COMPARABLE NONLINEAR CONTRACTIONS IN ORDERED METRIC SPACES

AFTAB ALAM*, QAMRUL HAQ KHAN AND MOHAMMAD IMDAD

Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India.
Email addresses: aafu.amu@gmail.com, qhkhan.ssitm@gmail.com, mhimdad@yahoo.co.in

Abstract. In this article, we generalize some frequently used metrical notions such as: completeness, closedness, continuity, g-continuity and compatibility to order-theoretic setting especially in ordered metric spaces and utilize these relatively weaker notions to prove some existence and uniqueness results on coincidence points for g-comparable mappings satisfying Boyd-Wong type nonlinear contractivity conditions. We also furnish some illustrative examples to demonstrate our results. Finally, as an application of our certain newly proved results, we establish the existence and uniqueness of solution of an integral equation.

Keywords: ordered metric space; *TCC* property; g-comparable mappings; g-admissible mappings; termwise monotone sequence.

AMS Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

The classical Banach contraction principle and its applications are well known. It has so many different generalizations with different approaches. One of the remarkable generalizations, known as φ-contraction, was given by Browder [1] in 1968, wherein author assumed φ to be right continuous and increasing control function and utilized the same to generalize Banach contraction principle. Later, many authors generalized Browder's fixed point theorem by varying the properties of control function φ. In 1969, Boyd and Wong [2] observed that it is sufficient to assume merely the right-upper semicontinuity of φ (without monotonicity requirement on φ) and extended Browder's fixed point theorem by introducing the following family of control functions:

$$\Psi = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \varphi \text{ is right} - \text{upper semicontinuous}\Big\}.$$

Inspired by Boyd and Wong [2], in 1977, Mukherjea [3] slightly modified Browder's fixed point theorem by introducing the following family of control functions:

$$\Theta = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \varphi \text{ is right continuous}\Big\}.$$

The following family of control functions available in the existing literature is more natural:

$$\Im = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \varphi \text{ is continuous}\Big\}.$$

*Correspondence: aafu.amu@gmail.com.

The following family of control functions is essentially due to Lakshmikantham and Ćirić [4]:

$$\Phi = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \lim_{r\to t^+} \varphi(r) < t \text{ for each t} > 0\Big\}.$$

The following family of control functions is ontained in Boyd and Wong [2] which was later utilized in Jotic [5]:

$$\Omega = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r\to t^+} \varphi(r) < t \text{ for each t} > 0\Big\}.$$

Recently, Alam *et al*. [22] studied the following relation among earlier described classes of control functions.

Proposition 1 [22]. The class Ω enlarges the classes Ψ, Θ, $\Im$ and Φ under the following inclusion relation:

$$\Im \subset \Theta \subset \Psi \subset \Omega \text{ and } \Im \subset \Theta \subset \Phi \subset \Omega.$$

Throughout the manuscript, $\mathbb{N}_0$ denotes the set of nonnegative integers (*i.e.* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). In what follows, by the pair $(X, \preceq)$, we mean a nonempty set X equipped with a partial order $\preceq$, often called an ordered set. We denote $\succeq$ by the dual order of $\preceq$ (*i.e.* $x \succeq y$ means $y \preceq x$). Two elements x and y in an ordered set $(X, \preceq)$ are said to be comparable if either $x \preceq y$ or $y \preceq x$ and we denote it as: $x \prec\succ y$. A subset E of an ordered set is called totally ordered if $x \prec\succ y$ for all $x, y \in E$. For a pair of self-mappings f and g defined on an ordered set $(X, \preceq)$, we say that f is g-increasing (resp. g-decreasing) if for any $x, y \in X$ $g(x) \preceq g(y) \Rightarrow f(x) \preceq f(y)$(resp. f(x) $\succeq$ f(y)). Also, f is called g-monotone if f is either g-increasing or g-decreasing. Further, under the restriction $g = I$, the identity mapping on X, the notions of g-increasing, g-decreasing and g-monotone mappings respectively reduce to increasing, decreasing and monotone mappings. Following O'Regan and Petruşel [11], the triplet $(X, d, \preceq)$, is called ordered metric space wherein a nonempty set X is equipped with a metric d and a partial order $\preceq$. If, in addition, d is a complete metric on X, then we say that $(X, d, \preceq)$ is ordered complete metric space.

