APPLICATION OF DOUBLE LAPLACE DECOMPOSITION METHOD FOR SOLVING SINGULAR ONE DIMENSIONAL SYSTEM OF HYPERBOLIC EQUATIONS

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Abstract. In this paper, the A domain decomposition methods and double Laplace transform methods are combined to solve linear and nonlinear singular one dimensional system of hyperbolic equations. In addition, we prove the convergence of double Laplace transform decomposition method applied to our problems. Furthermore, we illustrate our proposed methods using some examples.

1. introduction

Many applications in Sciences are modeled by linear and nonlinear partial differential equations. The hyperbolic partial differential equations as one of this applications arise in physical sciences as models of waves, such as acoustic, elastic, electromagnetic, or gravitational waves. However, it is very difficult to find explicit solutions of nonlinear partial differential equations generally. The Adomain decomposition method is the most transparent method for solutions of linear and nonlinear problem see [12, 13, 14, 18, 19] ; however, this method is involved in the calculation of complicated Adomain's polynomials which narrow down its application. Recently, many researchers and engineers have done excellent work, such as Laplace decomposition algorithm [7, 6]. The convergence of Adomian's method has been studied by several authors [8, 9, 10, 11]. The aim of this paper is to solve linear and nonlinear singular one dimensional system hyperbolic equations by using the combined domain decomposition techniques and double Laplace transform methods and also we study the sufficient condition of convergence of our methods. The main aim of this method is that it can be used directly without using restrictive assumptions or linearization. Now, we recall the following definitions which are given by [15, 16, 17]. The double Laplace transform is defined as

$$L_x L_t\left[f(x,t)\right] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)dt\,dx, \tag{1.1}$$

where $x, t > 0$ and p, s are complex values, and the double Laplace transform of the first order partial derivatives is given by

$$L_x L_t\left[\frac{\partial u(x,t)}{\partial x}\right] = pU(p,s) - U(0,s). \tag{1.2}$$

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Similarly, the double Laplace transform for second partial derivative with respect to x and t is defined as follows

$$\begin{aligned} L_xL_t\left[\frac{\partial^2 u(x,t)}{\partial^2 x}\right] &= p^2U(p,s)-pU(0,s)-\frac{\partial U(0,s)}{\partial x}, \\ L_xL_t\left[\frac{\partial^2 u(x,t)}{\partial^2 t}\right] &= s^2U(p,s)-sU(p,0)-\frac{\partial U(p,0)}{\partial t}. \end{aligned} \tag{1.3}$$

The following basic lemma of the double Laplace transform is given and is used in this paper.

Lemma 1. *Double Laplace transform of the non constant coefficient second order partial derivative $x^r\frac{\partial^2 u}{\partial t^2}$ and the function $x^r f(x,t)$ are given by*

$$L_xL_t\left(x^r\frac{\partial^2 u}{\partial t^2}\right)=(-1)^r\frac{d^r}{dp^r}\left[s^2U(p,s)-sU(p,0)-\frac{\partial U(p,0)}{\partial t}\right], \tag{1.4}$$

and

$$L_xL_t\left(x^r f(x,t)\right)=(-1)^r\frac{d^r}{dp^r}\left[L_xL_t\left(f(x,t)\right)\right]=(-1)^n\frac{d^rF\left(p,s\right)}{dp^r}. \tag{1.5}$$

where $r=1,2,3,...$

One can prove this lemma by using the definition of double Laplace transform in Eq.(1.1), Eq.(1.2) and Eq.(1.3). The main aim of this part is to discuss the use of modified double Laplace decomposition method for solving singular one dimensional coupled system of hyperbolic equations.

