EXTENSION OF THE FRACTIONAL DERIVATIVE OPERATOR OF THE RIEMANN-LIOUVILLE

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Abstract. In this paper, by using the generalized beta function, we extend the definition of the fractional derivative operator of the Riemann-Liouville and discuses its properties . Moreover, we establish the some relations to extended special functions of two and three variables via generating functions.

1. Introduction, Definitions and Preliminaries

In recent years, fractional derivative operators and their extensions have received considerable attention. There are many definitions of generalized fractional derivatives involving extended beta and hypergeometric functions [2, 3, 7, 10]. In continuation, Özarslan and Özergin [5] was introduced and studied the extended fractional derivative operator.

Definition 1. *The extended fractional derivative operator defined by:*

$$\mathcal{D}_z^{\eta,p}\left\{f(z)\right\} := \begin{cases} \dfrac{1}{\Gamma(-\eta)}\displaystyle\int_0^z (z-t)^{-\eta-1}\; e^{\left(\frac{-pz^2}{(z-t)t}\right)} f(t)\,\mathrm{d}t & \big(Re(\eta)<0\big) \\[2ex] \dfrac{\mathrm{d}^m}{\mathrm{d}z^m}\Big\{\mathcal{D}_z^{\eta-m}\left\{f(z)\right\}\Big\} & \big(m-1 \leqq Re(\eta) < m \;\; (m\in\mathbb{N})\big) \end{cases} \tag{1.1}$$

Clearly, the special case of (1.1), when $p = 0$ reduce immediately to Riemann–Liouville fractional derivative(see,[12, 13]).

In recent years number of researchers has been systematically study the extended fractional derivative operators and discussed their applications in different fields (see, [3, 4, 5, 7]). In view of the effectiveness of the above works, here by using the generalized beta function due to Choi *et al.* [1], we extend the definition of the fractional derivative operator of the Riemann-Liouville and discuses its various properties. Moreover, we establish the some relations to extended special functions of two and three variables via generating functions.

For our purpose we recall the some earlier works and definitions.

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Definition 2. *The generalized beta function defined by* [see Choi et al. [1]]:

$$B_{p,q}(x,\, y\,) := \int_0^1 t^{x-1}(1-t)^{y-1}\; e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\mathrm{d}t \tag{1.2}$$

$$\big(\min\{\Re(x),\Re(y)\} > 0;\; \min\{Re(p), Re(q)\} \geqq 0\big)$$

Definition 3. *The generalized hypergeometric function defined as* [see Choi et al. [1]]:

$${}_2F_{1;p,q}\,(a,\, b;\, c;\, z) := \sum_{n=0}^{\infty} (a)_n\, \frac{B_{p,q}(b+n,\, c-b\,)}{B(b,\, c-b)}\, \frac{z^n}{n!} \qquad \big(p \geqq 0,\; q \geqq 0;\; |z|<1;\; Re(c) > Re(b) > 0\big) \tag{1.3}$$

2. **Extension of Hypergeometric Functions and Integral Representations**

By making use of (1.2), we consider another extensions of Appell's and the Lauricella functions of one, two and three variables.

Definition 4. *The extension of the hypergeometric functions of two and three variables are defined as:*

$$F_1(a,b,c;d;x,y;p,q) = \sum_{m,n=0}^{\infty} (b)_m (c)_n\, \frac{B_{p,q}(a+m+n, d-a)}{B(a,d-a)}\, \frac{x^m}{m!}\, \frac{y^n}{n!} \tag{2.1}$$

$$\big(Re(p) > 0,\; Re(q) > 0;\; |x|<1, |y|<1\big)$$

$$F_2(a,b,c;d,e;x,y;p,q) = \sum_{m,n=0}^{\infty} (a)_{m+n}\, \frac{B_{p,q}(b+m,d-b)B_{p,q}(c+n,e-c)}{B(b,d-b)B(c,e-c)}\, \frac{x^m}{m!}\, \frac{y^n}{n!} \tag{2.2}$$

$$\big(Re(p) > 0,\; Re(q) > 0;\; |x|+|y|<1\big)$$

$$F_D^3(a,b,c,d;e;x,y,z;p,q) = \sum_{m,n,r=0}^{\infty} \frac{B_{p,q}(a+m+n+r,e-a)(b)_m(c)_n(d)_r}{B(a,e-a)}\, \frac{x^m}{m!}\, \frac{y^n}{n!}\, \frac{z^r}{r!} \tag{2.3}$$

$$\big(Re(p) > 0,\; Re(q) > 0; |x|<1, |y|<1, |z|<1\big)$$

Remark 1. *For $p=q$ above definitions are similar to Özarslan and Özergin* [5] *and for $p=0=q$ similar to* [12].