In the last decades, there has been a growing interest in studying the existence of fixed points for monotone contractive mappings in ordered metric spaces. This trend was initiated by Turinici [6, 7]. Later, Ran and Reurings [8] proved a slightly more natural version of the corresponding fixed point theorems of Turinici (cf.[6, 7]) for continuous monotone mappings with some applications to matrix equations. In subsequent papers many authors extended and refined the fixed point theorems of Ran and Reurings [8] and proved various fixed point theorems in ordered metric spaces (e.g. [9]-[23]). All such results involve some variant of monotone mappings.

To avoid the necessity of monotonicity, several authors like as: Nieto and Rodríguez-López [25], Turinici [26, 27] and Dorić *et al*. [28] assumed the property that the underlying mapping maps comparable elements to comparable elements, which is relatively weaker than monotonicity requirement on the mapping and often easy to check. Recently, Alam and Imdad [24] termed such mapping as comparable mappings and also generalized this idea for a pair of mappings by introducing the notion of g-comparable mappings.

Definition 1 [24]. Let $(X, \preceq)$ be an ordered set and f and g two self-mappings on X. We say that f is g-comparable (or weakly g-monotone or $(g, \prec\succ)$-preserving) if for any $x, y \in X$,

$$g(x) \prec\succ g(y) \Rightarrow f(x) \prec\succ f(y).$$

Notice that if we set $g = I$, the identity mapping on X in Definition 1, then f is called comparable (or weakly monotone or $\prec\succ$-preserving) mapping. Clearly every g-monotone mapping is g-comparable.

Definition 2 [30, 31]. Let X be a nonempty set and f and g two self-mappings on X. Then,

(i) an element $x \in X$ is called a coincidence point of f and g if

$$g(x) = f(x),$$

(ii) if $x \in X$ is a coincidence point of f and g and $\overline{x} \in X$ such that $\overline{x} = g(x) = f(x)$, then $\overline{x}$ is called a point of coincidence of f and g,

(iii) if $x \in X$ is a coincidence point of f and g such that $x = g(x) = f(x)$, then x is called a common fixed point of f and g,

(iv) f and g are said to be commuting if

$$g(fx) = f(gx) \quad \forall\, x \in X \text{ and}$$

(v) f and g are said to be weakly compatible (or partially commuting or coincidentally commuting) if f and g commute at their coincidence points, *i.e.*, for any $x \in X$,

$$g(x) = f(x) \Rightarrow g(fx) = f(gx).$$

Definition 3 [32, 33]. Let (X, d) be a metric space and f and g two self-mappings on X. Then

(i) f and g are said to be weakly commuting if

$$d(gfx, fgx) \leq d(gx, fx) \quad \forall\, x \in X \text{ and}$$

(ii) f and g are said to be compatible if for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = z \Rightarrow \lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

Definition 4 [23]. Let $(X, \preceq)$ be an ordered set and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ is said to be termwise bounded if there is an element $z \in X$ such that each term of $\{x_n\}$ is comparable with z, *i.e.*,

$$x_n \prec\succ z \qquad \forall\, n \in \mathbb{N}_0$$

so that z is a c-bound of $\{x_n\}$ and

(ii) $\{x_n\}$ is said to termwise monotone (or $\prec\succ$-preserving) if consecutive terms of $\{x_n\}$ are comparable, *i.e.*,

$$x_n \prec\succ x_{n+1} \;\; \forall\, n \in \mathbb{N}_0.$$

Clearly all bounded above as well as bounded below sequences are termwise bounded and all monotone sequences are termwise monotone.

Definition 5 [34]. Let $(X, d, \preceq)$ be called an ordered metric space. A nonempty subset Y of X is called a subspace of X if Y itself is an ordered metric space equipped with the metric d_Y and partial order $\preceq_Y$ defined by:

$$d_Y(x, y) = d(x, y) \ \forall \ x, y \in Y$$

and

$$x \preceq_Y y \Leftrightarrow x \preceq y \ \forall \ x, y \in Y.$$

Conventionally, we write d and $\preceq$ instead of d_Y and $\preceq_Y$ respectively.

Let $(X, d, \preceq)$ be an ordered metric space and $\{x_n\}$ a sequence in X. If $\{x_n\}$ is termwise monotone and $x_n \xrightarrow{d} x$; then we denote it symbolically by $x_n \updownarrow x$.

The following notion is formulated by using a suitable property on ordered metric space (in order to avoid the necessity of continuity requirement in Ran-Reurings Theorem) utilized by Nieto and Rodríguez-López [25].