Statement of the problem: We consider a singular one dimensional system hyperbolic equations with initial conditions in the form:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}-\frac{1}{x}\left(x\frac{\partial u}{\partial x}\right)_x-v &= f\left(x,t\right), \\ \frac{\partial^2 v}{\partial t^2}-\frac{1}{x}\left(x\frac{\partial v}{\partial x}\right)_x-u &= g\left(x,t\right), \end{aligned} \tag{1.6}$$

subject to

$$\begin{aligned} u\left(x,0\right) &= f_1\left(x\right),\ \frac{\partial u\left(x,0\right)}{\partial t}=f_2\left(x\right), \\ v\left(x,0\right) &= g_1\left(x\right),\ \frac{\partial v\left(x,0\right)}{\partial t}=g_2\left(x\right), \end{aligned} \tag{1.7}$$

where, the linear term, $\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$ and $\frac{1}{x}\left(x\frac{\partial v}{\partial x}\right)_x$ are called Bessel's operators and $f\left(x,t\right),\ g\left(x,t\right),\ f_1\left(x\right),\ f_2\left(x\right),\ g_1\left(x\right)$ and $g_2\left(x\right)$ are known functions. In order to obtain the solution of Eq.(1.6), we use modified double Laplace decomposition methods as follow:

Step 1: Multiply both sides of Eq.(1.6) by x.

Step 2: Using Lemma 1 and definition of the double Laplace transform of partial derivatives for equations in step 1 and single Laplace transform for initial condition, we get

$$\begin{aligned} \frac{dU\left(p,s\right)}{dp} &= \frac{1}{s}\frac{dF_1\left(p\right)}{dp}+\frac{1}{s^2}\frac{dF_2\left(p\right)}{dp}+\frac{1}{s^2}\frac{dF\left(p,s\right)}{dp} \\ &\quad -\frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+xv\right], \end{aligned} \tag{1.8}$$

and

$$\begin{aligned}\frac{dV(p,s)}{dp} &= \frac{1}{s}\frac{dG_1(p)}{dp}+\frac{1}{s^2}\frac{dG_2(p)}{dp}+\frac{1}{s^2}\frac{dG(p,s)}{dp}\\ &-\frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right)+xu\right],\end{aligned} \tag{1.9}$$

where $F_1(p)$, $F_2(p)$, $F(p,s)$, $G_1(p)$, $G_2(p)$and $G(p,s)$ are Laplace transform of the functions $f_1(x)$, $f_2(x)$, $f(x,t)$, $g_1(x)$, $g_2(x)$ and $g(x,t)$ respectively.
Step 3: By integrating both sides of Eq.(1.8) and Eq.(1.9) from 0 to p with respect to p, we have

$$\begin{aligned}U(p,s) &= \frac{F_1(p)}{s}+\frac{F_2(p)}{s^2}+\frac{F(p,s)}{s^2}-\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+xv\right]dp,\\ V(p,s) &= \frac{G_1(p)}{s}+\frac{G_2(p)}{s^2}+\frac{G(p,s)}{s^2}\\ &-\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right)+xu\right]dp,\end{aligned} \tag{1.10}$$

Step 4: Using double Laplace a domain decomposition methods to define the solution of the system as $u(x,t)$ and $v(x,t)$ by infinite series as follows:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \quad v(x,t)=\sum_{n=0}^{\infty}v_n(x,t), \tag{1.11}$$

Step 5: Operating the inverse double Laplace transform for both sides of Eq.(1.10) and use Eq.(1.11), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}u_n(x,t) &= f_1(x)+tf_2(x)+L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right]\\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n\right)\right]dp\right]\\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\int_0^p\left(x\sum_{n=0}^{\infty}v_n\right)dp\right]\right],\end{aligned} \tag{1.12}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}v_n(x,t) &= g_1(x)+tg_2(x)+L_p^{-1}L_s^{-1}\left[\frac{G(p,s)}{s^2}\right]\\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}v_n\right)\right]dp\right]\\ &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\int_0^p\left(x\sum_{n=0}^{\infty}u_n\right)dp\right]\right].\end{aligned} \tag{1.13}$$

