Multidimensional Backward Doubly Stochastic Differential Equations with Integral Non-Lipschitz Coefficients

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Abstract: The paper is devoted to solving multidimensional backward doubly stochastic differential equations under integral non-Lipschitz conditions in general spaces. By stochastic analysis and constructing approximation sequence, a new set of sufficient conditions for multidimensional backward doubly stochastic differential equations is obtained. The results generalize the recent results on this issue. Finally, an example is given to illustrate the advantage of the main results.

***Keywords*:** Backward doubly stochastic differential equations; Existence and uniqueness; Integral non-Lipschitz

2010 Mathematics Subject Classification. Primary 60H10, 60H30, 60H99.

1 Introduction

Motivated by the probabilistic interpretation of solutions to a class of quasilinear parabolic partial differential equations(PDEs in short), Pardoux and Peng [1] introduced nonlinear backward stochastic differential equations(BSDEs in short). In the past decades, the theory of BSDEs have been extensively developed and gradually become an important tool in financial problems [2, 3], stochastic control [4] and stochastic games [5] and so on. One highlight of the theory is relaxing the conditions of existence and uniqueness of the solutions. Mao [6] has proved the existence and uniqueness of the multidimensional BSDEs with non-Lipschitz coefficients. Lepeltier and San Martin [7] have relaxed the generator with continuous conditions. S. Hamadène [9] investigated the existence of the multidimensional BSDEs where the generator satisfies uniformly continuous conditions. Recently, Fan *et al.* [7] discussed the existence and uniqueness of the multidimensional BSDEs with Osgood hypothesis where the method is different from [6]. Hu and Tang [10] studied the same problem with diagonally quadratic generators.

In 1994, Pardoux and Peng [11] studied the backward doubly stochastic differential equations(BDSDEs in short). They proved the existence and uniqueness under Lipschitz conditions, and also, discussed the probabilistic representation of solution of quasilinear stochastic PDEs. Furthermore, Shi *et al.* [12] obtained the existence of the BDSDEs with continuous coefficients.

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Lin [13, 14] made further efforts to establish the existence or uniqueness of solutions with non-Lipschitz. Even recently, Wang *et al.* [15] obtained the result where the first generator satisfied Osgood hypothesis, the second non-Lipschitz conditions. As a matter of fact, those results are obtained which the generators are uniform on t. To the best of our knowledge, the multidimensional BDSDEs with generators of integral non-Lipschitz assumptions in general spaces have rarely been reported.

The structure of this paper is organized as follows. In section 2, we present some basic notions and assumptions which will be needed in the sequel. Section 3 is devoted to investigating the existence and uniqueness of solutions for multidimensional BDSDEs in general space. Finally, we give an example to show the effectiveness of the main result.

2 Notations

Let $T > 0$ be a fixed terminal time. $|\cdot|$ denotes the Eulclidean norm of $\mathbb{R}^k$, $< x, y >$ denotes the inner product of $x, y \in \mathbb{R}^k$. For any $z \in \mathbb{R}^{k\times d}$, its norm is defined by $\|z\| = \sqrt{Trace(zz^*)}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{B_t,\}_{t\in[0,T]}$ and $\{W_t,\}_{t\in[0,T]}$ are two mutually independent standard Brownian motions with values respectively in $\mathbb{R}^l$ and $\mathbb{R}^d$. $\mathcal{N}$ denotes the totality of $P-$null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_{0,t}^W \bigvee \mathcal{F}_t^B,$$

where for any process η_t, $\mathcal{F}_{0,t}^{\eta} = \sigma\{\eta_r - \eta_s, s \le r \le t\} \vee \mathcal{N}$, $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$. For a deterministic integrable function $a(t)$, we define $A(t) = \int_0^t a^2(s)\mathrm{d}s$.

Let us introduce some spaces for $\beta > 0$ which will be carried out in the following parts.