Theorem 1. *The following integral holds true for* (2.1)*:*

$$F_1(a,b,c;d;x,y;p,q) = \frac{1}{B(a,d-a)} \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c} e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\mathrm{d}t \tag{2.4}$$

Proof. To prove the above Theorem, we start by assuming that

$$\mathfrak{I}=\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\mathrm{d}t.$$

Using the binomial series expansion for $(1-xt)^{-b}$ and $(1-yt)^{-c}$ and interchanging the order of summation and integration, we get

$$\begin{aligned}\mathfrak{I}&=\int_0^1 t^{a-1}(1-t)^{d-a-1}e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\left\{\sum_{n=0}^{\infty}(b)_n\frac{(xt)^n}{n!}\sum_{m=0}^{\infty}(c)_m\frac{(yt)^m}{m!}\right\}\mathrm{d}t\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(b)_n(c)_m\left\{\int_0^1 t^{a+m+n-1}(1-t)^{d-a-1}e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\mathrm{d}t\right\}\frac{x^n}{n!}\frac{y^m}{m!},\end{aligned}$$

by applying (1.2) and (2.1), we get the desired representation. □

Theorem 2. *The following integral holds true for* (2.2)*:*

$$\begin{aligned}F_2(a,b,c;d,e;x,y;p,q)&=\frac{1}{B(b,d-b)B(c,e-c)}\\&=\int_0^1\int_0^1\frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}e^{\left(-\frac{p}{t}-\frac{q}{1-t}-\frac{p}{s}-\frac{q}{1-s}\right)}\mathrm{d}t\mathrm{d}s\end{aligned}$$

Proof. We start by expanding $(1-xt-ys)^{-a}$ we have

$$\begin{aligned}&\int_0^1\int_0^1\frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}e^{\left(-\frac{p}{t}-\frac{q}{1-t}-\frac{p}{s}-\frac{q}{1-s}\right)}\mathrm{d}t\mathrm{d}s\\&=\int_0^1\int_0^1 t^{b-1}(1-t)^{d-b-1}e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}s^{c-1}(1-s)^{e-c-1}e^{\left(-\frac{p}{s}-\frac{q}{1-s}\right)}\sum_{N=0}^{\infty}(a)_N\frac{(xt+ys)^N}{N!}\mathrm{d}t\mathrm{d}s\end{aligned}$$

Using the summation formula

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}f(l+r)\frac{x^r}{r!}\frac{y^l}{l!}$$

we get

$$\begin{aligned}&\int_0^1\int_0^1\frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}e^{\left(-\frac{p}{t}-\frac{q}{1-t}-\frac{p}{s}-\frac{q}{1-s}\right)}\mathrm{d}t\mathrm{d}s\\&=\int_0^1\int_0^1 t^{b-1}(1-t)^{d-b-1}e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}s^{c-1}(1-s)^{e-c-1}e^{\left(-\frac{p}{s}-\frac{q}{1-s}\right)}\qquad(2.5)\\&\qquad\times\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}(a)_{r+l}\frac{(xt)^r}{r!}\frac{(ys)^l}{l!}\mathrm{d}t\mathrm{d}s\end{aligned}$$

Here, involving series and the integrals are convergent, then by interchanging the order of summation and integration, we get

$$\int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} e^{\left(-\frac{p}{t}-\frac{q}{1-t}-\frac{p}{s}-\frac{q}{1-s}\right)}\mathrm{d}t\mathrm{d}s$$

$$= \sum_{r=0}^{\infty}\sum_{l=0}^{\infty}(a)_{l+r}\frac{x^r}{r!}\frac{y^l}{l!}\int_0^1 t^{b+r-1}(1-t)^{d-b-1}e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\mathrm{d}t$$

$$\int_0^1 s^{c+l-1}(1-s)^{e-c-1}e^{\left(-\frac{p}{s}-\frac{q}{1-s}\right)}\mathrm{d}s,$$

by applying (1.2) and (2.2), we get the desired representation.