In particular, we have

$$\begin{aligned} u_0 &= f_1(x) + t f_2(x) + L_p^{-1} L_s^{-1}\left[\frac{F(p,s)}{s^2}\right] \\ v_0 &= g_1(x) + t g_2(x) + L_p^{-1} L_s^{-1}\left[\frac{G(p,s)}{s^2}\right], \end{aligned} \tag{1.14}$$

and the rest terms can be written as follows

$$\begin{aligned} u_{n+1} &= -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} u_n\right)\right] dp\right] \\ &\quad -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2} L_x L_t\left[\int_0^p \left(x\sum_{n=0}^{\infty} v_n\right) dp\right]\right], \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} v_{n+1} &= -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} v_n\right)\right] dp\right] \\ &\quad -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2} L_x L_t\left[\int_0^p \left(x\sum_{n=0}^{\infty} u_n\right) dp\right]\right]. \end{aligned} \tag{1.16}$$

where $L_x L_t$ is double Laplace transform with respect to x, t and double inverse Laplace transform is denoted by $L_p^{-1} L_s^{-1}$ with respect to p, s. Here, we provide double inverse Laplace transform with respect to p and s exist, for each terms in the right hand side of Eqs.(1.14), (1.15) and (1.16). In order to confirm our method for solving the singular one dimensional coupled hyperbolic equations, we consider the following example:

Example 1. *Consider the following nonhomogeneous form of a singular one dimensional system of hyperbolic equations:*

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{1}{x}\left(x\frac{\partial u}{\partial x}\right)_x - v &= -x^2\sin t - 4\sin t - x^2\cos t, \\ \frac{\partial^2 v}{\partial t^2} - \frac{1}{x}\left(x\frac{\partial v}{\partial x}\right)_x - u &= -x^2\cos t - 4\cos t - x^2\sin t, \end{aligned} \tag{1.17}$$

subject to the initial condition

$$u(x,0) = 0,\ \frac{\partial u(x,0)}{\partial t} = x^2,\quad v(x,0) = x^2,\ \frac{\partial v(x,0)}{\partial t} = 0, \tag{1.18}$$

By using the above steps, we obtain

$$\begin{aligned} u_{n+1} &= x^2\sin t + 4\sin t + x^2\cos t - 4t - x^2 \\ &\quad -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\left(x\left(\sum_{n=0}^{\infty} u_{nx}(x,t)\right)_x\right)_x\right] dp\right] \\ &\quad -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\left(\sum_{n=0}^{\infty} v_{nx}(x,t)\right)\right] dp\right] \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} v_{n+1} &= x^2\cos t+4\cos t+x^2\sin t-4-x^2t \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\left(x\left(\sum_{n=0}^{\infty}v_{nx}(x,t)\right)_x\right)_x\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\left(\sum_{n=0}^{\infty}u_{nx}(x,t)\right)\right]dp\right] \end{aligned} \tag{1.20}$$

By using Eqs.(1.14), (1.15)and (1.16) the components are given by

$$\begin{aligned} u_0 &= x^2\sin t+4\sin t+x^2\cos t-4t-x^2 \\ v_0 &= x^2\cos t+4\cos t+x^2\sin t-4-x^2t \end{aligned}$$

and

$$\begin{aligned} u_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_0\right)+xv_0\right]dp\right] \\ &= L_p^{-1}L_s^{-1}\left[\frac{4}{ps^2(s^2+1)}+\frac{8}{ps(s^2+1)}+\frac{2}{p^3s(s^2+1)}+\frac{4}{p^3s^2(s^2+1)}-\frac{8}{ps^3}-\frac{2}{p^3s^4}\right] \\ u_1 &= 4t-4\sin t-8\cos t+x^2-x^2\cos t+8+x^2+x^2t-x^2\sin t-4t^2-\frac{1}{6}x^2t^3 \end{aligned}$$

and

$$\begin{aligned} v_1 &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v_0\right)+xu_0\right]dp\right] \\ v_1 &= 4-8\sin t-4\cos t+x^2-x^2\cos t+8t+x^2+x^2t-x^2\sin t-\frac{4}{3}t^3-\frac{1}{2}x^2t^2 \end{aligned}$$

In the same manner, we obtain that

$$\begin{aligned} u_2 &= 4t^2+8\cos t-8+12\sin t-12t+2t^3-\frac{1}{10}t^5 \\ &\quad +\frac{1}{2}x^2t^2-x^2t+x^2\cos t-x^2+x^2\sin t+\frac{1}{6}x^2t^3-\frac{1}{24}x^2t^4 \end{aligned}$$

and

$$\begin{aligned} v_2 &= \frac{4}{3}t^3+8\sin t-8t+12\cos t-12+6t^2-\frac{1}{2}t^4 \\ &\quad +\frac{1}{2}x^2t^2-x^2t+x^2\cos t-x^2+x^2\sin t+\frac{1}{6}x^2t^3-\frac{1}{120}x^2t^5 \end{aligned}$$