• $L^2(\beta, a, T, \mathbb{R})$ denotes the set of all $\mathcal{F}_T-$ measurable $\mathbb{R}^k-$ valued random variable ξ such that

$$\|\xi\|^2 = \mathbb{E}(e^{\beta A(T)}|\xi|^2) < +\infty.$$

• $L^2(\beta, a)$ denotes the collection of the $\mathcal{F}_t-$ adapted, $\mathbb{R}^k-$valued continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y_t\|_\beta^2 = \mathbb{E}\int_0^T e^{\beta A(t)}|Y_t|^2\mathrm{d}t < +\infty.$$

• $L^{2,a}(\beta, a)$ denotes the set of the $\mathcal{F}_t-$ adapted, $\mathbb{R}^k-$valued continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y_t\|_{\beta,a}^2 = \mathbb{E}\int_0^T e^{\beta A(t)}a^2(t)|Y_t|^2\mathrm{d}t < +\infty.$$

• $\mathcal{S}^2(\beta, a)$ denotes the space of the $\mathcal{F}_t-$ adapted, $\mathbb{R}^k-$valued continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y_t\|_{\mathcal{S}^2}^2 = \mathbb{E}(\sup_{t\in[0,T]} e^{\beta A(t)}|Y_t|^2) < +\infty.$$

• $\mathcal{M}^2(\beta, a)$ denotes the space of the $\mathcal{F}_t-$ adapted, $\mathbb{R}^{k\times d}-$valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z_t\|_{\mathcal{M}^2}^2 = \mathbb{E}\int_0^T e^{\beta A(t)}\|Z_t\|^2\mathrm{d}t < +\infty.$$

- $\mathcal{M}^{2,a} := L^{2,a}(\beta,a)\times\mathcal{M}^2(\beta,a)$ denotes the Banach space with the norm

$$\|Y,Z\|_\beta^2=\|Y\|_{\beta,a}^2+\|Z\|_{\mathcal{M}^2}^2.$$

- $\mathcal{M}^{2,c} := (\mathcal{S}^2(\beta,a)\bigcap L^{2,a}(\beta,a))\times\mathcal{M}^2(\beta,a)$ denotes the Banach space with the norm

$$\|Y,Z\|_{\beta,c}^2=\|Y\|_{\mathcal{S}^2}^2+\|Y\|_{\beta,a}^2+\|Z\|_{\mathcal{M}^2}^2.$$

In this paper, we consider the backward doubly stochastic differential equations

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\mathrm{d}s+\int_t^T g(s,Y_s,Z_s)\mathrm{d}B_s-\int_t^T Z_s\mathrm{d}W_s,\ t\in[0,T], \tag{2.1}$$

where the integral with respect to B_t is the classical backward Itô integral and the integral with respect to W_t is standard forward Itô integral. The equations are often abbreviated by BDSDEs(ξ,f,g).

With the above preparations, we introduce the definition of solution of (2.1).

Definition 1 *A pair of process $(Y_t,Z_t)_{t\in[0,T]}\in\mathcal{M}^{2,c}$ is a solution to (2.1), if it satisfies (2.1).*

In order to get the solution of (2.1), we propose the following assumptions:

(H1) The terminal value $\xi\in L^2(\beta,a,T,\mathbb{R})$;

(H2) (i) The coefficients $f:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k, g:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^{k\times l}$, are progressively measurable for any $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$ such that $\frac{f(\cdot,0,0)}{a(\cdot)},\ g(\cdot,0,0)\in L^2(\beta,a)$.

(ii) There exist some integrable functions $p(t),\ q(t),\ u(t):[0,T]\to\mathbb{R}^+$ such that for any $t\in[0,T],\ y_1,\ y_2\in\mathbb{R}^k, z_1,z_2\in\mathbb{R}^{k\times d}$,

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|\le p(t)\rho(|y_1-y_2|)+q(t)\|z_1-z_2\|,$$

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|^2\le p(t)|y_1-y_2|\rho(|y_1-y_2|)+u(t)\|z_1-z_2\|^2,$$

where $\rho(x)$ is a concave and nondecreasing function with $\rho(0)=0$ and $\int_{0^+}\frac{\mathrm{d}u}{\rho(u)}=+\infty$.

(iii) There exists a constant $0<\alpha<1$ such that $u(t)\le\alpha$, for all $t\ge 0$.

Remark 1 *For the above given spaces, if $a(t)=C$, C is a nonnegative constant, we can easily find that the spaces degenerate into the classical spaces.*

Before giving our main results, we introduce some technical tools. The first Lemma appears in [17].