Definition 6 [23]. Let $(X, d, \preceq)$ be an ordered metric space. We say that $(X, d, \preceq)$ has *TCC* (termwise monotone-convergence-c-bound) property if every termwise monotone convergent sequence $\{x_n\}$ in X has a subsequence, which is termwise bounded by the limit (of the sequence) as a c-bound, *i.e.*,

$$x_n \updownarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } \ x_{n_k} \prec\succ x \ \ \forall \ k \in \mathbb{N}_0.$$

Definition 7 [23]. Let $(X, d, \preceq)$ be an ordered metric space and g a self-mapping on X. We say that $(X, d, \preceq)$ has g-*TCC* property if every termwise monotone convergent sequence $\{x_n\}$ in X has a subsequence, whose g-image is termwise bounded by g-image of limit (of the sequence) as a c-bound, *i.e.*,

$$x_n \updownarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } \ g(x_{n_k}) \prec\succ g(x) \ \ \forall \ k \in \mathbb{N}_0.$$

Notice that under the restriction $g = I$, the identity mapping on X, Definition 7 reduces to Definition 6.

Definition 8 [35]. Let (X, d) be a metric space, f and g two self-mappings on X and $x \in X$. We say that f is g-continuous at x if for any sequence $\{x_n\} \subset X$,

$$g(x_n) \xrightarrow{d} g(x) \Rightarrow f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called g-continuous if it is g-continuous at each point of X.
Notice that particularly, at $g = I$, the identity mapping on X, Definition 8 reduces to the definition of continuity.

Definition 9 [23]. An ordered set $(X, \preceq)$ is called sequentially chainable if range of every termwise monotone sequence in X remains a totally ordered subset of X.

Proposition 2 [23]. The following are equivalent:
(i) $(X, \preceq)$ is sequentially chainable,
(ii) $\prec\succ$ is transitive on range of every termwise monotone sequence in X,
(iii) for every termwise monotone sequence $\{x_n\}$ in X,

$$x_n \prec\succ x_m \ \ \forall\ n, m \in \mathbb{N}_0.$$

Inspired by Jleli *et al*. [19], Alam and Imdad [24] defined the following notion:

Definition 10 [24]. Let $(X, \preceq)$ be an ordered set and f and g two self-mappings on X. We say that $(X, \preceq)$ is (f, g)-directed if for each pair $x, y \in X$, $\exists\ z \in X$ such that $f(x) \prec\succ g(z)$ and $f(y) \prec\succ g(z)$.
In cases $g = I$ and $f = g = I$ (where I denotes identity mapping on X), $(X, \preceq)$ is called f-directed and directed respectively.

Inspired by Turinici [26, 27], Alam and Imdad [24] defined the following notion:

Definition 11 [24]. Let $(X, \preceq)$ be an ordered set, $E \subseteq X$ and $a, b \in E$. A finite subset $\{e_1, e_2, ..., e_k\}$ of E is called $\prec\succ$-chain between a and b in E if
(i) $k \geq 2$,
(ii) $e_1 = a$ and $e_k = b$,
(iii) $e_i \prec\succ e_{i+1}$ for each i $(1 \leq i \leq k-1)$.

We need the following known results in the proof of our main result.

Lemma 1 [22]**.** Let $\varphi \in \Omega$. If $\{a_n\} \subset (0, \infty)$ is a sequence such that $a_{n+1} \leq \varphi(a_n)\ \forall\ n \in \mathbb{N}_0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2 [19, 36, 37]. Let (X, d) be a metric space and $\{x_n\}$ a sequence in X. If $\{x_n\}$ is not a Cauchy, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that
(i) $k \leq m_k < n_k \ \ \forall\ k \in \mathbb{N}$,
(ii) $d(x_{m_k}, x_{n_k}) > \epsilon \ \ \forall\ k \in \mathbb{N}$,
(iii) $d(x_{m_k}, x_{n_{k-1}}) \leq \epsilon \ \ \forall\ k \in \mathbb{N}$.
Moreover, suppose that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, then
(iv) $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,
(v) $\lim\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon$.
Lemma 3 [38]**.** Let X be a nonempty set and g a self-mapping on X. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-one.

Lemma 4 [22]**.** Let f and g be two self-mappings defined on a nonempty set X. If f and g are weakly compatible, then every point of coincidence of f and g is also a coincidence point of f and g.

Most recently, Alam and Imdad [24] proved the following result in ordered metric spaces for linear contractivity conditions without involving g-monotone mappings.