It is obvious that the self-cancelling some terms appear between various components and the connected by coming terms, then we have,

$$u(x,t)=u_0+u_1+u_2+...,\quad v(x,t)=v_0+v_1+v_2+...$$

Therefore, the exact solution is given by

$$u(x,t)=x^2\sin t \quad \textit{and } v(x,t)=x^2\cos t$$

2. Singular nonlinear one dimensional system of hyperbolic equations

In this section, we discuss the use of modified double Laplace to solve the singular nonlinear one dimensional system of hyperbolic equations which is given by

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}-\frac{1}{x}\left(x\frac{\partial u}{\partial x}\right)_x-v\frac{\partial u}{\partial x} &= f(u), \\ \frac{\partial^2 v}{\partial t^2}-\frac{1}{x}\left(x\frac{\partial v}{\partial x}\right)_x-u\frac{\partial v}{\partial x} &= g(u), \end{aligned} \tag{2.1}$$

subject to Eq.(1.7) where $f(u)$ and $g(u)$ are nonlinear functions. To obtain the solution of Eq.(2.1) we apply our method as follows.
Step 1: Multiplying equation Eq.(2.1) by x

Step 2: Using Lemma 1 and definition of the double Laplace transform of partial derivatives for equations in step 1 and single Laplace transform for initial condition.
Step 3: Integrating the obtained equations with respect to p, from 0 to p

Step 4: Operating the inverse double Laplace transform for equations. We obtain

$$\begin{aligned} u(x,t) &= f_1(x)+tf_2(x)-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[xf(u)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\int_0^p\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+xv\frac{\partial u}{\partial x}dp\right]\right], \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v(x,t) &= g_1(x)+tg_2(x)-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[xg(u)\right]dp\right] \\ &\quad -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}L_xL_t\left[\int_0^p\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right)+xu\frac{\partial v}{\partial x}dp\right]\right]. \end{aligned} \tag{2.3}$$

The modified double Laplace decomposition methods (MDLDM) which define the solution of the singular one dimensional system of hyperbolic equations that can be represented as a power series are defined by Eq.(1.11). The nonlinear operators can be defined as follows

$$N_1=\sum_{n=0}^{\infty}A_n, \text{ and } N_2=\sum_{n=0}^{\infty}B_n, \tag{2.4}$$

where A_n and B_n are given by:

$$\begin{aligned} A_n &= \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{i=0}^{\infty}(\lambda^n u_n)\right]\right)_{\lambda=0}, \\ B_n &= \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2\sum_{i=0}^{\infty}(\lambda^n v_n)\right]\right)_{\lambda=0}. \end{aligned} \tag{2.5}$$

Here, A domain's polynomials A_n and B_n are given by:

$$\begin{aligned} A_0 &= v_0u_{0x} \\ A_1 &= v_0u_{1x}+v_1u_{0x} \\ A_2 &= v_0u_{2x}+v_1u_{1x}+v_2u_{0x}. \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} B_0 &= u_0 v_{0x} \\ B_1 &= u_0 v_{1x} + u_1 v_{0x} \\ B_2 &= u_0 v_{2x} + u_1 v_{1x} + u_2 v_{0x}. \end{aligned} \tag{2.7}$$

By substituting Eqs.(2.4) and (2.5) into Eqs.(2.2) and (2.3) we obtain

$$u_0 = f_1(x) + t f_2(x), \quad v_0 = g_1(x) + t g_2(x), \tag{2.8}$$

and the rest terms can be written as follows

$$\begin{aligned} u_{n+1} &= -L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[x f\left(\sum_{n=0}^{\infty} u_n \right) \right] dp \right] \\ &\quad - L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial}{\partial x} \left(x \frac{\partial}{\partial x} \sum_{n=0}^{\infty} u_n \right) \right] dp \right] \\ &\quad - L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} L_x L_t \left[\int_0^p \left(x \sum_{n=0}^{\infty} A_n \right) dp \right] \right], \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} v_{n+1} &= -L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[x g\left(\sum_{n=0}^{\infty} u_n \right) \right] dp \right] \\ &\quad - L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial}{\partial x} \left(x \frac{\partial}{\partial x} \sum_{n=0}^{\infty} v_n \right) \right] dp \right] \\ &\quad - L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} L_x L_t \left[\int_0^p \left(x \sum_{n=0}^{\infty} B_n \right) dp \right] \right]. \end{aligned} \tag{2.10}$$