Lemma 1 *If $\rho(u)$ is a concave and nondecreasing function with $\rho(0)=0$ and $\int_{0^+}\frac{\mathrm{d}u}{\rho(u)}=+\infty$, there exists a concave nondecreasing function $\phi(u)$ with $\phi(0)=0$ and $\int_{0^+}\frac{\mathrm{d}u}{\phi(u)}=+\infty$, moreover, $a\sqrt{u}\rho(\sqrt{u})\le\phi(u)\le 2a\sqrt{u}\rho(\sqrt{u})$, where $a>0$ is a constant.*

Lemma 2 *Assume that the generator f satisfies (H2). Let $f^{(i)}$ denote the ith component of the generator f, we define a series of functions $f_n=(f_n^{(1)},f_n^{(2)},\cdots,f_n^{(k)})$ with $f_n^{(i)}$ as follows*

$$f_n^{(i)}(t,y,z)=\inf_{u\in\mathbb{R}^k}\{f^{(i)}(t,u,z)+(n+A)p(t)|y-u|\}.$$

Then, it satisfies

(i) For any $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$, $|f_n(t,y,z)-f(t,y,z)|\le kp(t)\rho(\frac{2A}{n})$.

(ii) For any $(y_i,z_i)\in\mathbb{R}^k\times\mathbb{R}^{k\times d},\ i=1,\ 2,$ *we have*

$$|f_n(t,y_1,z_1)-f_n(t,y_2,z_2)|\le k(n+A)[p(t)|y_1-y_2|+q(t)||z_1-z_2||],$$
$$|f_n(t,y_1,z_1)-f_n(t,y_2,z_2)|\le kp(t)\rho(|y_1-y_2|)+kq(t)||z_1-z_2||.$$

(iii) $\frac{f_n(t,0,0)}{a(t)}\in L^{2,a}(\beta,a)$.

Proof: The proof is similar to the process step 1 of Theorem 1 in [9], we omit it.

3 Existence and uniqueness

In this section, we begin with establishing a priori estimate on the solutions of (2.1). Because $\rho(x)$ is a concave, there exists a nonnegative constant A such that $\rho(x)\le A(x+1)$. Furthermore, we let $a^2(t)=p(t)+q^2(t)$ in following parts.

Proposition 1 *Assume that (H1) and (H2) hold,* (Y_t,Z_t) *be a solution of (2.1), then for enough large* β, *there exists a constant* $d_{\beta,T}$ *which depends on* β *and* T *such that*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le s\le T}e^{\beta A(s)}|Y_s|^2\right]+\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{ds}+\mathbb{E}\int_{\mathrm{t}}^{\mathrm{T}}\mathrm{e}^{\beta\mathrm{A(s)}}||\mathrm{Z_s}||^2\mathrm{ds}\\&\le d_{\beta,T}\left\{\mathbb{E}e^{\beta A(T)}|\xi|^2+\mathbb{E}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+\mathbb{E}\int_t^T e^{\beta A(s)}|g(s,0,0)|^2\mathrm{d}s+\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\phi(|Y_s|^2)\mathrm{d}s\right\}.\end{aligned}$$

Proof: Applying Itô formula to $e^{\beta A(t)}|Y_t|^2$ yields that, for any $t\in[0,T]$

$$\begin{aligned}e^{\beta A(t)}|Y_t|^2 \quad + \quad &\beta\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{ds}+\int_{\mathrm{t}}^{\mathrm{T}}\mathrm{e}^{\beta\mathrm{A(s)}}||\mathrm{Z_s}||^2\mathrm{ds}\\&=e^{\beta A(T)}|\xi|^2+2\int_t^T e^{\beta A(s)}Y_sf(s,Y_s,Z_s)\mathrm{ds}+2\int_t^T e^{\beta A(s)}Y_sg(s,Y_s,Z_s)\mathrm{d}B_s\\&+\int_t^T e^{\beta A(s)}|g(s,Y_s,Z_s)|^2\mathrm{ds}-2\int_t^T e^{\beta A(s)}Y_sZ_s\mathrm{d}W_s.\end{aligned}\tag{3.1}$$