□

Theorem 3. *The following integral holds true for* (2.3)*:*

$$F_D^3(a,b,c,d;e;x,y,z;p,q)$$

$$= \frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}(1-yt)^{-d}\, e^{\left(-\frac{p}{t}-\frac{q}{1-t}\right)}\mathrm{d}t$$

Proof. The proof of Theorem 3 is as similar to proof of Theorem 1. Therefore, we omit its detail here. □

3. Extended Riemann-Liouville Fractional Derivative Operator

Here, we introduce new extended Riemann-Liouville type fractional derivative operator as follows:

Definition 5. *The extended Riemann-Liouville type fractional derivative operator defined by*

$$\mathcal{D}_z^{\eta}\{f(z);p,q\} := \begin{cases} \dfrac{1}{\Gamma(-\eta)}\displaystyle\int_0^z (z-t)^{-\eta-1}\; e^{\left(\frac{-pz}{t}-\frac{qz}{z-t}\right)} f(t)\,\mathrm{d}t & \big(Re(\eta)<0\big) \\ \dfrac{\mathrm{d}^m}{\mathrm{d}z^m}\Big\{\mathcal{D}_z^{\eta-m}\{f(z);p,q\}\Big\} & \big(m-1 \leqq Re(\eta) < m\ \ (m\in\mathbb{N})\big) \end{cases} \tag{3.1}$$

where $Re(p)>0, Re(q)>0$ *and the path of integration is a line from* 0 *to* z *in the complex* t*-plane.*

Clearly for $p=q$, (3.1) reduces to (1.1) and for $p=0=q$, we obtain its classical form (see, for details [2, 12, 13]).

Now, we establishing some theorems involving the extended fractional derivatives.

Theorem 4. *The following representation for* (3.1) *holds true:*

$$D_z^{\eta}[z^{\lambda};p,q] = \frac{B_{p,q}(\lambda+1,-\eta)}{\Gamma(-\eta)}z^{\lambda-\eta} \quad (Re(\eta)<0) \tag{3.2}$$

Proof. Using (3.1) and (1.2), we get

$$D_z^\eta[z^\lambda; p, q] = \frac{1}{\Gamma(-\eta)} \int_0^z t^\lambda (z-t)^{-\eta-1} e^{\left(-\frac{pz}{t} - \frac{qz}{z-t}\right)} \mathrm{d}t$$

replacing $t = uz$, we have

$$\begin{aligned} D_z^\eta[z^\lambda; p, q] &= \frac{1}{\Gamma(-\eta)} \int_0^1 (uz)^\lambda (z-uz)^{-\eta-1} e^{\left(-\frac{pz}{uz} - \frac{qz}{z-uz}\right)} z\mathrm{d}u \\ &= \frac{z^{\lambda-\eta}}{\Gamma(-\eta)} \int_0^1 u^\lambda (1-u)^{-\eta-1} e^{\left(-\frac{p}{u} - \frac{q}{1-u}\right)} \mathrm{d}u. \end{aligned} \tag{3.3}$$

By applying Definition (1.2) to yield (5.4) directly. □

Theorem 5. *Let $\Re(\eta) < 0$ and suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < \rho$) for some $\rho \in \mathbb{R}^+$. Then we have*

$$D_z^\eta [f(z); p, q] = \sum_{n=0}^{\infty} a_n D_z^\eta [z^n; p, q] \tag{3.4}$$

Proof. We begin from Definition 5 to the function $f(z)$ with its series expansion, we get

$$D_z^\eta [f(z); p, q] = \frac{1}{\Gamma(-\eta)} \int_0^z \sum_{n=0}^{\infty} a_n t^n (z-t)^{-\eta-1} e^{\left(-\frac{pz}{t} - \frac{qz}{z-t}\right)} \mathrm{d}t$$

Since the power series converges uniformly on any closed disk centered at the origin with its radius smaller than ρ, so does the series on the line segment from 0 to a fixed z for $|z| < \rho$. This fact guarantees term-by-term integration as follows:

$$\begin{aligned} D_z^\eta [f(z); p, q] &= \sum_{n=0}^{\infty} a_n \left\{ \frac{1}{\Gamma(-\eta)} \int_0^z t^n (z-t)^{-\eta-1} e^{\left(-\frac{pz}{t} - \frac{qz}{z-t}\right)} \mathrm{d}t \right\} \\ &= \sum_{n=0}^{\infty} a_n D_z^\eta [z^n; p, q] \end{aligned} \tag{3.5}$$

