r l^{k-r},$$

we get

$$\psi(z+l) - \psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\sum_{r=0}^{k-1} \binom{k}{r} B_r l^{k-r}\right] \frac{1}{z^k}.$$

Then we obtain the following result.

Lemma 3.1. *The following asymptotic series holds:*

$$G_h(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \left[B_r\left(\frac{h}{2}\right) - B_r\right] \frac{1}{x^n}, \qquad x \to \infty. \tag{21}$$

or

$$G_h(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\sum_{r=0}^{n-1} \binom{n}{r} 2^r B_r h^{n-r}\right] \frac{1}{x^n}, \qquad x \to \infty. \tag{22}$$

Remark 2. As a special case at $h = 1$, we obtain

$$G(x) \sim \frac{1}{x} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \left[B_r\left(\frac{1}{2}\right) - B_r\right] \frac{1}{x^n}, \qquad x \to \infty$$

and using the identities [1]

$$B_n\left(\frac{1}{2}\right) = \left(2^{1-n}-1\right) B_n, \qquad n = 0, 1, 2, ...$$

and

$$B_{2n+1} = 0, \qquad n = 1, 2, ...$$

then we get the asymptotic series (9).

In [12], Mahmoud and Agarwal studied the completely monotonic property of the function $G(x) - \frac{s}{x^r}$ for different values of the parameter s by the motivation of Mortici results [17] about Psi function. In the following result, we will generalize this result to the function $G_h(x)$.

Lemma 3.2. *The functions*

$$\chi_{s,1}(x,h) = G_h(x) - \frac{s}{x} \qquad s \le h;\ x > 0;\ 0 < h < 2, \tag{23}$$

and

$$\chi_{s,r}(x,h) = G_h(x) - \frac{s}{x^r} \qquad s < 0;\ x > 0;\ 0 < h < 2;\ r = 2, 3, 4, ... \tag{24}$$

are strictly completely monotonic.

Proof. Using the relation (15) and the known formula

$$(r-1)!\ x^{-m} = \int_0^\infty v^{m-1} e^{-xv} dv, \qquad m \in \mathbb{N} \tag{25}$$

we get

$$(-1)^n \chi_{s,r}^{(n)}(x,h) = \int_0^\infty \phi_{h,s}(r,t) \frac{t^n e^{-xt}}{e^{2t}-1} dt, \qquad n = 0, 1, 2, 3, ... \tag{26}$$

where

$$\phi_{h,s}(r,t) = 2\left(e^{2t} - e^{(2-h)t}\right) - \frac{s\ t^{r-1}}{(r-1)!}\left(e^{2t}-1\right).$$

Then

$$\phi_{h,s}(r,t) = \sum_{k=1}^{\infty} \frac{2^{k+1} t^k}{k!} P_{h,s}(r,t),$$

where

$$P_{h,s,k}(r,t) = 1 - \left(1 - \frac{h}{2}\right)^k - \frac{s}{2(r-1)!} t^{r-1}.$$

Firstly, if $r = 1$, we obtain

$$P_{h,s,k}(1,t) = 1 - \left(1-\frac{h}{2}\right)^k - \frac{s}{2} > 0 \quad \text{iff} \quad \frac{s}{2} \le 1 - \left(1-\frac{h}{2}\right)^k \qquad k = 1, 2, 3, ...$$

But

$$\frac{h}{2} \le 1 - \left(1 - \frac{h}{2}\right)^k \qquad 0 < h < 2;\ k = 1, 2, 3, ...$$

and thus, $\phi_{h,s}(1,t) > 0$ for all $t \ge 0$ iff $s \le h$. Secondly, when $r = 2, 3, 4, ...$, then $P_{h,s,k}(r,t)$ is increasing as a function of t if $s < 0$ with $P_{h,s,k}(r,0) = 1 - \left(1 - \frac{h}{2}\right)^k > 0$ for $0 < h < 2$ and $k = 1, 2, 3, ...$. Thus $\phi_{h,s}(r,t) > 0$ for all $t \ge 0$, $r = 2, 3, ...$ iff $s < 0$. □

As a result of the strictly completely monotonicity of the function $\chi_{s,1}(x,h)$ and the relation (22), we obtain

$$\chi_{s,1}(x,h) > \lim_{x\to\infty}\left(\chi_{s,1}(x,h)\right) = 0, \qquad s \le h.$$

Hence, we have the following result:

Corollary 3.3. *The following inequality holds*

$$G_h(x) > \frac{h}{x}, \qquad x > 0;\ 0 < h < 2. \tag{27}$$

4 Some Bounds of the function $G_h(x)$.

Lemma 4.1.

$$G_h(x) < \frac{2}{x} + \frac{h(2-h)}{2x^2}, \qquad x > 0;\ 0 < h < 2. \tag{28}$$

Proof. By using the formulas (15) and (25), we get for $x > 0$ that

$$\begin{aligned} G_h(x) - \frac{2}{x} - \frac{h(2-h)}{2x^2} &= \int_0^\infty \left(2\left(e^{2t} - e^{(2-h)t}\right) - 2\left(e^{2t}-1\right) - \frac{h(2-h)}{2}\left(e^{2t}-1\right)t\right)\frac{e^{-xt}}{e^{2t}-1}dt \\ &= \int_0^\infty \left(2\left(1 - e^{(2-h)t}\right) - \frac{h(2-h)}{2}\left(e^{2t}-1\right)t\right)\frac{e^{-xt}}{e^{2t}-1}dt \\ &< 0 \quad \text{for } 0 < h < 2. \end{aligned}$$

