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# On the generalized Ulam-Hyers-Rassias stability for quartic functional equation in modular spaces

Kittipong Wongkum[a,c], Poom Kumam[a,c,*], Yeol Je Cho[b,*], Parin Chaipunya[a,c]

[a] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140 Thailand.*

[b] *Department of Mathematics Education, Gyeongsang Natoinal University , Jinju 660-701, Korea.*

[c] *Theoretical and Computational Science (TaCS) Center, Science Laboratory Building, Faculty of Science, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.*

**Abstract**

In this paper, we prove the generalized UHR stability of a quartic functional equations $f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$ via the extensive studies of fixed point theory. Our results are obtained in the framework of modular spaces by the modular is l.s.c. and convex.

*Keywords:* quartic mapping, generalized UHR Stability, Modular space

## 1. Introduction

The investigation the stability of functional equations began in 1940 by S.M. Ulam [19]. During a conference at Wisconsin University, S.M. Ulam posted the question asking about the stability of group homomorphisms as following:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $f:G_1\to G_2$ satisfies the inequality $d(f(xy),f(x)f(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $g:G_1\to G_2$ with $d(f(x),g(x))<\epsilon$ for all $x\in G_1$?

That is, we are interested in the situation when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a exact homomorphism close to it. In 1941, D. H. Hyers [8] gave a partial solution of Ulam 's problem under the assumption that $G_1$ and $G_2$ are Banach spaces.This is the provenance of Hyers-Ulam stability of functional equation. In 1950, Aoki [2] generalized Hyers' theorem for approximately additive functions. In 1978, Th. M. Rassias [15] generalized the theorem of Hyers to an

*Corresponding authors.
*Email addresses:* kittipong.wong@mail.kmutt.ac.th (Kittipong Wongkum), poom.kum@kmutt.ac.th (Poom Kumam), yjcho@gnu.ac.kr (Yeol Je Cho), yjchomath@gmail.com (Yeol Je Cho), parin.cha@mail.kmutt.ac.th (Parin Chaipunya)

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unbounded case. The result of Rassias has had a lot of influence in the development of what we now call Ulam-Hyers-Rassias stability of functional equations. The stability problem of the functional equations has been investigated by many mathematicians and you can see in [5, 6, 4, 1, 3, 17].

The first author who treated the stability of the quartic equation was J. M. Rassias [14] for function $F : X \to Y$, where $X$ is a linear space and $Y$ is a Banach space. The stability problem of the quartic equation has been investigated by many mathematicians. For the example, S. H. Lee, S. M. Im and I. S. Hwang [10] solved quartic functional equation and proved the stability of a quartic functional equation for function $f : X \to Y$, where $X, Y$ are real vector spaces. Y. S. Lee and S. Y. Chung [11] presented the general solution of a generalization of quartic functional equation in the class of functions between real vector spaces.

In this work, we introduce the following function equation in modular spaces:

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y), \tag{1.1}$$

where a mapping in (1.1) is called a quartic functional equation and every solution of the quartic functional equation (in form : $f(x) = cx^4$) is said to be quartic function. We consider the case where $X$ is a linear space and $Y$ is a $\rho$-complete modular space, whose the scalar fields are arbitrary. Our main results are obtained by using the fixed point method under the assumptions that the modular is lower semicontinuous (in brief l.s.c.) and convex.

## 2. Preliminaries

In this section, we recall some basic definitions and properties of a modular space. In addition, we use $\mathbb{R}, \mathbb{C}$, and $\mathbb{N}$ to denote respectively the set of all reals, complexes, and nonnegative integers, respectively.

**Definition 2.1.** Let $X$ be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized functional $\rho : X \to [0, \infty]$ is called a *modular* if for arbitary $x, y \in X$,

(m1) $\rho(x) = 0$ if and only if $x = 0$,

(m2) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $|\alpha| = 1$,

(m3) $\rho(z) \le \rho(x) + \rho(y)$ whenever $z$ is a convex combination of $x$ and $y$.

The corresponding *modular space*, denoted by $X_\rho$, is then defined by

$$X_\rho := \{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}.$$

**Remark 1.** Note that for a fixed $x \in X_\rho$, the valuation $\gamma \in \mathbb{K} \mapsto \rho(\gamma x)$ is increasing.

Continuous or convex in a modular is not necessary. However, it is often occur that some weaker form of them are assumed.