This completes the proof. □

Theorem 6. *The following representation holds true:*

$$D_z^{\lambda-\eta}[z^{\lambda-1}(1-z)^{-\alpha}; p, q] = \frac{\Gamma(\lambda)}{\Gamma(\eta)} z^{\eta-1} {}_2F_{1;p,q}(\alpha, \lambda; \eta; z) \tag{3.6}$$

$$(Re(\eta) > Re(\lambda) > 0 \text{ and } \mid z \mid < 1)$$

Proof. Direct calculations yield

$$D_z^{\lambda-\eta}[z^{\lambda-1}(1-z)^{-\alpha};p,q] = \frac{1}{\Gamma(\eta-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\alpha}e^{\left(-\frac{pz}{t}-\frac{qz}{z-t}\right)}(z-t)^{\eta-\lambda-1}\mathrm{d}t$$
$$= \frac{z^{\eta-\lambda-1}}{\Gamma(\eta-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\alpha}\left(1-\frac{t}{z}\right)^{\eta-\lambda-1} e^{\left(-\frac{pz}{t}-\frac{qz}{z-t}\right)}\mathrm{d}t$$
$$= \frac{z^{\eta-\lambda-1}z^{\lambda}}{\Gamma(\eta-\lambda)}\int_0^1 u^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\eta-\lambda-1}e^{\left(-\frac{p}{u}-\frac{q}{1-u}\right)}\mathrm{d}u$$

Using (1.3) and after little simplification, we have the (3.6). This completes the proof. □

Theorem 7. *The following representation for holds true:*

$$D_z^{\lambda-\eta}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,q] = \frac{\Gamma(\lambda)}{\Gamma(\eta)}z^{\eta-1}F_1(\lambda,\alpha,\beta;\eta;az,bz;p,q) \tag{3.7}$$

$$(Re(\eta) > Re(\lambda) > 0, Re(\alpha) > 0, Re(\beta) > 0; \mid az \mid < 1 and \mid bz \mid < 1)$$

More generally, we have

$$D_z^{\lambda-\eta}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma};p,q] = \frac{\Gamma(\lambda)}{\Gamma(\eta)}z^{\eta-1}F_D^3(\lambda,\alpha,\beta,\gamma;\eta;az,bz,cz;p,q) \tag{3.8}$$

$$(Re(\eta) > \Re(\lambda) > 0, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, \mid az \mid < 1, \mid bz \mid < 1 \text{ and } \mid cz \mid < 1)$$

Proof. To prove (3.7), using the following power series expansion for $(1-az)^{-\alpha}$ and $(1-bz)^{-\beta}$

$$(1-az)^{-\alpha}(1-bz)^{-\beta} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(\alpha)_l(\beta)_k\frac{(az)^l}{l!}\frac{(bz)^k}{k!},$$

then applying Theorem 4, we obtain

$$D_z^{\lambda-\eta}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,q]$$
$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(\alpha)_l(\beta)_k\frac{(a)^l}{l!}\frac{(b)^k}{k!}D_z^{\lambda-\eta}[z^{\lambda+l+k-1};p,q] \tag{3.9}$$
$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(\alpha)_l(\beta)_k\frac{(a)^l}{l!}\frac{(b)^k}{k!}\frac{B_{p,q}(\lambda+l+k,\eta-\lambda)}{\Gamma(\eta-\lambda)}z^{l+k+\eta-1}. \tag{3.10}$$

Now, applying (2.1), we get

$$D_z^{\lambda-\eta}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta};p,q]$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\eta)}z^{\eta-1}F_1(\lambda,\alpha,\beta;\eta;az,bz;p,q). \tag{3.11}$$

Similarly, as in the proof of (3.7), taking the binomial theorem for $(1-az)^{-\alpha}, (1-bz)^{-\beta}$ and $(1-cz)^{-\gamma}$, then applying Theorem 4 and (2.3) into account, one can easily prove (3.8). Therefore, we omit the details of its proof. This completes the proof. □

Theorem 8. *The following representation holds true:*

$$D_z^{\lambda-\eta}\left[z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right);p,q\right]=\frac{1}{B(\beta,\gamma-\beta)\Gamma(\eta-\lambda)}z^{\eta-1}F_2(\alpha,\beta,\lambda;\gamma,\eta;x,z;p,q) \tag{3.12}$$

$(Re(\mu)>Re(\lambda)>0, Re(\alpha)>0, Re(\beta)>0, Re(\gamma)>0; \mid \frac{x}{1-z} \mid<1$ *and* $\mid x\mid+\mid z\mid<1)$