Following the assumptions (H1) and (H2), elementary inequality, we have

$$\begin{aligned}2Y_tf(t,Y_t,Z_t)&=2Y_t[f(t,Y_t,Z_t)-f(t,Y_t,0)+f(t,Y_t,0)-f(t,0,0)+f(t,0,0)]\\&\le 2|Y_t||f(t,Y_t,Z_t)-f(t,Y_t,0)|+2|Y_t||f(t,Y_t,0)-f(t,0,0)|+2|Y_t||f(t,0,0)|\\&\le(\frac{2}{1-\alpha}+\frac{\beta}{8})a^2(t)|Y_t|^2+\frac{1-\alpha}{2}||Z_t||^2+a^2(t)\phi(|Y_t|^2)+\frac{8}{\beta a^2(t)}|f(t,0,0)|^2, \quad (3.2)\\|g(t,Y_t,Z_t)|^2&=|g(t,Y_t,Z_t)-g(t,0,0)+g(t,0,0)|^2\\&\le\alpha(1+\frac{1}{\gamma})||Z_t||^2+(1+\frac{1}{\gamma})a^2(t)\phi(|Y_t|^2)+(1+\gamma)|g(t,0,0)|^2, \quad (3.3)\end{aligned}$$

where γ is a nonnegative constant.

Take expectation on both sides of (3.1), by (3.2) and (3.3), we have

$$\begin{aligned}
&\mathbb{E}e^{\beta A(t)}|Y_t|^2+\beta\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s+\mathbb{E}\int_\mathrm{t}^\mathrm{T}\mathrm{e}^{\beta\mathrm{A(s)}}\|\mathrm{Z_s}\|^2\mathrm{ds}\\
&\le \mathbb{E}e^{\beta A(T)}|\xi|^2+(\frac{2}{1-\alpha}+\frac{\beta}{8})\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s+[\frac{1-\alpha}{2}+\alpha(1+\frac{1}{\gamma})]\mathbb{E}\int_\mathrm{t}^\mathrm{T}\mathrm{e}^{\beta\mathrm{A(s)}}\|\mathrm{Z_s}\|^2\mathrm{ds}\\
&+\frac{8}{\beta}\mathbb{E}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+(1+\gamma)\mathbb{E}\int_t^T e^{\beta A(s)}|g(s,0,0)|^2\mathrm{d}s\\
&+(2+\frac{1}{\gamma})\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\phi(|Y_s|^2)\mathrm{d}s.
\end{aligned}$$

Let $\gamma=\frac{4\alpha}{1-\alpha}$, we deduce

$$\begin{aligned}
&\mathbb{E}e^{\beta A(t)}|Y_t|^2+(\frac{7\beta}{8}-\frac{2}{1-\alpha})\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s+\frac{1-\alpha}{4}\mathbb{E}\int_\mathrm{t}^\mathrm{T}\mathrm{e}^{\beta\mathrm{A(s)}}||\mathrm{Z_s}||^2\mathrm{ds}\\
&\le \mathbb{E}e^{\beta A(T)}|\xi|^2+(1+\gamma)\mathbb{E}\int_t^T e^{\beta A(s)}|g(s,0,0)|^2\mathrm{d}s+\frac{8}{\beta}\mathbb{E}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s\\
&+(2+\frac{1}{\gamma})\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\phi(|Y_s|^2)\mathrm{d}s.
\end{aligned}$$

Let β enough large, there exists a nonnegative constant $C_{\beta,T}$ such that

$$\mathbb{E}e^{\beta A(t)}|Y_t|^2+\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s+\mathbb{E}\int_\mathrm{t}^\mathrm{T}\mathrm{e}^{\beta\mathrm{A(s)}}||\mathrm{Z_s}||^2\mathrm{ds}\le \mathrm{C_{\beta,T}X_t}. \tag{3.4}$$

where $X_t=\mathbb{E}e^{\beta A(T)}|\xi|^2+\mathbb{E}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+\mathbb{E}\int_t^T e^{\beta A(s)}|g(s,0,0)|^2\mathrm{d}s+\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\phi(|Y_s|^2)\mathrm{d}s$.

By the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
2\mathbb{E}\left[\sup_{r\in[t,T]}|\int_r^T e^{\beta A(s)}Y_s g(s,Y_s,Z_s)\mathrm{d}B_s|\right]&\le 12\mathbb{E}\left[\sup_{r\in[t,T]}[e^{\frac{\beta A(r)}{2}}Y_r](\int_t^T e^{\beta A(s)}|g(s,Y_s,Z_s|^2\mathrm{d}s)^{1/2}\right]\\
&\le\frac{1}{4}\mathbb{E}\left[\sup_{r\in[t,T]}[e^{\beta A(r)}|Y_r|^2]\right]+144\mathbb{E}\int_t^T e^{\beta A(s)}|g(s,Y_s,Z_s|^2\mathrm{d}s. \qquad (3.5)
\end{aligned}$$

$$2\mathbb{E}\left[\sup_{r\in[t,T]}|\int_s^T e^{\beta A(s)}Y_s Z_s)\mathrm{d}W_s|\right]\le\frac{1}{4}\mathbb{E}\left[\sup_{r\in[t,T]}[e^{\beta A(r)}|Y_r|^2]\right]+144\mathbb{E}\int_r^T e^{\beta A(s)}||Z_s||^2\mathrm{d}s. \tag{3.6}$$