**Remark 2.** In case a modular $\rho$ is convex, one has $\rho(x) \le \delta\rho(\frac{1}{\delta}x)$ for all $x \in X_\rho$, provided that $0 < \delta \le 1$.

**Definition 2.2.** Let $X_\rho$ be a modular space and $\{x_n\}$ be a sequence in $X_\rho$. Then,

(i) $\{x_n\}$ is *$\rho$-convergent* to a point $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is called *$\rho$-Cauchy* if for any $\epsilon > 0$, we have $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$.

(iii) A subset $K \subset X_\rho$ is called *$\rho$-complete* if any $\rho$-Cauchy sequence is $\rho$-convergent.

In modular spaces, the convergence of a sequence $\{x_n\}$ to $x$ does not imply that $\{cx_n\}$ converges to $cx$, where $c$ is chosen from the corresponding scalar field. So, it is necessary to add some conditions. Such preferences are referred to mostly under the term related to the $\Delta_2$-conditions.

A modular $\rho$ is said to satisfy the $\Delta_2$-condition if there exists $\kappa \geq 2$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in X_\rho$. Some authors varied the notion so that only $\kappa > 0$ is required and called it the $\Delta_2$-type condition. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\Delta_2$-conditions.

**Remark 3.** The averaged sequence $\{\frac{1}{2^k}\sum_{i=1}^{2k} x_n^i\}$ $\rho$-converges to $\frac{1}{2^k}\sum_{i=1}^{2k} x^i$, where we suppose that $\{x_n^1\}, \{x_n^2\}, \cdots, \{x_n^{2k}\}$, for some $k \in \mathbb{N}$, are sequences in $X_\rho$ in which they $\rho$-converge to the points $x^1, x^2, \cdots,$ $x^{2k} \in X_\rho$, respectively.

In [9], Khamsi proved a series of fixed point theorems in modular spaces. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed. Moreover, if it happens to have more than one fixed points, say $x$ and $y$, then it must be the case that $\rho(x - y) = \infty$.

**Definition 2.3.** Given a modular space $X_\rho$, a nonempty subset $C \subset X_\rho$, and a mapping $T : C \to C$. The *orbit* of $T$ around a point $x \in C$ is the set

$$\mathcal{O}(x) := \{x, Tx, T^2x, \cdots\}.$$

The quantity $\delta_\rho(x) := \sup\{\rho(u - v) : u, v \in \mathcal{O}(x)\}$ is then associated and is called the *orbital diameter* of $T$ at $x$. In particular, if $\delta_\rho(x) < \infty$, we say that $T$ has a bounded orbit at $x$.

**Lemma 2.4** ( [9]). *Let $X_\rho$ be a modular space whose the induced modular is l.s.c. and $C \subset X_\rho$ be a $\rho$-complete subset. If $T : C \to C$ is a $\rho$-contraction, i.e., there is a constant $k \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq k\rho(x - y), \quad \forall x, y \in C,$$

*and $T$ has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is $\rho$-convergent to a point $w \in C$.*

## 3. Main results

This section describes details stability of quartic functional equation in modular spaces. The prove method requires no $\Delta_2$-conditions which are different from other researches.

**Definition 3.1.** For a constant $L \geq 0$ and a linear space $V$, we define $\Phi_{V,L}$ to be the collection of all nonnegative real-valued functions $\psi$ defined on $V$ with the following properties for all $x, y \in V$:

$$\begin{cases} \lim_{n\to\infty} \psi(2^n x, y)/16^n = 0, \\ \psi(2x, 0) \leq 16L\psi(x, 0). \end{cases}$$

**Theorem 3.2.** *Let $V$ be linear space, $X_\rho$ be a $\rho$-complete modular space where $\rho$ is l.s.c. and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. Suppose that for each $x, y \in V$, the following dominating condition holds:*

$$\rho(f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y)) \leq \phi(x, y), \tag{3.1}$$

*where $\phi \in \Phi_{V,L}$ with $0 < L < 1$. Then, there exists a quartic mapping $w : V \to X_\rho$ such that*

$$\rho(w(x) - f(x)) \leq \frac{1}{16(1 - L)}\phi(x, 0) \tag{3.2}$$

*for all $x \in V$. Equivalently, the quartic mapping is generalized UHR stable.*

To prove this stability result, we shall need the following lemma:

**Lemma 3.3.** *Suppose that every assumptions of Theorem 3.2 hold. Then, the following statements hold:*

*(S1) The set* $\mathcal{M} := \{g : V \to X_\rho : g(0) = 0\}$ *is a linear space.*

*(S2) A generalized function* $\tilde{\rho}$ *defined for each* $g \in \mathcal{M}$ *by*

$$\tilde{\rho}(g) := \inf\{c > 0 : \rho(g(x)) \le c\phi(x, 0), \ \forall x \in V\}$$

*is a convex modular on* $\mathcal{M}$*.*

*(S3) The corresponding modular space* $\mathcal{M}_{\tilde{\rho}}$ *is the whole space* $\mathcal{M}$ *and is* $\tilde{\rho}$*-complete.*

*(S4)* $\tilde{\rho}$ *is l.s.c.*

*Proof.* **(S1)** is trivial.

**(S2)** It is also easy to verify that $\tilde{\rho}$ satisfies the axioms (m1) and (m2) of a modular. We shall next show that $\tilde{\rho}$ is convex, and hence (m3) is satisfied. Let $\epsilon > 0$ be given. Then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\tilde{\rho}(g) \le c_1 \le \tilde{\rho}(g) + \epsilon \quad \text{and} \quad \tilde{\rho}(h) \le c_2 \le \tilde{\rho}(h) + \epsilon.$$

Consecutively, we have

$$\rho(g(x)) \le c_1\phi(x, 0) \quad \text{and} \quad \rho(h(x)) \le c_2\phi(x, 0).$$

Thus, if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$, we get

$$\rho(\alpha g(x) + \beta h(x)) \le \alpha\rho(g(x)) + \beta\rho(h(x)) \le (\alpha c_1 + \beta c_2)\phi(x, 0),$$

so that

$$\tilde{\rho}(\alpha g + \beta h) \le \alpha\tilde{\rho}(g) + \beta\tilde{\rho}(h) + (\alpha + \beta)\epsilon.$$

Hence, we have

$$\tilde{\rho}(\alpha g + \beta h) \le \alpha\tilde{\rho}(g) + \beta\tilde{\rho}(h).$$

This concludes that $\tilde{\rho}$ is a convex modular on $\mathcal{M}$.

**(S3)** The fact that the corresponding modular space $\mathcal{M}_{\tilde{\rho}}$ is the whole space $\mathcal{M}$ is trivial, so we only show that $\mathcal{M}_{\tilde{\rho}}$ is $\tilde{\rho}$-complete. Let $\{g_n\}$ be a $\tilde{\rho}$-Cauchy sequence in $\mathcal{M}_{\tilde{\rho}}$ and let $\epsilon > 0$ be given. There exists a positive integer $n_0 \in \mathbb{N}$ such that $\tilde{\rho}(g_n - g_m) < \epsilon$ for all $n, m \ge n_0$. By the definition, we may see that

$$\rho(g_n(x) - g_m(x)) \le \epsilon\phi(x, 0) \tag{3.3}$$

for all $x \in V$ and $n, m \ge n_0$. Thus, at each fixed $x \in V$, the sequence $\{g_n(x)\}$ is a $\rho$-Cauchy sequence. Since $X_\rho$ is $\rho$-complete, so $\{g_n(x)\}$ is $\rho$-convergent in $X_\rho$ for each $x \in V$. Hence, we can define a function $g : V \to X_\rho$ by

$$g(x) := \lim_{n\to\infty} g_n(x)$$

for any $x \in V$. Since $\rho$ is l.s.c., it follows from (3.3) that

$$\rho(g_n(x) - g(x)) \le \liminf_{m\to\infty} \rho(g_n(x) - g_m(x)) \le \epsilon\phi(x, 0),$$

provided that $n \ge n_0$. Thus, $\{g_n\}$ $\tilde{\rho}$-converges, so that $\mathcal{M}_{\tilde{\rho}}$ is $\tilde{\rho}$-complete.