Proof. Applying (2.2) on the LHS of (3.12), we get

$$\begin{aligned} &D_z^{\lambda-\eta}\left[z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right);p,q\right]\\ &\quad=D_z^{\lambda-\eta}\left[z^{\lambda-1}(1-z)^{-\alpha}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_nB_{p,q}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)\,n!}\left(\frac{x}{1-z}\right)^n\right\};p,q\right]\\ &\quad=\frac{1}{B(\beta,\gamma-\beta)}D_z^{\lambda-\eta}\left[z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_nB_{p,q}(\beta+n,\gamma-\beta)\frac{x^n}{n!}\left\{(1-z)^{-\alpha-n}\right\};p,q\right]. \end{aligned}$$

Using power series expansion for $(1-z)^{-\alpha-n}$, applying Theorem 4 and (2.2), we get

$$\begin{aligned} &D_z^{\lambda-\eta}\left[z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right);p,q\right]\\ &\quad=D_z^{\lambda-\eta}\left[z^{\lambda-1}(1-z)^{-\alpha}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_nB_{p,q}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)\,n!}\left(\frac{x}{1-z}\right)^n\right\};p,q\right]\\ &\quad=\frac{1}{B(\beta,\gamma-\beta)}D_z^{\lambda-\eta}\left[z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_nB_{p,q}(\beta+n,\gamma-\beta)\frac{x^n}{n!}\left\{(1-z)^{-\alpha-n}\right\};p,q\right]. \end{aligned}$$

This completes the proof. □

4. **Mellin Transform Representations**

The double Mellin transforms [8, p. 293, Eq. (7.1.6)] of a suitable classes of integrable function $f(x,y)$ with index r and s is defined by

$$\mathfrak{M}\left\{f(x,y):x\to r,\,y\to s\right\}:=\int_0^{\infty}\int_0^{\infty}x^{r-1}y^{s-1}\,f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{4.1}$$

provided that the improper integral in (4.1) exists.

Theorem 9. *The following Mellin transform formula holds true:*

$$\mathfrak{M}\left\{D_z^{\mu,p,q}(z^{\lambda}):p\to r,\;q\to s\right\}:=\frac{\Gamma(r)\Gamma(s)}{\Gamma(-\mu)}B(\lambda+r+1,s-\mu)z^{\lambda-\mu} \tag{4.2}$$

$$(\Re(\lambda)>-1,\;\Re(\mu)<0,\;\Re(s)>0\;\Re(r)>0)$$

Proof. Applying definition (4.1) on to (3.1), we get

$$\begin{aligned}
\mathfrak{M}\left\{D_z^{\mu,p,q}(z^\lambda) : p \to r,\ q \to s\right\} &:= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} D_z^{\mu,p,q}(z^\lambda) \mathrm{d}p\mathrm{d}q \\
&= \frac{1}{\Gamma(-\mu)} \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left[\int_0^z t^\lambda (z-t)^{-\mu-1} \exp\left(-\frac{pz}{t} - \frac{qz}{z-t}\right) \mathrm{d}t\right] \mathrm{d}p\mathrm{d}q \\
&= \frac{z^{-\mu-1}}{\Gamma(-\mu)} \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left[\int_0^z t^\lambda \left(1-\frac{t}{z}\right)^{-\mu-1} \exp\left(-\frac{pz}{t} - \frac{qz}{z-t}\right) \mathrm{d}t\right] \mathrm{d}p\mathrm{d}q \\
&= \frac{z^{-\mu-1}}{\Gamma(-\mu)} \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left[\int_0^1 u^\lambda z^\lambda (1-u)^{-\mu-1} \exp\left(-\frac{p}{u} - \frac{q}{1-u}\right) \mathrm{d}t\right] \mathrm{d}p\mathrm{d}q \\
&= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left[\int_0^1 u^\lambda (1-u)^{-\mu-1} \exp\left(-\frac{p}{u} - \frac{q}{1-u}\right) \mathrm{d}t\right] \mathrm{d}p\mathrm{d}q \\
&= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^\lambda (1-u)^{-\mu-1} \left(\int_0^\infty p^{r-1} \exp\left(\frac{-p}{u}\right) \mathrm{d}p\right) \left(\int_0^\infty q^{s-1} \exp\left(\frac{-q}{1-u}\right) \mathrm{d}q\right) \mathrm{d}u
\end{aligned}$$

where we have changed the order of integration by absolutely convergent under the stated conditions. Using the definition of gamma function, we have

$$\begin{aligned}
\mathcal{M}\left\{D_z^{\mu,p,q}(z^\lambda) : p \to r,\ q \to s\right\} &:= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^\lambda (1-u)^{-\mu-1} u^r \Gamma(r) (1-u)^s \Gamma(s) du \\
&= \frac{z^{\lambda-\mu}\Gamma(r)\Gamma(s)}{\Gamma(-\mu)} \int_0^1 u^{\lambda+r} (1-u)^{s-\mu-1} du \\
&= \frac{z^{\lambda-\mu}\Gamma(r)\Gamma(s)}{\Gamma(-\mu)} B(\lambda+r+1, s-\mu)
\end{aligned}$$