From (3.1), (3.5) and (3.6), it follows that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[t,T]}e^{\beta A(r)}|Y_r|^2\right]+\beta\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s+\mathbb{E}\int_\mathrm{t}^\mathrm{T}\mathrm{e}^{\beta\mathrm{A(s)}}||\mathrm{Z_s}||^2\mathrm{ds}\\
&\le \mathbb{E}e^{\beta A(T)}|\xi|^2+2\mathbb{E}\int_t^T e^{\beta A(s)}Y_s f(s,Y_s,Z_s)\mathrm{d}s+\int_t^T e^{\beta A(s)}|g(s,Y_s,Z_s)|^2\mathrm{d}s\\
&+2\mathbb{E}\left[|\sup_{r\in[t,T]}\int_r^T e^{\beta A(s)}Y_s g(s,Y_s,Z_s)\mathrm{d}B_s|\right]+2\mathbb{E}\sup_{r\in[t,T]}|\int_r^T e^{\beta A(s)}Y_s Z_s\mathrm{d}W_s|\\
&\le \mathbb{E}e^{\beta A(T)}|\xi|^2+\frac{1}{2}\mathbb{E}\left[\sup_{r\in[t,T]}e^{\beta A(r)}|Y_r|^2\right]+[144+\frac{1-\alpha}{2}+145\alpha(1+\frac{1}{\gamma})]\mathbb{E}\int_t^T e^{\beta A(s)}||Z_s||^2\mathrm{d}s\\
&+\frac{8}{\beta}\mathbb{E}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}\mathrm{d}s+(\frac{2}{1-\alpha}+\frac{\beta}{8})\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s\\
&+145(1+\gamma)\mathbb{E}\int_t^T e^{\beta A(s)}|g(s,0,0)|^2\mathrm{d}s+(146+\frac{145}{\gamma})\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\phi(|Y_s|^2)\mathrm{d}s. \qquad (3.7)
\end{aligned}$$

From (3.4) and (3.7), we can derive the result.

Theorem 1 *Assume (H1) and (H2) hold. Then, there exists a unique solution* $(Y_t, Z_t) \in \mathcal{M}^{2,c}$ *satisfying (2.1).*

Proof: Uniqueness. Let $(Y_t^i, Z_t^i) \in \mathcal{M}^{2,c}(i = 1, 2)$ be solutions of (2.1), we have

$$Y_t^1 - Y_t^2 = \int_t^T [f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)]\mathrm{d}s + \int_t^T [g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)]\mathrm{d}B_s - \int_t^T (Z_s^1 - Z_s^2)\mathrm{d}W_s. \quad (3.8)$$

Applying Itô formula to $e^{\beta A(t)}|Y_t^1 - Y_t^2|^2$,

$$\begin{aligned}
&e^{\beta A(t)}|Y_t^1 - Y_t^2|^2 + \beta \int_t^T e^{\beta A(s)} a^2(s)|Y_s^1 - Y_s^2|^2 \mathrm{d}s + \int_t^T e^{\beta A(s)}\|Z_s^1 - Z_s^2\|^2 \mathrm{d}s \\
&= 2\int_t^T e^{\beta A(s)}(Y_s^1 - Y_s^2)(f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2))\mathrm{d}s + \int_t^T e^{\beta A(s)}|(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 \mathrm{d}s. \\
&+2\int_t^T e^{\beta A(s)}(Y_s^1 - Y_s^2)(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2))\mathrm{d}B_s - 2\int_t^T e^{\beta A(s)}(Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)\mathrm{d}W_s. \quad (3.9)
\end{aligned}$$