**(S4)** Suppose that $\{g_n\}$ is a sequence in $\mathcal{M}_{\tilde{\rho}}$ which is $\tilde{\rho}$-convergent to an element $g \in \mathcal{M}_{\tilde{\rho}}$. Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, let $c_n$ be a constant such that

$$\tilde{\rho}(g_n) \le c_n \le \tilde{\rho}(g_n) + \epsilon.$$

Again, we have

$$\rho(g_n(x)) \leq c_n\phi(x,0), \quad \forall x \in V.$$

Now, observe from the lower semicontinuity of $\rho$ that

$$\begin{aligned}\rho(g(x)) &\leq \liminf_{n\to\infty} \rho(g_n(x)) \\ &\leq \liminf_{n\to\infty} c_n\phi(x,0) \\ &\leq \left[\liminf_{n\to\infty} \tilde{\rho}(g_n) + \epsilon\right]\phi(x,0).\end{aligned}$$

Thus, we have

$$\tilde{\rho}(g) \leq \liminf_{n\to\infty} \tilde{\rho}(g_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we can finally conclude that $\tilde{\rho}$ is l.s.c. □

Next, we show that a self mapping $T : \mathcal{M}_{\tilde{\rho}} \to \mathcal{M}_{\tilde{\rho}}$ defined by

$$Tg(x) := \frac{1}{16}g(2x), \quad \forall g \in \mathcal{M}_{\tilde{\rho}}, \forall x \in V. \tag{3.4}$$

has some fixed point accordingly to Lemma 2.4.

**Remark 4.** It is useful to notice that $T$ is a linear mapping.

**Lemma 3.4.** *Suppose that every assumptions of Theorem 3.2 hold and $T$ is defined as in (3.4). Then, $T$ has some fixed point.*

*Proof.* We first show that $T$ is a $\tilde{\rho}$-contraction. Let $x \in V$, $g, h \in \mathcal{M}_{\tilde{\rho}}$ and $c \in [0,\infty]$ be an arbitrary constant with $\tilde{\rho}(g-h) \leq c$. Observe that we have

$$\rho(g(x) - h(x)) \leq c\phi(x,0),$$

so that

$$\rho\left(\frac{g(2x)}{16} - \frac{h(2x)}{16}\right) \leq \frac{1}{16}\rho(g(2x) - h(2x)) \leq \frac{1}{16}c\phi(2x,0) \leq Lc\phi(x,0).$$

Therefore, we have $\tilde{\rho}(Tg - Th) \leq L\tilde{\rho}(g-h)$. Since $g, h \in \mathcal{M}_{\tilde{\rho}}$ are arbitrary, $T$ is a $\tilde{\rho}$-strict contraction w.r.t. $L$.

Next, we show that $T$ has a bounded orbit at $f$, where $f$ is taken from the assumption. Let $x \in V$ be arbitrary and set $y = 0$ in (3.1), we get

$$\rho(2f(2x) - 32f(x)) \leq \phi(x,0), \quad \forall x \in V. \tag{3.5}$$

Set $x = 2x$ in (3.5), we get

$$\rho(2f(4x) - 32f(2x)) \leq \phi(2x,0), \quad \forall x \in V. \tag{3.6}$$

Since $\rho$ is convex, we obtain

$$\begin{aligned}\rho\left(f(2x) - 16f(x)\right) &= \rho\left(\frac{2}{2}\cdot f(2x) - \frac{2}{2}\cdot 16f(x)\right) \\ &= \rho\left(\frac{1}{2}\Big(2f(2x) - 32f(x)\Big) + \frac{1}{2}\cdot 0\right) \\ &\leq \frac{1}{2}\rho\Big(2f(2x) - 32f(x)\Big) \\ &\leq \frac{1}{2}\phi(x,0), \quad \forall x \in V.\end{aligned}$$

Moreover,

$$\begin{aligned}
\rho\left(\frac{f(2^2x)}{16}-16f(x)\right) &= \rho\left(\frac{2}{2}\cdot\frac{f(2^2x)}{16}-\frac{2}{2}f(2x)+\frac{2}{2}f(2x)-\frac{2}{2}\cdot 16f(x)\right)\\
&= \rho\left[\frac{1}{2}\left(\frac{2f(2^2x)}{16}-2f(2x)\right)+\frac{1}{2}\Big(2f(2x)-32f(x)\Big)\right]\\
&\le \frac{1}{2}\rho\left(\frac{2f(2^2x)}{16}-2f(2x)\right)+\frac{1}{2}\rho\Big(2f(2x)-32f(x)\Big)\\
&= \frac{1}{2}\rho\left(\frac{1}{16}\Big(2f(2^2x)-32f(2x)\Big)\right)+\frac{1}{2}\rho\Big(2f(2x)-32f(x)\Big)\\
&\le \frac{1}{2}\cdot\frac{1}{16}\rho\Big(2f(2^2x)-32f(2x)\Big)+\frac{1}{2}\rho\Big(2f(2x)-32f(x)\Big)\\
&\le \frac{1}{2}\left(\frac{1}{16}\phi(2x,0)+\phi(x,0)\right), \quad \forall x\in V.
\end{aligned}$$