Which completes the proof. □

Theorem 10. *The following formula for (4.3) holds true:*

$$\mathfrak{M}\left\{D_z^{\mu,p,q}((1-z)^{-\alpha}) : p \to r,\ q \to s\right\} := \frac{\Gamma(r)\Gamma(s) z^{-\mu}}{\Gamma(-\mu) B(r+1, s-\mu)} F(\alpha, r+1; r+s-\mu+1; z) \tag{4.3}$$

$$(\Re(\mu) < 0, \Re(s) > 0, \Re(r) > 0, \Re(\alpha) > 0 and \mid z \mid < 1)$$

Proof. Applying Theorem 9 with $\lambda = n$, we can write that

$$\mathfrak{M}\left\{D_z^{\mu,p,q}((1-z)^{-\alpha}) : p \to r,\ q \to s\right\} := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \mathfrak{M}\left\{D_z^{\mu,p,q}((1-z)^{-\alpha}) : p \to r,\ q \to s\right\}$$

$$= \frac{\Gamma(r)\Gamma(s)}{\Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} B(n+r+1, s-\mu) z^{n-\mu}$$

$$= \frac{\Gamma(r)\Gamma(s) z^{-\mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} B(n+r+1, s-\mu) \frac{(\alpha)_n z^n}{n!}$$

$$= \frac{\Gamma(r)\Gamma(s) z^{-\mu}}{\Gamma(-\mu) B(r+1, s-\mu)} F(\alpha, r+1; r+s-\mu+1; z)$$

Which completes the proof. □

5. **Generating Relations and Further Results**

Here, we obtain some generating relations of linear and bilinear type for the extended hypergeometric functions.

Theorem 11. *The following generating relation hold true:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_{1;p,q}(\lambda+n, \alpha; \beta; x) t^n = (1-t)^{-\lambda}_2 F_{1;p,q}\left(\lambda, \alpha; \beta; \frac{x}{1-t}\right)$$

$$(|\ x\ | < \min(1, |\ 1-t\ |) and \Re(\lambda) > 0, \Re(\beta) > \Re(\alpha) > 0)$$

Proof. Let us consider the elementary identity

$$[(1-x)-t]^{-\lambda} = (1-t)^{-\lambda}\left[1 - \frac{x}{1-t}\right]^{-\lambda},$$

Using power series expension, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-x)^{-\lambda} \left(\frac{t}{1-x}\right)^n = (1-t)^{-\lambda}\left[1 - \frac{x}{1-t}\right]^{-\lambda}$$

Now, multiplying both sides of the above equality by $x^{\alpha-1}$ and applying the operator $D_x^{\alpha-\beta,p,q}$ on both sides, we can get

$$D_x^{\alpha-\beta,p,q}\left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-x)^{-\lambda} \left(\frac{t}{1-x}\right)^n x^{\alpha-1}\right] = (1-t)^{-\lambda} D_x^{\alpha-\beta,p,q}\left[x^{\alpha-1}\left(1 - \frac{x}{1-t}\right)^{-\lambda}\right]$$

Interchanging the order, which is valid under the stated conditions, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_x^{\alpha-\beta,p,q}\left[x^{\alpha-1}(1-x)^{-\lambda-n}\right] t^n = (1-t)^{-\lambda} D_x^{\alpha-\beta,p,q}\left[x^{\alpha-1}\left(1 - \frac{x}{1-t}\right)^{-\lambda}\right]$$

Using Theorem 5, we get the desired result. □

Theorem 12. *The following generating relation holds true:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_{1;p,q}(\rho-n,\alpha;\beta;x)t^n=(1-t)^{-\lambda}F_1\left(\alpha,\rho,\lambda;\beta;x,\frac{-xt}{1-t};p,q\right)$$

$$(\Re(\beta)>\Re(\alpha)>0,\Re(\rho)>0,\Re(\lambda)>0;\mid t\mid<\frac{1}{1+\mid x\mid}$$

Proof. To prove above theorem we use the elementary identity

$$[1-(1-x)t]^{-\lambda}=(1-t)^{-\lambda}\left[1+\frac{xt}{1-t}\right]^{-\lambda}$$