Taking expectation on both sides of (3.9), from (H1), (H2) and elementary inequality $2ab \leq \theta a^2 + \frac{1}{\theta}b^2, \theta > 0$, we have

$$\begin{aligned}
&\mathbb{E}e^{\beta A(t)}|Y_t^1 - Y_t^2|^2 + \beta\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)|Y_s^1 - Y_s^2|^2 \mathrm{d}s + \mathbb{E}\int_t^T e^{\beta A(s)}\|Z_s^1 - Z_s^2\|^2 \mathrm{d}s \\
&= 2\mathbb{E}\int_t^T e^{\beta A(s)}(Y_s^1 - Y_s^2)(f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2))\mathrm{d}s + \mathbb{E}\int_t^T e^{\beta A(s)}|(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)|^2 \mathrm{d}s \\
&\leq 3\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)\phi(|Y_s^1 - Y_s^2|^2)\mathrm{d}s + \frac{\beta}{4}\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)|Y_s^1 - Y_s^2|^2 \mathrm{d}s \\
&+(\frac{4}{\beta} + \alpha)\mathbb{E}\int_t^T e^{\beta A(s)}\|Z_s^1 - Z_s^2\|^2 \mathrm{d}s.
\end{aligned}$$

By Lemma 1, taking β enough large, there exists a nonnegative constant C_1 such that

$$\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)|Y_s^1 - Y_s^2|^2 \mathrm{d}s + \mathbb{E}\int_t^T e^{\beta A(s)}\|Z_s^1 - Z_s^2\|^2 \mathrm{d}s \leq C_1\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)\phi(|Y_s^1 - Y_s^2|^2)\mathrm{d}s. \quad (3.10)$$

From (3.9), (3.10) and Burkholder-Davis-Gundy inequality, there exists a a nonnegative constant C_2 such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq r\leq T} e^{\beta A(r)}|Y_r^1 - Y_r^2|^2\right] + \mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)|Y_s^1 - Y_s^2|^2 \mathrm{d}s + \mathbb{E}\int_t^T e^{\beta A(s)}\|Z_s^1 - Z_s^2\|^2 \mathrm{d}s \\
&\leq C_2\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)\phi(|Y_s^1 - Y_s^2|^2)\mathrm{d}s \leq C_2\int_t^T a^2(s)\phi(\mathbb{E}\sup_{s\leq r\leq T} e^{\beta A(r)}|Y_r^1 - Y_r^2|^2)\mathrm{d}s.
\end{aligned}$$

By Bihari inequality, we can obtain $Y_s^1 = Y_s^2, Z_s^1 = Z_s^2, \mathrm{d}P - a.s.$

Existence. By the definition of f_n, Lemma 1 and Lemma 2, we can easily deduce BDSDEs(ξ, f_n, g) is a special case in [16]. Therefore, BDSDEs(ξ, f_n, g) have a unique solution denoted by (Y_t^n, Z_t^n).

Applying Itô formula to $e^{\beta A(t)}|Y_t^n - Y_t^m|^2$,

$$\begin{aligned}
&e^{\beta A(t)}|Y_t^n - Y_t^m|^2 + \beta \int_t^T e^{\beta A(s)} a^2(s)|Y_s^n - Y_s^m|^2 \mathrm{d}s + \int_t^T e^{\beta A(s)}\|Z_s^n - Z_s^m\|^2 \mathrm{d}s \\
&= 2\int_t^T e^{\beta A(s)}(Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))\mathrm{d}s \\
&+2\int_t^T e^{\beta A(s)}(Y_s^n - Y_s^m)(g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m))\mathrm{d}B_s - 2\int_t^T e^{\beta A(s)}(Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\mathrm{d}W_s \\
&+ \int_t^T e^{\beta A(s)}|(g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)|^2 \mathrm{d}s. \quad (3.11)
\end{aligned}$$

Taking suitable β, by Lemma 2 and Burkholder-Davis-Gundy inequality, there exist a nonnegative constant C_3 such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\le T} e^{\beta A(r)}|Y_r^n-Y_r^m|^2\right]+\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|Y_s^n-Y_s^m|^2\mathrm{d}s+\mathbb{E}\int_t^T e^{\beta A(s)}||Z_s^n-Z_s^m||^2\mathrm{d}s\\
&\le C_3\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\psi(|Y_s^n-Y_s^m|^2)\mathrm{d}s\\
&\le C_3\int_t^T a^2(s)\psi(\mathbb{E}\sup_{s\le r\le T} e^{\beta A(r)}|Y_r^n-Y_r^m|^2)\mathrm{d}s, \qquad (3.12)
\end{aligned}$$

where $\psi(u)$ is a concave and nondecreasing function with $\psi(0)=0$ and $\int_{0+}\frac{\mathrm{d}u}{\psi(u)}=+\infty, ku\rho(u)\le \psi(u)\le 2ku\rho(u), k>0$.