By inductively, we may deduce for all $j\in N$ that

$$\rho\left(\frac{f(2^jx)}{16^{j-1}}-16f(x)\right) \le \frac{1}{2}\Sigma_{j=1}^{j}\frac{1}{16^{j-1}}\phi(2^{j-1}x,0), \quad \forall x\in V.$$

Moreover, we get

$$\begin{aligned}
\rho\left(\frac{f(2^jx)}{16^{j-1}}-16f(x)\right) &\le \frac{1}{2}\Sigma_{j=1}^{j}\frac{1}{16^{j-1}}\phi(2^{j-1}x,0), \quad \forall x\in V. \qquad (3.7)\\
&= \frac{1}{2}\left[\frac{1}{16^{j-1}}\phi(2^{j-1}x,0)+\frac{1}{16^{j-2}}\phi(2^{j-2}x,0)+...+\frac{1}{16^{j-j}}\phi(2^{j-j}x,0)\right]\\
&= \frac{1}{2}\left[\phi(x,0)+\frac{1}{16}\phi(2x,0)+...+\frac{1}{16^{j-1}}\phi(2^{j-1}x,0)\right]\\
&\le \frac{1}{2}\Big[\phi(x,0)+\phi(x,0)\cdot L+\phi(x,0)\cdot L^2+...+\phi(x,0)\cdot L^{j-1}\Big]\\
&= \frac{1}{2}\phi(x,0)\left[1+L+L^2+...+L^{j-1}\right]\\
&\le \frac{1}{2}\phi(x,0)\left[1+L+L^2+...\right]\\
&= \frac{1}{2(1-L)}\phi(x,0), \quad \forall x\in V.
\end{aligned}$$

Now, for each $m,n\in\mathbb{N}$, we have

$$\begin{aligned}
\rho\left(\frac{f(2^nx)}{16^n}-\frac{f(2^mx)}{16^m}\right) &= \rho\left(\frac{2}{2}\cdot\frac{f(2^nx)}{16^n}-\frac{2}{2}\cdot f(x)+\frac{2}{2}\cdot f(x)-\frac{2}{2}\cdot\frac{f(2^mx)}{16^m}\right)\\
&= \rho\left(\frac{1}{2}\left(\frac{2f(2^nx)}{16^n}-2f(x)\right)+\frac{1}{2}\left(2f(x)-\frac{2f(2^mx)}{16^m}\right)\right)\\
&\le \frac{1}{2}\left[\rho\left(\frac{2f(2^nx)}{16^n}-2f(x)\right)+\rho\left(2f(x)-\frac{2f(2^mx)}{16^m}\right)\right]\\
&\le \frac{1}{2}\left[\frac{1}{8}\rho\left(\frac{f(2^nx)}{16^{n-1}}-16f(x)\right)+\frac{1}{8}\rho\left(16f(x)-\frac{f(2^mx)}{16^{m-1}}\right)\right]\\
&\le \frac{1}{2}\left[\frac{1}{8}\cdot\frac{1}{2(1-L)}\phi(x,0)+\frac{1}{8}\cdot\frac{1}{2(1-L)}\phi(x,0)\right]\\
&= \frac{1}{16(1-L)}\phi(x,0), \quad \forall x\in V.
\end{aligned}$$

By the definition of $\tilde{\rho}$, we conclude that

$$\tilde{\rho}(T^n f - T^m f) \leq \frac{1}{16(1-L)} < \infty,$$

which implies the boundedness of an orbit of T at $f$. According to Lemma 2.4, the sequence $\{T^n f\}$ $\rho$-converges to some element, say $w \in \mathcal{M}_{\tilde{\rho}}$.