Theorem 1 [24]**.** Let $(X, d, \preceq)$ be an ordered metric space and f and g two self-mappings on X. Suppose that the following conditions hold:
(a) $f(X) \subseteq g(X)$,
(b) f is g-comparable,
(c) there exists $x_0 \in X$ such that $g(x_0) \prec\succ f(x_0)$,
(d) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha\; d(gx, gy), \;\; \forall\; x, y \in X \text{ with } g(x) \prec\succ g(y),$$

(e) (e1) (X, d) is complete,
(e2) f and g are compatible,
(e3) g is continuous,
(e4) either f is continuous or $(X, d, \preceq)$ has g-TCC property,
or
(e') (e'1) either (fX, d) or (gX, d) is complete,
(e'2) either f is g-continuous or f and g are continuous or $(gX, d, \preceq)$ has TCC property.
Then f and g have a coincidence point.

The aim of this paper is to extend Theorem 1 under nonlinear contractivity condition due to Boyd and Wong [2]. Particularly, we observe that in our results neither the whole space X nor the range subspaces ($f(X)$ or $g(X)$) are required to be necessarily complete.

2. Results on Coincidence Points

Firstly, we adopt several well-known metrical notions such as: completeness, closedness, continuity, g-continuity and compatibility with respect to relation $\prec\succ$.

Definition 12. An ordered metric space $(X, d, \preceq)$ is called $\prec\succ$-complete if every termwise monotone Cauchy sequence in X converges.

Remark 1. If (X, d) is a complete metric space, then for each partial order $\preceq$ defined on X, the ordered metric space $(X, d, \preceq)$ is $\prec\succ$-complete.

Definition 13. Let $(X, d, \preceq)$ be an ordered metric space. A subset E of X is called $\prec\succ$-closed if for any sequence $\{x_n\} \subset E$,

$$x_n \updownarrow x \Rightarrow x \in E.$$

Remark 2. Every closed subset of an ordered metric space is $\prec\succ$-closed.

Proposition 3. A $\prec\succ$-complete subspace of an ordered metric space is $\prec\succ$-closed.
Proof. Let $(X, d, \preceq)$ be an ordered metric space. Suppose that Y is $\prec\succ$-complete subspace of X. Take a sequence $\{x_n\} \subset Y$ such that $x_n \updownarrow x \in X$. As each convergent sequence is Cauchy, $\{x_n\}$ is a termwise monotone Cauchy sequence in Y. Hence, $\prec\succ$-completeness of Y implies that the limit of $\{x_n\}$ must lie in Y, *i.e.*, $x \in Y$. Therefore, Y is $\prec\succ$-closed.

Proposition 4. A $\prec\succ$-closed subspace of a $\prec\succ$-complete ordered metric space is $\prec\succ$-complete.
Proof. Let $(X, d, \preceq)$ be a $\prec\succ$-complete metric space. Suppose that Y is $\prec\succ$-closed subspace of X. Take a termwise monotone Cauchy sequence $\{x_n\}$ in Y. As X is $\prec\succ$-complete, $\exists\, x \in X$ such that $x_n \xrightarrow{d} x$ and so $x_n \updownarrow x$. Hence, $\prec\succ$-closeness of Y implies that $x \in Y$. Therefore, Y is $\prec\succ$-complete.

Definition 14. Let $(X, d, \preceq)$ be an ordered metric space, f a self-mapping on X and $x \in X$. We say that f is $\prec\succ$-continuous at x if for any sequence $\{x_n\} \subset X$,

$$x_n \updownarrow x \Rightarrow f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called $\prec\succ$-continuous if it is $\prec\succ$-continuous at each point of X.

Remark 3. Every continuous mapping defined on an ordered metric space is $\prec\succ$-continuous.

Definition 15. Let $(X, d, \preceq)$ be an ordered metric space, f and g two self-mappings on X and $x \in X$. We say that f is $(g, \prec\succ)$-continuous at x if for any sequence $\{x_n\} \subset X$,

$$g(x_n) \updownarrow g(x) \Rightarrow f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called $(g, \prec\succ)$-continuous if it is $(g, \prec\succ)$-continuous at each point of X. Notice that particularly, at $g = I$, the identity mapping on X, Definition 15 reduces to Definition 14.

Remark 4. Every g-continuous mapping defined on an ordered metric space is $(g, \prec\succ)$-continuous.

Definition 16. Let $(X, d, \preceq)$ be an ordered metric space and f and g two self-mappings on X. We say that f and g are $\prec\succ$-compatible if if for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$g(x_n) \updownarrow z \text{ and } f(x_n) \updownarrow z \Rightarrow \lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

Remark 5. In an ordered metric space, commutativity $\Rightarrow$ weak commutativity $\Rightarrow$ compatibility $\Rightarrow$ $\prec\succ$-compatibility $\Rightarrow$ weak compatibility.