From Bihari inequality, (3.12), we have (Y_t^n,Z_t^n) is a Cauchy sequence in $\mathcal{M}^{2,c}$.

On the other hand,

$$\begin{aligned}
&\int_t^T e^{\beta A(s)}|f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|\mathrm{d}s\\
&\le \mathbb{E}\int_t^T e^{\beta A(s)}\left[kp(s)\rho(\frac{2A}{n})+p(s)\rho(|Y_s^n-Y_s|)+q(s)\|Z_s^n-Z_s\|\right]\mathrm{d}s\\
&\le \mathbb{E}\int_t^T e^{\beta A(s)}kp(s)\left[\rho(\frac{2A}{n})+\rho(\frac{2A}{m+A})\right]\mathrm{d}s+(m+A)\mathbb{E}\int_t^T e^{\beta A(s)}p(s)|Y_s^n-Y_s|\mathrm{d}s\\
&\quad+\mathbb{E}\int_t^T e^{\beta A(s)}q(s)\|Z_s^n-Z_s\|\mathrm{d}s\\
&\le \mathbb{E}\int_t^T e^{\beta A(s)}kp(s)\left[\rho(\frac{2A}{n})+\rho(\frac{2A}{m+A})\right]\mathrm{d}s+\mathbb{E}\left\{\left[\int_t^T e^{\beta A(s)}a^2(s)\mathrm{d}s\right]^{\frac12}\left[\int_t^T e^{\beta A(s)}\|Z_s^n-Z_s\|^2\mathrm{d}s\right]^{\frac12}\right\}\\
&\quad+(m+A)\mathbb{E}\left\{\left[\int_t^T e^{\beta A(s)}a^2(s)\mathrm{d}s\right]^{\frac12}\left[\int_t^T e^{\beta A(s)}a^2(s)|Y_s^n-Y_s|^2\mathrm{d}s\right]^{\frac12}\right\}. \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\int_t^T e^{\beta A(s)}|g_n(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2\mathrm{d}s\\
&\le \mathbb{E}\int_t^T e^{\beta A(s)}\left[|Y_s^n-Y_s|p(s)\rho(|Y_s^n-Y_s|)+\alpha\|Z_s^n-Z_s\|^2\right]\mathrm{d}s\\
&\le \int_t^T p(s)\rho(\mathbb{E}\sup_{s\le r\le T} e^{\beta A(r)}|Y_r^n-Y_r|^2)\mathrm{d}s+\alpha\mathbb{E}\int_t^T e^{\beta A(s)}\|Z_s^n-Z_s\|^2\mathrm{d}s. \qquad (3.14)
\end{aligned}$$

From (3.13), (3.14), we have

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\mathrm{d}s+\int_t^T g(s,Y_s,Z_s)\mathrm{d}B_s-\int_t^T Z_s\mathrm{d}W_s,\ t\in[0,T],$$

Then, $(Y_t,Z_t)_{t\in[0,T]}$ is a solution of (2.1).

Example: For convenience, let $k=1$, and $f(t,y,z)=\frac{1}{\sqrt{t}}h(|y|)+\frac{1}{\sqrt[6]{t}}|z|+|B_t|, g(t,y,z)=\frac{1}{\sqrt[4]{t}}\sin|y|+\frac{1}{2\sqrt{2+t^2}}|z|+|B_t|$, and δ is a enough small nonnegative constant,

$$h(x)=\begin{cases} -x\ln x, & x\le\delta,\\ h'(\delta-)(x-\delta)+h(\delta), & x>\delta,\\ 0, & othercases.\end{cases}$$

We choose $p(t)=\frac{1}{\sqrt{t}}, q(t)=\frac{1}{\sqrt[6]{t}}, u(t)=\frac{1}{2+t^2}$, then $|f(t,y_1,z_1)-f(t,y_2,z_2)|\le p(t)h(|y_1-y_2|)+q(t)\|z_1-z_2\|,\ |g(t,y_1,z_1)-g(t,y_2,z_2)|^2\le[\frac{1}{\sqrt[4]{t}}|y_1-y_2|+\frac{1}{2\sqrt{2+t^2}}\|z_1-z_2\||]^2\le 3p(t)|y_1-y_2|^2+$