Expanding the left hand side, we have

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^n t^n=(1-t)^{-\lambda}\left[1-\frac{-xt}{1-t}\right]^{-\lambda}$$

Now, multiplying both sides of the above equality by $x^{\alpha-1}(1-x)^{-\rho}$ and applying the operator $D_x^{\alpha-\beta,p,q}$ on both sides, we get

$$D_x^{\alpha-\beta,p,q}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^{\alpha-1}(1-x)^{-\rho+n}t^n\right]=(1-t)^{-\lambda}D_x^{\alpha-\beta,p,q}\left[x^{\alpha-1}(1-x)^{-\rho}\left((1-\frac{-xt}{1-t}\right)^{-\lambda}\right]$$

Interchanging the order, which is valid for $Re(\alpha)>0$ and $\mid xt\mid<\mid 1-t\mid$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\alpha-\beta,p,q}\left[x^{\alpha-1}(1-x)^{-\rho+n}\right]t^n=(1-t)^{-\lambda}D_x^{\alpha-\beta,p,q}\left[x^{\alpha-1}(1-x)^{-\rho}\left((1-\frac{-xt}{1-t}\right)^{-\lambda}\right]$$

Using Theorem 5, we get the desired result. □

Theorem 13. *The following bilinear generating relation holds true:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_{1;p,q}(\gamma,-n;\delta;y){}_2F_{1;p,q}(\lambda+n,\alpha;\beta;x)t^n=(1-t)^{-\lambda}F_2\left(\lambda,\alpha,\gamma;\beta,\delta;\frac{x}{1-t},\frac{-yt}{1-t};p,q\right)$$

$$(\Re(\delta)>\Re(\gamma)>0,\Re(\alpha)>0,\Re(\lambda)>0,\Re(\beta)>0;\mid t\mid<\frac{1-\mid x\mid}{1+\mid y\mid}and\mid x\mid<1)$$

Proof. Replacing $t\to(1-y)t$ in Theorem 11, multiplying the resulting equality by $y^{\gamma-1}$ and then applying the operator $D_y^{\gamma-\delta,p,q}$, we get

$$D_y^{\gamma-\delta,p,q}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}y_2^{\gamma-1}F_{1;p,q}(\lambda+n,\alpha;\beta;x)(1-y)^n t^n\right]$$

$$=D_y^{\gamma-\delta,p,q}\left[(1-(1-y)t)^{-\lambda}y_2^{\gamma-1}F_{1;p,q}\left(\lambda,\alpha;\beta;\frac{x}{1-(1-y)t}\right)\right]$$

Interchanging the order, which is valid under stated conditions, we can write that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_y^{\gamma-\delta,p,q}\left[y^{\gamma-1}(1-y)^n\right]_2F_{1;p,q}(\lambda+n,\alpha;\beta;x)t^n$$
$$=(1-t)^{-\lambda}D_y^{\gamma-\delta,p,q}\left[y^{\gamma-1}\left(1-\frac{-yt}{1-t}\right)^{-\lambda}_2F_{1;p,q}\left(\lambda,\alpha;\beta;\frac{\frac{x}{1-t}}{1-\frac{-yt}{1-t}}\right)\right]$$

Using Theorems 5 and 6, we get the result. □

Remark 2. *For $p=0=q$, the results presented here would reduce to the corresponding well-known results (see, for details,* [2, 10, 12, 13]*).*

Theorem 14. *Let $\Re(p)>0,\Re(q)>0,\Re(\mu)>\Re(\lambda)>0$; $\gamma,\delta\in\mathbb{C}$ and the extended Riemann-Liouville fractional derivative* (3.1)*. Then there holds the formula:*

$$D_z^{\lambda-\mu,p,q}\left[z^{\lambda-1}\,E^{\mu}_{\gamma,\delta}(z)\right]=\frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma\,n+\delta)}B_{p,q}(\lambda+n,\mu-\lambda)\frac{z^n}{n!},\tag{5.1}$$

where $E^{\mu}_{\gamma,\delta}(z)$ is a well known generalized Mittag-Leffler function due to Prabhakar [9] *defined as:*

$$E^{\mu}_{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma\,n+\delta)}\frac{z^n}{n!}\ \ (\gamma,\delta,\mu\in\mathbb{C};\Re(\gamma)>0).\tag{5.2}$$

Proof. Applying (5.2) to (5.1) and using Theorem 8 and 4, we get

$$\begin{aligned}D_z^{\lambda-\mu,p,q}[z^{\lambda-1}\,E^{\mu}_{\gamma,\delta}(z)]&=D_z^{\lambda-\mu,p,q}\left[z^{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma\,n+\delta)}\frac{z^n}{n!}\right\}\right]\\&=\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma\,n+\delta)n!}\left\{D_z^{\lambda-\mu,p,q}\left[z^{\lambda+n-1}\right]\right\}\\&=\sum_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\gamma\,n+\delta)n!}\left\{\frac{B_{p,q}(\lambda+n,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu+n-1}\right\}.\end{aligned}\tag{5.3}$$

□

Remark 3. *If we set $p=q$ in* (5.1)*, we get the interesting known result given by Özarslan and Yilmaz* [6, Theorem 9].