Definition 17. Let f and g be two self-mappings defined on a nonempty set X. We say that f is g-admissible if for any $x, y \in X$,

$$g(x) = g(y) \Rightarrow f(x) = f(y).$$

Proposition 5. Let f and g be two self-mappings defined on an ordered set $(X, \preceq)$. If f is g-monotone, then f is g-admissible.
Proof. Take $x, y \in X$ such that $g(x) = g(y)$. On using reflexivity of $\preceq$, we have

$$g(x) \preceq g(y) \text{ and } g(x) \succeq g(y).$$

Suppose that f is g-increasing (resp. g-decreasing), we have

$$f(x) \preceq f(y) \text{ and } f(x) \succeq f(y) (\text{ resp. } f(x) \succeq f(y) \text{ and } f(x) \preceq f(y)),$$

which, in both the cases (owing to anti-symmetric property of $\preceq$) gives rise that

$$f(x) = f(y).$$

Hence f is g-admissible.

We are equipped to prove the following result, which guarantees the existence of alleast one coincidence point under φ-contractivity condition.

Theorem 2. Let $(X, d, \preceq)$ be an ordered metric space and f and g two self-mappings on X. Let Y be a $\prec\succ$-complete subspace of X such that $(Y, \preceq)$ is sequentially chainable. Suppose that the following conditions hold:

(a) $f(X) \subseteq g(X) \cap Y$,
(b) f is g-comparable and g-admissible,
(c) there exists $x_0 \in X$ such that $g(x_0) \prec\succ f(x_0)$,
(d) there exists $\varphi \in \Omega$ such that
$$d(fx, fy) \leq \varphi(d(gx, gy)) \ \ \forall\, x, y \in X \text{ with } g(x) \prec\succ g(y),$$
(e) $(e1)$ f and g are $\prec\succ$-compatible,
$(e2)$ g is $\prec\succ$-continuous,
$(e3)$ either f is $\prec\succ$-continuous or $(Y, d, \preceq)$ has g-TCC property,
or alternately
(e') $(e'1)$ $Y \subseteq g(X)$,
$(e'2)$ either f is $(g, \prec\succ)$-continuous or f and g are continuous or $(Y, d, \preceq)$ has TCC property.

Then f and g have a coincidence point.

Proof. Firstly, we notice that the assumption (a) is equivalent to saying that $f(X) \subseteq g(X)$ and $f(X) \subseteq Y$. Now, in view of assumption (c) if $g(x_0) = f(x_0)$, then x_0 is a coincidence point of f and g and hence proof is completed. Otherwise, using assumption $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $g(x_1) = f(x_0)$. Again from $f(X) \subseteq g(X)$, we can choose $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process, we define a sequence $\{x_n\} \subset X$ of joint iterates such that

$$g(x_{n+1}) = f(x_n) \ \ \forall\, n \in \mathbb{N}_0. \tag{1}$$

Now, we claim that $\{gx_n\}$ is a termwise monotone sequence, *i.e.*,

$$g(x_n) \prec\succ g(x_{n+1}) \ \ \forall\, n \in \mathbb{N}_0. \tag{2}$$

We prove this fact by mathematical induction. On using equation (1) with $n = 0$ and assumption (c), we have

$$g(x_0) \prec\succ f(x_0) = g(x_1)$$

Thus, (2) holds for $n = 0$. Suppose that (2) holds for $n = r > 0$, *i.e.*,

$$g(x_r) \prec\succ g(x_{r+1}),$$

which on using (1) and g-comparability of f gives rise

$$g(x_{r+1}) = f(x_r) \prec\succ f(x_{r+1}) = g(x_{r+2}).$$

i.e., (2) holds for $n = r + 1$. Hence, by induction, (2) holds for all $n \in \mathbb{N}_0$.

In view of (1) and (2), the sequence $\{fx_n\}$ is also a termwise monotone sequence, *i.e.*,

$$f(x_n) \prec\succ f(x_{n+1}) \ \ \forall\, n \in \mathbb{N}_0. \tag{3}$$

If $g(x_{n_0}) = g(x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then using (1), we have $g(x_{n_0}) = f(x_{n_0})$, *i.e.*, x_{n_0} is a coincidence point of f and g so that we are through. On the other hand, if $g(x_n) \neq g(x_{n+1})$ for each $n \in \mathbb{N}_0$, we can define a sequence $\{d_n\}_{n=0}^{\infty} \subset (0, \infty)$, where