□

Theorem 1.

$$G_h(x) < \frac{h}{x} + \frac{h(2-h)}{2x^2}, \qquad x > 0;\ 1 \le h < 2. \tag{29}$$

Proof. Using the two formulas (15) and (25), we have

$$G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2} = \int_0^\infty \rho_h(t)\frac{e^{-xt}}{e^{2t}-1}dt,$$

where

$$\rho_h(t) = 2\left(e^{2t} - e^{(2-h)t}\right) - h\left(e^{2t}-1\right) - \frac{h(2-h)}{2}\left(e^{2t}-1\right)t \qquad t > 0.$$

Then

$$\rho_h''(t) = 2(h-2)e^{(2-h)t}Q_h(t)$$

with

$$Q_h(t) = 2 - h + e^{ht}(h - 2 + ht).$$

The function $Q_h(t)$ is convex function with minimum value at $t_0 = \frac{1-h}{h}$, which is non positive for $1 \le h < 2$ and $Q_h(0) = 0$. Hence $Q_h(t) > 0$ for $1 \le h < 2$. Hence $\rho_h(t)$ is concave for $1 \le h < 2$ and its has maximum value at $t = 0$. Then

$$\rho_h(t) < 0, \qquad 1 \le h < 2;\ t > 0.$$

Then the function $G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2}$ is strictly increasing function for $1 \le h < 2$ and $x > 0$ and using the asymptotic expansion (22), we get

$$\lim_{x\to\infty}\left(G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2}\right) = 0,$$

which complete the proof. □

Remark 3. In case of $h = 1$, the inequality (29) will prove the right-hand side of the inequality (10).

To obtain our next result, we will apply the following monotone form of L'Hôpital's rule [10] (see also [24] and [31]).

Theorem 2. *Let $-\infty < \alpha < \beta < \infty$ and $L, U : [\alpha, \beta] \to \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable on (α, β). Let $U'(x) \neq 0$ on (α, β). If $L'(x)/U'(x)$ is increasing (decreasing) on (α, β), then so are*

$$\frac{L(x) - L(\alpha)}{U(x) - U(\alpha)} \quad \text{and} \quad \frac{L(x) - L(\beta)}{U(x) - U(\beta)}. \tag{30}$$

If $L'(x)/U'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Theorem 3.

$$G_h(x) > \frac{h}{x} + \frac{h(2-h)}{2\left(x^2 + 3h^2\right)}, \qquad x > 0;\ 0 < h < 2. \tag{31}$$

Proof. Using the two formulas (15) and (25) and the Laplace transformation of the sine function, we get

$$G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2\left(x^2 + 3h^2\right)} = \int_0^\infty \xi_h(t) \frac{e^{-xt}}{6\left(e^{2t} - 1\right)} dt,$$

where

$$\xi_h(t) = 6\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right) + \sqrt{3}e^{ht}\left(-1 + e^{2t}\right)(-2+h)\sin\left(\sqrt{3}ht\right).$$

Now consider the function

$$\tau_h(t) = \frac{2\sqrt{3}e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right)}{\left(-1 + e^{2t}\right)(2-h)} \qquad t > 0;\ 0 < h < 2.$$

The function

$$\frac{2\sqrt{3}\frac{d}{dt}\left(e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right)\right)}{\frac{d}{dt}\left(\left(-1 + e^{2t}\right)(2-h)\right)} = 2\sqrt{3}e^{-ht}(-1 + e^{ht})$$

is increasing function for $t > 0$. Using the monotone form of L'Hôpital's rule, we get that the function $\tau_h(t)$ is increasing. Similarly, the function

$$H_h(t) = \frac{\tau_h(t)}{\sqrt{3}ht}, \qquad t > 0;\ 0 < h < 2.$$

is increasing function and

$$\lim_{t\to\infty} H_h(t) = 1.$$

Then

$$2\sqrt{3}e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right) > ht\left(-1+e^{2t}\right)(2-h), \qquad t>0;\ 0<h<2$$

and using Jordan's inequality

$$\frac{2z}{\pi} \le \sin z \le z, \qquad x \in [0, \pi/2]$$

we have

$$2\sqrt{3}e^{-ht}\left(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}\right) > ht\left(-1+e^{2t}\right)(2-h)\sin\left(\sqrt{3}ht\right), \qquad t>0;\ 0<h<2.$$

Hence

$$\xi_h(t) > 0, \qquad t>0;\ 0<h<2.$$

Then the function $G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2(x^2+3h^2)}$ is strictly decreasing function for $0 \le h < 2$ and $x > 0$. Also, using the asymptotic expansion (22), we get

$$\lim_{x\to\infty}\left(G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2\left(x^2+3h^2\right)}\right) = 0,$$

which complete the proof. □

Remark 4. In case of $h = 1$, the inequality (31) will prove the left-hand side of the inequality (10).

Remark 5. Using the inequalities (28), (29) and (31) with the relation (19), we get the following estimations

$$\frac{1}{2x} + \frac{2-h}{4\left(x^2+3h^2\right)} < \sum_{n=0}^{\infty}\frac{1}{(x+2n)(x+h+2n)} < \frac{1}{2x} + \frac{2-h}{4x^2}, \qquad x>0;\ 1\le h<2$$

and

$$\frac{1}{2x} + \frac{2-h}{4\left(x^2+3h^2\right)} < \sum_{n=0}^{\infty}\frac{1}{(x+2n)(x+h+2n)} < \frac{1}{hx} + \frac{2-h}{4x^2}, \qquad x>0;\ 0\le h<2.$$

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