Theorem 15. *Let $\Re(p)>0,\Re(q)>0,\Re(\mu)>\Re(\lambda)>0$; $\gamma,\delta\in\mathbb{C}$ and the extended Riemann-Liouville fractional derivative* (3.1)*. Then there holds the formula:*

$$D_z^{\lambda-\mu,p,q}\left[z^{\lambda-1}\,{}_m\Psi_n\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,m}\\(b_j,\beta_j)_{1,n}\end{matrix}\right.\right]\right]=\frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma\left(a_i+\alpha_ik\right)}{\prod_{j=1}^{n}\Gamma\left(b_j+\beta_jk\right)}B_{p,q}(\lambda+k,\mu-\lambda)\frac{z^k}{k!},\tag{5.4}$$

where ${}_p\Psi_q(z)$ *is the Fox-Wright function defined by* (*see* [2, pp. 56–58])

$$ {}_m\Psi_n(z) = {}_m\Psi_n\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,m}\\(b_j,\beta_j)_{1,n}\end{matrix}\right.\right] := \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{n}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!}. \tag{5.5}$$

Proof. Applying the result in Theorem 4 to the (5.5) and using same process as similar to Theorem 14, we get desired result.

□

Remark 4. *If we set* $p = q$ *in* (5.4)*, we get the interesting known result given by Shrma and Devi* [11, p. 49, Theorem 8]*.*

6. Conclusion

The fractional derivative operator $\mathcal{D}_z^{\mu,p,q}\{f(z)\}$ in (3.1) is defined for $\{\Re(p),\Re(q)\} \geq 0$. The extended fractional derivatives for the some elementary functions are given by Theorem 4-8. The Mellin transform of the (3.1) and generating relations of linear and bilinear type for the extended hypergeometric functions are given by Theorem 11 to 13, respectively. All of this show that this paper has the distinctive advantage in the field of applied mathematics.

References

[1] Choi J., Rathie A.K. and Parmar R.K., Extension of extended beta, hypergeometric and confluent hypergeometric functions, *Honam Math. J.* **36(2)** (2014) 339367.

[2] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands, 2006; Volume 204.

[3] Luo M. J., Milovanovic G. V. and Agarwal P., Some results on the extended beta and extended hypergeometric functions, *Appl. Math. Comput.* **248** (2014), 631–651.

[4] Olver F. W. J., Lozier D. W., Boisvert R. F. and Clark C. W. (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[5] Özarslan M.A., Özergin E., Some generating relations for extended hypergeometric function via generalized fractional derivative operator, *Math. Comput. Modelling* **52** (2010), 1825–1833.

[6] Özarslan M. A. and Yilmaz B., The extended Mittag-Leffer function and its properties, *J. Inequal Appl.*(2014). doi:10.1186/1029-242X-2014-85.

[7] Parmar R. K., Some generating relations for generalized extended hypergeometric functions involving generalized fractional derivative operator, *J. Concr. Appl. Math.* **12** (2014), 217–228.

[8] Paris R.B. and Kaminski D., *Asymptotics and Mellin-Barnes Integrals*, Cambridge University Press, Cambridge, 2001.

[9] Prabhakar T. R., A Singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math J* **19** (1971), 7–15.

[10] Samko S.G., Kilbas A.A., Marichev O.I., *Fractional Integrals and Derivatives*: *Theory and Applications*, Translated from the Russian: *Integrals and Derivatives of Fractional Order and Some of Their Applications* ("Nauka i Tekhnika", Minsk, 1987); Gordon and Breach Science Publishers: Reading, UK, 1993.

[11] Sharma S. C. and Devi M.,Certain Properties of Extended Wright Generalized Hypergeometric Function, *Annals of Pure and Applied Mathematics* **9**(1) (2014), 45–51.

[12] Srivastava H.M. and Karlsson P. W., *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[13] Srivastava H.M. and Manocha H. L., *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

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