$$d_n := d(gx_n, gx_{n+1}). \tag{4}$$

On using (1), (2), (4) and assumption (d), we obtain

$$\begin{aligned} d_{n+1} &= d(gx_{n+1}, gx_{n+2}) \\ &= d(fx_n, fx_{n+1}) \\ &\leq \varphi(d(gx_n, gx_{n+1})) \\ &= \varphi(d_n) \end{aligned}$$

so that

$$d_{n+1} \leq \varphi(d_n).$$

Hence by Lemma 1, we obtain

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{5}$$

Next, we show that $\{gx_n\}$ is a Cauchy sequence. On contrary suppose that $\{gx_n\}$ is not a Cauchy, then owing to Lemma 2, there exist $\epsilon > 0$ and two subsequences $\epsilon > 0$ and two subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ of $\{gx_n\}$ such that $k \leq m_k < n_k$ and $d(gx_{m_k}, gx_{n_k}) > \epsilon \geq d(gx_{m_k}, gx_{n_{k-1}})$ for all $k \in \mathbb{N}$. Further, in view of (5), Lemma 2 assures us that

$$\lim_{k\to\infty} d(gx_{m_k}, gx_{n_k}) = \lim_{k\to\infty} d(gx_{m_k+1}, gx_{n_k+1}) = \epsilon. \tag{6}$$

Denote $r_k := d(gx_{m_k}, gx_{n_k})$. Owing to (1), we have $\{gx_n\} \subset f(X) \subseteq Y$ so that $\{gx_n\}$ is termwise monotone sequence in Y (due to (2)). As $(Y, \preceq)$ is sequentially chainable, by using Proposition 2, we obtain $g(x_{m_k}) \prec\succ g(x_{n_k})$. On using (1) and assumption (d), we obtain

$$\begin{aligned} d(gx_{m_k+1}, gx_{n_k+1}) &= d(fx_{m_k}, fx_{n_k}) \\ &\leq \varphi(d(gx_{m_k}, gx_{n_k})). \\ &= \varphi(r_k) \end{aligned}$$

so that

$$d(gx_{m_k+1}, gx_{n_k+1}) \leq \varphi(r_k). \tag{7}$$

On taking limit superior as $k \longrightarrow \infty$ in (7) and using (6) and the definition of Ω, we have

$$\epsilon = \limsup_{k\to\infty} d(gx_{m_k+1}, gx_{n_k+1}) \leq \limsup_{k\to\infty} \varphi(r_k) = \limsup_{r_k\to\epsilon^+} \varphi(r_k) < \epsilon,$$

which is a contradiction yielding thereby $\{gx_n\}$ is a Cauchy sequence.

Therefore $\{gx_n\}$ is a termwise monotone Cauchy sequence in Y. As Y is $\prec\succ$-complete, there exists $z \in Y$ such that $\lim_{n\to\infty} g(x_n) = z$, which combining with (2), gives rise

$$g(x_n) \updownarrow z. \tag{8}$$

On using (1), (3) and (8), we obtain

$$f(x_n) \updownarrow z. \tag{9}$$

Now, we use assumptions (e) and (e') to accomplish the proof. Assume that (e) holds. Using assumption $(e2)$ ($i.e.$ $\prec\succ$-continuity of g) in (8) and (9), we have

$$\lim_{n\to\infty} g(gx_n) = g(z). \tag{10}$$

$$\lim_{n\to\infty} g(fx_n) = g(z). \tag{11}$$

On using (8), (9) and assumption $(e1)$($i.e.$ $\prec\succ$-compatibility of f and g), we obtain

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0. \tag{12}$$

Now, we show that z is a coincidence point of f and g. To accomplish this, we use assumption $(e3)$. Suppose that f is $\prec\succ$-continuous. On using (8) and $\prec\succ$-continuity of f, we obtain

$$\lim_{n\to\infty} f(gx_n) = f(z). \tag{13}$$

On using (11), (12), (13) and continuity of d, we obtain

$$\begin{aligned} d(gz, fz) &= d(\lim_{n\to\infty} gfx_n, \lim_{n\to\infty} fgx_n) \\ &= \lim_{n\to\infty} d(gfx_n, fgx_n) \\ &= 0 \end{aligned}$$

so that

$$g(z) = f(z).$$

Thus $z \in X$ is a coincidence point of f and g and hence we are through. Alternately, suppose that $(Y, d, \preceq)$ has g-TCC property, then due to availability of (8), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(gx_{n_k}) \prec\succ g(z)\ \forall\ k \in \mathbb{N}_0. \tag{14}$$

On using (14) and assumption (d), we obtain

$$d(fgx_{n_k}, fz) \leq \varphi(d(ggx_{n_k}, gz))\ \ \forall\ k \in \mathbb{N}_0.$$

Now, we asserts that

$$d(fgx_{n_k}, fz) \leq d(ggx_{n_k}, gz)\ \ \forall\ k \in \mathbb{N}. \tag{15}$$

On account of two different possibilities arising here, we consider a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$, $i.e.$, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

(i)$d(ggx_{n_k}, gz) = 0\ \ \forall\ k \in \mathbb{N}^0$,

(ii)$d(ggx_{n_k}, gz) > 0\ \ \forall\ k \in \mathbb{N}^+$.