Now, by the $\tilde{\rho}$-contractivity of $T$, one has

$$\tilde{\rho}(Tw - T^{n+1} f) \leq L\tilde{\rho}(w - T^n f).$$

Passing $n$ towards $\infty$ and apply the lower semicontinuity of $\tilde{\rho}$, we obtain that

$$\tilde{\rho}(Tw - w) \leq \liminf_{n\to\infty} \tilde{\rho}(Tw - T^{n+1} f) \leq \liminf_{n\to\infty} L\tilde{\rho}(w - T^n f) = 0.$$

Therefore, $w$ is a fixed point of $T$. □

Now, with the two lemmas above, we can finally give a simple proof to our main stability result, namely Theorem 3.2.

**Proof of Theorem 3.2.** Since $2^n x$ is in $V$ provided that $x \in V$, we deduce from (3.1) that

$$\begin{aligned} \rho\Big( & f\big(2^n(2x+y)\big) + f\big(2^n(2x-y)\big) - 4f\big(2^n(x+y)\big) - 4f\big(2^n(x-y)\big) \\ & -24f\big(2^n(x)\big) + 6f\big(2^n(y)\big)\Big) \\ \leq \; & \phi(2^n x, 2^n y), \quad \forall x, y \in V. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \rho\Big( & \frac{f\big(2^n(2x+y)\big)}{16^n} + \frac{f\big(2^n(2x-y)\big)}{16^n} - \frac{4f\big(2^n(x+y)\big)}{16^n} - \frac{4f\big(2^n(x-y)\big)}{16^n} \\ & -\frac{24f\big(2^n(x)\big)}{16^n} + \frac{6f\big(2^n(y)\big)}{16^n}\Big) \\ \leq \; & \frac{1}{16^n}\rho\Big( f\big(2^n(2x+y)\big) + f\big(2^n(2x-y)\big) - 4f\big(2^n(x+y)\big) - 4f\big(2^n(x-y)\big) \\ & -24f\big(2^n(x)\big) + 6f\big(2^n(y)\big)\Big) \\ \leq \; & \frac{\phi(2^n x, 2^n y)}{16^n}, \quad \forall x, y \in V. \end{aligned}$$

As from Lemma 3.4 and Remark 3, letting $n \longrightarrow \infty$ and apply the lower semicontinuity of $\rho$, we deduce that

$$w(2x+y) + w(2x-y) = 4w(x+y) + 4w(x-y) + 24w(x) - 6w(y), \quad \forall x, y \in V.$$

That is, $w$ is a quadratic mapping. Since every quadratic map is a fixed point of $T$, Lemma 3.4 also guarantee the uniqueness of $w$. On the other hand, it follows from inequality (3.7) that

$$\tilde{\rho}(w - f) \leq \frac{1}{16(1-L)}.$$

□

**Corollary 3.5.** *Let $V$ be a linear space, $X_\rho$ be a $\rho$-complete modular space where $\rho$ is l.s.c. and convex, and $f : V \to X_\rho$ be a mapping with $f(0) = 0$. If there exist a constant $\delta > 0$ such that*

$$\rho(f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)) \leq \delta, \quad \forall x, y \in V,$$

*then there exists a unique quadratic mapping $w : V \to X_\rho$ such that*

$$\rho(w(x) - f(x)) \leq \frac{\delta}{15}, \quad \forall x \in V.$$

For the next two corollaries, we shall consider the case where $\rho$ is actually a norm.

**Corollary 3.6.** *Let $V$ be a linear space, $(X, \cdot)$ be a Banach space, and $f : V \to X$ be a mapping with $f(0) = 0$. Suppose that for each $x, y \in V$, there holds the inequality*

$$f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) \leq \phi(x, y),$$

*where $\phi \in \Phi_{V,L}$ with $0 < L < 1$. Then, there exists a unique quartic mapping $w : V \to X$ such that*

$$w(x) - f(x) \leq \frac{1}{16(1-L)}\phi(x, 0), \quad \forall x \in V.$$

**Corollary 3.7.** *Let $V$ be a linear space, $(X, \cdot)$ be a Banach space, and $f : V \to X$ be a mapping with $f(0) = 0$. It there exists a constant $\delta > 0$ such that*

$$f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) \leq \delta, \quad \forall x, y \in V,$$

*then there exists a unique quartic mapping $w : V \to X$ such that*

$$w(x) - f(x) \leq \frac{\delta}{15}, \quad \forall x \in V.$$

## Concluding remarks

Our results guarantee the stability of quartic mappings, whose codomain is equipped with a convex and l.s.c. modular, in both generalized and original senses. Our proofs contain different techniques to avoid the usage of $\Delta_2$-conditions.

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