$\frac{1}{2+t^2}\|z_1-z_2\|^2$. Let $\rho(x)=h(x)+3x$, we can deduce $\rho(x)$ is a concave function, $\int_{0^+}\frac{1}{\rho(x)}\mathrm{d}x=+\infty$. According to above analysis, the functions $f(t,y,z),g(t,y,z)$ satisfied (H1) and (H2), the equation (2.1) has a unique solution. Obviously, $f(t,y,z),g(t,y,z)$ do not satisfy the assumptions in [11-15]. Let $p(t),q(t)=C,u(t)=\alpha(0<\alpha<1)$, C is a nonnegative constant, then, the results generalize the results in [11, 15]. Moreover, if $g(t,y,z)=0$, [1, 6, 9] are special cases of our main results.

Acknowledgement

The author would like to thank his/her referee for the valuable comments. The work is partially supported Key projects of domestic and overseas visits to outstanding young (gxfxZD2016261), the NSF of Anhui Province (KJ2011B176) and the Scientific Platform of Suzhou University(2010YKF11).

REFERENCES

[1] E. Pardoux, S. Peng, Adapted solution of backward stochastic differential equations, Systems Control Letters, 1990, 14(1): 51-61.

[2] N. El Karoui, S. Peng, M. C. Quenez, Backward stochastic differential equations in finance, Mathematical Finance, 1997, 7(1): 1-71.

[3] N. El Karoui, M. C. Quenez, Non-linear pricing theory and backward stochastic differential equations, in: W. J. Runggaldier (Ed.), Financial Mathematics, in: Lecture Notes in Math., vol. 1656, Springer-Verlag, Berlin, 1997, pp. 191-246.

[4] S. Hamadène, BSDEs and risk sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations, Stochastic Processes and their Applications, 2003, 107(1): 145-169.

[5] S. Hamadène, J. P. Lepeltier, Zero-sum stochastic differential games and BSDEs, Systems Control Letters, 1995, 24(4): 259-263.

[6] Xuerong Mao, Adapted solutions of backward stochastic differential equations with non- Lipschitz coefficients, Stochastic Processes and their Applications, 1995, 58(2): 281-292.

[7] J. P. Lepeltier, J. S. Martin, Backward stochastic differential equations with continuous coefficients. Statistic and Probability Letters, 1997, 34(4): 425-430.

[8] S. Hamadène, Multidimensional backward stochastic differential equations with uniformly continuous coefficients. Bernoulli, 2003, 9(3): 517-534.

[9] Shengjun Fan, Long, Jiang, Matt Davison, existence and uniqueness result for multidimensional BSDEs with generators of Osgood type, Front. Math. China, 2013, 8(4): 811-824.

[10] Ying Hu, Shanjian Tang. Multi-dimensional backward stochastic differential equations of diagonally quadratic generators, Stochastic Processes and their Applications, 2016, 126(4): 1066-1086.

[11] E. Pardoux, S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs, Probability Theory and Related Fields, 1994, 98(2): 209-227.

[12] Shi Yufeng, Gu Yanling, Liu Kai. Comparison theorem of backward doubly stochastic differential equations and applications. Stochastic Analysis and Application, 2005, 23(1): 97-110.

[13] Qian Lin. A class of backward doubly stochastic differential equations with non-Lipschitz coefficients. Statistic and Probability Letters, 2009, 79(20): 2223-2229.

[14] Qian Lin. A generalized existence theorem of backward doubly stochastic differential equations. Acta Mathematica Sinica, English Series, 2010, 26(8): 1525-1534.

[15] Xianfei Wang , Long Jiang, Jiaojiao Ma, Multidimensional backward doubly stochastic differential equations with generators of Osgood type, Journal of Shandong University(Natural Science), 2015 50(8): 24-33.

[16] Lanying Hu, reflected backward doubly stochastic differential equations driven by a Lévy process with stochastic Lipschitz conditon, Applied Mathematics and Computation, 2012, 219(3): 1153-1157.

[17] Shengjun Fan, Long Jiang, Multi-dimensional BSDEs with weak Monotonicity and general growth generators, Acta Mathematica Sinica, English Series, 2013, 23(10): 1885-1906.