In case (i), on using g-admissibility of f, we get $d(fgx_{n_k}, fz) = 0\ \forall\ k \in \mathbb{N}^0$ and hence (15) holds for all $k \in \mathbb{N}^0$. In case (ii), owing to the definition of Ω, we have $d(fgx_{n_k}, fz) \leq \varphi(d(ggx_{n_k}, gz)) < d(ggx_{n_k}, gz)\ \forall\ k \in \mathbb{N}^+$ and hence (15) holds for all $k \in \mathbb{N}^+$. Thus (15) holds for all $k \in \mathbb{N}$.

On using triangular inequality, (10), (11), (12) and (15), we get

$$\begin{aligned} d(gz, fz) &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + d(fgx_{n_k}, fz) \\ &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + d(ggx_{n_k}, gz) \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

so that

$$g(z) = f(z).$$

Thus $z \in X$ is a coincidence point of f and g.

Now, assume that (e') holds. Owing to assumption $(e'1)$ (*i.e.*, $Y \subseteq g(X)$), we can find some $u \in X$ such that $z = g(u)$. Hence, (8) and (9) respectively reduce to

$$g(x_n) \updownarrow g(u). \tag{16}$$

$$f(x_n) \updownarrow g(u). \tag{17}$$

Now, we show that u is a coincidence point of f and g. To accomplish this, we use assumption $(e'2)$. Firstly, suppose that f is $(g, \prec\succ)$-continuous, then using (16), we get

$$\lim_{n\to\infty} f(x_n) = f(u). \tag{18}$$

On using (17) and (18), we get

$$g(u) = f(u).$$

Hence, we are done. Secondly, suppose that f and g are continuous. Owing to Lemma 3, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-one. Now, define $T : g(E) \to g(X)$ by

$$T(ga) = f(a) \ \ \forall\, g(a) \in g(E) \text{ where } a \in E. \tag{19}$$

As $g : E \to X$ is one-one and $f(X) \subseteq g(X)$, T is well defined. Again since f and g are continuous, it follows that T is continuous. Using $g(X) = g(E)$, assumptions (a) and $(e'1)$ reduce to respectively $f(X) \subseteq g(E) \cap Y$ and $Y \subseteq g(E)$, which follows that, without loss of generality, we are able to construct $\{x_n\}_{n=1}^{\infty} \subset E$ satisfying (1) and to chose $u \in E$. On using (16), (17), (19) and continuity of T, we get

$$f(u) = T(gu) = T(\lim_{n\to\infty} gx_n) = \lim_{n\to\infty} T(gx_n) = \lim_{n\to\infty} f(x_n) = g(u).$$

Thus $u \in X$ is a coincidence point of f and g and hence we are through. Finally, suppose that $(Y, d, \preceq)$ has TCC property, then due to availability of (16), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(x_{n_k}) \prec\succ g(u)\ \forall\ k \in \mathbb{N}_0. \tag{20}$$

On using (1), (20) and assumption (d), we obtain

$$d(gx_{n_k+1}, fu) = d(fx_{n_k}, fu) \le \varphi(d(gx_{n_k}, gu)) \ \ \forall\ k \in \mathbb{N}_0.$$

We asserts that

$$d(gx_{n_k+1}, fu) \le d(gx_{n_k}, gu) \ \ \forall\ k \in \mathbb{N}. \tag{21}$$

On account of two different possibilities arising here, we consider a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$, *i.e.*, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

(i)$d(gx_{n_k}, gu) = 0 \ \ \forall\ k \in \mathbb{N}^0$,

(ii)$d(gx_{n_k}, gu) > 0 \ \ \forall\ k \in \mathbb{N}^+$.