where $L_x L_t$ is double Laplace transform with respect to x, t and double inverse Laplace transform is denoted by $L_p^{-1} L_s^{-1}$ with respect to p, s. Here, we provide double inverse Laplace transform with respect to p and s exist for each terms in the right hand side of Eqs.(2.9) and (2.10).

3. Convergence analysis of the method

In this part, we will discuss the convergence analysis of the modified double Laplace decomposition methods for the singular nonlinear one dimensional system of hyperbolic equations which is given by

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{1}{x} \left(x \frac{\partial u}{\partial x} \right)_x - v \frac{\partial u}{\partial x} &= f(u), \\ \frac{\partial^2 v}{\partial t^2} - \frac{1}{x} \left(x \frac{\partial v}{\partial x} \right)_x - u \frac{\partial v}{\partial x} &= g(u), \end{aligned} \tag{3.1}$$

We propose to extend this idea given in [4], for all $u, v \in H$. We define H as $H = L^2_\mu((a,b)\times\ [0,T])$, where $a \ll 0$ and

$$\begin{aligned} u &: (a,b)\times[0,T] \to \mathbb{R}\times\mathbb{R}, \text{ with } \|u\|_H^2 = \int_Q xu^2(x,t)\,dxdt \\ (u,v) &= \int_Q xu(x,t)\,v(x,t)\,dxdt, \end{aligned}$$

where $Q = (a,b)\times[0,T]$ and

$$H = \left\{ \begin{array}{c} (u,v) : (a,b)\times[0,T], \text{ with} \\ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[u(x,t)\right](p,s)\,dp\right](x,t) < \infty \end{array} \right\}.$$

Multiplying both sides of Eq.(3.1) by x, and write the equation in the operator form as follows:

$$\begin{aligned} L(u) &= x\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + xv\frac{\partial u}{\partial x} + xf(u), \\ L(v) &= x\frac{\partial^2 v}{\partial t^2} = \frac{\partial v}{\partial x} + x\frac{\partial^2 v}{\partial x^2} + xu\frac{\partial v}{\partial x} + xg(v), \end{aligned} \tag{3.2}$$

where $|x| \le b$. For L is hemicontinuous operator, consider the following hypothesis:
1.(H1) $(L(u) - L(w), u - w) \ge k\|u-w\|^2$ and $(L(v) - L(w), v - w) \ge k\|v-w\|^2$; $k > 0, \forall u, v, w \in H$
2.(H2) whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, w \in H$ with $\|u\| \le M, \|v\| \le M, \|w\| \le M$ we have:

$$(L(u) - L(w), z) \le C(M)\|u-z\|\|w\|, \text{ and } (L(v) - L(z), w) \le C(M)\|v-z\|\|w\|$$

for every $w, z \in H$. In the next Theorem we follows [3, 1, 2].

Theorem 1. *(Sufficient condition of convergence) The Modified double Laplace decomposition methods applied to the singular nonlinear one dimensional system of hyperbolic equations Eq.(3.2) without initial and boundary conditions, converges towards a particular solution.*

Proof. First, we verify the convergence hypothesis H1 for the operator $L(u)$, $L(v)$ of Eq.(3.2). we use the definition of our operator L, and then we have

$$\begin{aligned} L(u) - L(w) &= \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}\right) + \left(x\frac{\partial^2 u}{\partial x^2} - x\frac{\partial^2 w}{\partial x^2}\right) \\ &\quad + \left(xv\frac{\partial u}{\partial x} - xv\frac{\partial w}{\partial x}\right) + x(f(u) - f(w)) \\ &= \frac{\partial}{\partial x}(u-w) + x\frac{\partial^2}{\partial x^2}(u-w) \\ &\quad + xv\frac{\partial}{\partial x}(u-w) + x(f(u) - f(w)), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
L(v) - L(w) &= \left(\frac{\partial v}{\partial x} - \frac{\partial w}{\partial x}\right) + \left(x\frac{\partial^2 v}{\partial x^2} - x\frac{\partial^2 w}{\partial x^2}\right) \\
&\quad + \left(xu\frac{\partial v}{\partial x} - xu\frac{\partial w}{\partial x}\right) + x\left(g(v) - f(w)\right) \\
&= \frac{\partial}{\partial x}(v - w) + x\frac{\partial^2}{\partial x^2}(v - w) \\
&\quad + xu\frac{\partial}{\partial x}(v - w) + x\left(g(v) - f(w)\right), \qquad (3.4)
\end{aligned}$$

therefore,

$$\begin{aligned}
\left(L(u) - L(w), u - w\right) &= \left(\frac{\partial}{\partial x}(u - w), u - w\right) \\
&\quad + \left(x\frac{\partial^2}{\partial x^2}(u - w), u - w\right) \\
&\quad + \left(xv\frac{\partial}{\partial x}(u - w), u - w\right) \\
&\quad + \left(x\left(f(u) - f(w)\right), u - w\right). \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\left(L(v) - L(w), v - w\right) &= \left(\frac{\partial}{\partial x}(v - w), v - w\right) \\
&\quad + \left(x\frac{\partial^2}{\partial x^2}(v - w), v - w\right) \\
&\quad + \left(xu\frac{\partial}{\partial x}(v - w), v - w\right) \\
&\quad + \left(x\left(g(v) - f(w)\right), v - w\right). \qquad (3.6)
\end{aligned}$$

According to the coercive operator the differential operator $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ in H , then there exists constants $\alpha, \beta, \delta > 0$ such that

$$\left(\frac{\partial}{\partial x}(u - w), u - w\right) \geq \alpha \left\|u - w\right\|^2, \qquad (3.7)$$

and

$$\begin{aligned}
-\left(x\frac{\partial^2}{\partial x^2}(u - w), u - w\right) &\leq |x| \left\|\frac{\partial^2}{\partial x^2}(u - w)\right\| \left\|u - w\right\| \\
&\leq b\beta \left\|u - w\right\|^2, \\
&\Leftrightarrow \\
\left(x\frac{\partial^2}{\partial x^2}(u - w), u - w\right) &\geq -b\beta \left\|u - w\right\|^2 \qquad (3.8)
\end{aligned}$$

where $|x| \leq b$, and $\|u\| \leq M$, $\|v\| \leq M, \|w\| \leq M$, and according to the Schwartz inequality, we get

$$\begin{aligned} -\left(xv\frac{\partial}{\partial x}(u-w), u-w \right) &\leq |x|\,\|v\| \left\| \frac{\partial}{\partial x}(u-w) \right\| \|u-w\| \\ &\leq bM\delta \|u-v\| \|u-v\| \\ &\leq bM\delta \|u-w\|^2 \\ &\leq b\delta M \|u-w\|^2 \end{aligned}$$

hence,

$$\left(xv\frac{\partial}{\partial x}(u-w), u-w \right) \geq -b\delta M \|u-w\|^2 . \tag{3.9}$$

By using cauchy schwarz inequality, where $\sigma > 0$ and f is Lipschitzian function, we have

$$\begin{aligned} (-x(f(u)-f(w)), u-w) &\leq |x|\,\|f(u)-f(w)\| \|u-w\| \\ &\leq b\|f(u)-f(w)\| \|u-w\| \\ &\leq b\sigma \|u-w\|^2 \Leftrightarrow \\ (x(f(u)-f(w)), u-v) &\geq -b\sigma \|u-w\|^2 , \end{aligned} \tag{3.10}$$

substituting Eq.(3.7), Eq.(3.8), Eq.(3.9) and Eq.(3.10) into equation Eq.(3.5) give

$$\begin{aligned} (L(u)-L(w), u-w) &\geq (\alpha - b\beta - b\delta M - b\sigma)\|u-w\|^2 \\ (L(u)-L(w), u-w) &\geq k\|u-w\|^2 . \end{aligned}$$

where

$$k = \alpha - b\beta - b\delta M - b\sigma > 0.$$

By the same method for Eq.(3.6) there exists constants $\zeta, \eta,\ \lambda, \rho > 0$ we obtain that

$$\begin{aligned} (L(v)-L(w), v-w) &\geq (\zeta - b\eta - b\lambda M - b\rho)\|v-w\|^2 \\ (L(v)-L(w), v-w) &\geq k_1\|v-w\|^2 , \end{aligned}$$

where

$$k_1 = \zeta - b\eta - b\lambda M - b\rho > 0.$$

So the hypothesis (H1) holds. Now we verify the convergence hypotheses (H2) for the operator $L(u)$ and $L(v)$ for every $M > 0$, there exist a constant $C(M) > 0$ such that for $u, v, w \in H$ with $\|u\| \leq M$, $\|v\| \leq M$,

$$(L(u)-L(w), z_1) \leq C(M)\|u-w\| \|z_1\| ,$$

for every $z_1, z_2 \in H$. For that we have,

$$\begin{aligned} (L(u)-L(w), z_1) &= \left(\frac{\partial}{\partial x}(u-w), z_1 \right) \\ &\quad + \left(x\frac{\partial^2}{\partial x^2}(u-w), z_1 \right) \\ &\quad + \left(xv\frac{\partial}{\partial x}(u-w), z_1 \right) \\ &\quad + (x(f(u)-f(w)), z_1) . \end{aligned}$$

By using the cauchy Schwartz inequality and the fact that u and w are bounded, we obtain the following:

$$\left(\frac{\partial}{\partial x}(u-w), z_1\right) \leq \alpha_1 \|u-w\| \|z_1\|,$$

$$\left(x\frac{\partial^2}{\partial x^2}(u-w), z_1\right) \leq b\beta_1 \|u-w\| \|z_1\|,$$

$$\begin{aligned} \left(xv\frac{\partial}{\partial x}(u-w), z_1\right) &\leq \alpha_2 |x| \|v\| \|u-w\| \|z_1\| \\ &\leq b\alpha_2 M \|u-w\| \|z_1\|, \end{aligned}$$

and

$$(x(f(u)-f(w)), z_1) \leq b\sigma_1 \|u-w\| \|z_1\|$$

where $|x| \leq b$ and the constants $\alpha_1, \beta_1, \alpha_2, \sigma_1 > 0$, we have:

$$\begin{aligned} (L(u)-L(w), z_1) &\leq (\alpha_1 + b\beta_1 + b\alpha_2 M + b\sigma_1) \|u-w\| \|z_1\| \\ &= C(M) \|u-w\| \|z_1\|, \end{aligned}$$

where

$$C(M) = (\alpha_1 + b\beta_1 + b\alpha_2 M + b\sigma_1),$$

and

$$\begin{aligned} (L(v)-L(w), z_2) &= \left(\frac{\partial}{\partial x}(v-w), z_2\right) \\ &\quad + \left(x\frac{\partial^2}{\partial x^2}(v-w), z_2\right) \\ &\quad + \left(xu\frac{\partial}{\partial x}(v-w), z_2\right) \\ &\quad + (x(g(v)-f(w)), z_2). \end{aligned}$$

Similarly, we get,

$$\begin{aligned} (L(v)-L(w), z_2) &\leq (\zeta_1 + b\eta_1 + b\lambda_1 M + b\rho_1) \|v-w\| \|z_2\| \\ &= C(M) \|v-w\| \|z_2\|, \end{aligned}$$

where $C(M) = \zeta_1 + b\eta_1 + b\lambda_1 M + b\rho_1$ and $\zeta_1, \eta_1, \lambda_1, \rho_1 > 0$,have therefore (H2) holds. This completes the proof. □

Conclusion 1. *In this work, first a double Laplace transform algorithm which is based on the Adomian decomposition method is used for solving the linear and nonlinear singular one dimensional system of hyperbolic equations. Second, we presente a convergence proof of the (DLADM) applied to the nonlinear singular one dimensional system of hyperbolic equations.*

Competing interests

The author declare that they have no competing interests.

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