In case (i), on using g-admissibility of f, we get $d(fx_{n_k}, fu) = 0\ \forall\ k \in \mathbb{N}^0$, which in view of (1), gives rise $d(gx_{n_k+1}, fu) = 0\ \forall\ k \in \mathbb{N}^0$ and hence (21) holds for all $k \in \mathbb{N}^0$. In case (ii), by the definition of Ω, we have $d(gx_{n_k+1}, fu) \le \varphi(d(gx_{n_k}, gu)) < d(gx_{n_k}, gu)\ \forall\ k \in \mathbb{N}^+$ and hence (21) holds for all $n \in \mathbb{N}^+$. Thus (21) holds for all $k \in \mathbb{N}$.

On using (16), (21) and continuity of d, we get

$$\begin{aligned} d(gu, fu) &= d(\lim_{n\to\infty} gx_{n+1}, fu) \\ &= \lim_{k\to\infty} d(gx_{n_k+1}, fu) \end{aligned}$$

$$\leq \lim_{k\to\infty} d(gx_{n_k}, gu)$$
$$= 0$$

so that

$$g(u) = f(u).$$

Hence, u is a coincidence point of f and g. This completes the proof.

Remark 6. In view of Proposition 1, Theorem 2 remains true if we replace the class Ω by anyone of the classes Ψ, Θ, $\Im$ and Φ.

With a view to deduce a natural consequence, we particularize Theorem 2 by assuming the $\prec\succ$-completeness of whole space X.

Corollary 1. Let $(X, d, \preceq)$ be a $\prec\succ$-complete ordered metric space such that $(X, \preceq)$ is sequentially chainable and f and g two self-mappings on X. Suppose that the following conditions hold:

(a) $f(X) \subseteq g(X)$,
(b) f is g-comparable and g-admissible,
(c) there exists $x_0 \in X$ such that $g(x_0) \prec\succ f(x_0)$,
(d) there exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)) \ \ \forall\, x, y \in X \text{ with } g(x) \prec\succ g(y),$$

(e) ($e1$) f and g are $\prec\succ$-compatible,
($e2$) g is $\prec\succ$-continuous,
($e3$) either f is $\prec\succ$-continuous or $(X, d, \preceq)$ has g-TCC property,
or alternately
(e') ($e'1$) there exists a $\prec\succ$-closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$,
($e'2$) either f is $(g, \prec\succ)$-continuous or f and g are continuous or $(Y, d, \preceq)$ has TCC property.

Then f and g have a coincidence point.

Proof. The result corresponding to part (e) follows easily on setting $Y = X$ in Theorem 2, while the same (result) in the presence of part (e') follows using Proposition 4.

Corollary 2. Theorem 2 (also Corollary 1) remains true if we replace condition "f is g-admissible" by one of the following conditions besides retaining the rest of the hypotheses

(i) $\varphi(0) = 0$,
(ii) g is one-one.

Proof. Suppose that (i) holds. Take $x, y \in X$ such that $g(x) = g(y)$, then $g(x) \preceq g(y)$ and $g(x) \succeq g(y)$. On applying these points to the contractivity condition (d), we get

$$d(fx, fy) \leq \varphi(d(gx, gy)) = \varphi(0) = 0,$$

which implies that $f(x) = f(y)$. It follows that f is g-admissible.
Suppose that (ii) holds. Take $x, y \in X$ such that $g(x) = g(y)$. As g is one-one, we get $x = y$, which implies that $f(x) = f(y)$. Hence, f is g-admissible.

Using the fact that g-monotonicity implies g-comparability and Proposition 5, the following consequence of Theorem 2 and Corollary 1 trivially holds:

Corollary 3. Theorem 2 (also Corollary 1) remains true if we replace condition (b) by the following condition (besides retaining the rest of the hypotheses):

$(b)'$ f is g-monotone.

Remark 7. Notice that Corollary 3 improves the main result of Alam and Imdad [23].

Now, we prove the corresponding result of Theorem 2 under linear contractivity condition, as follows:

Theorem 3. Let $(X, d, \preceq)$ be an ordered metric space and f and g two self-mappings on X. Let Y be a $\prec\succ$-complete subspace of X. Suppose that the following conditions hold:

(a) $f(X) \subseteq g(X) \cap Y$,

(b) f is g-comparable,

(c) there exists $x_0 \in X$ such that $g(x_0) \prec\succ f(x_0)$,

(d) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha\ d(gx, gy)\ \ \forall\ x, y \in X \text{ with } g(x) \prec\succ g(y),$$

(e) $(e1)$ f and g are $\prec\succ$-compatible,

$(e2)$ g is $\prec\succ$-continuous,

$(e3)$ either f is $\prec\succ$-continuous or $(Y, d, \preceq)$ has g-TCC property,

or alternately

(e') $(e'1)$ $Y \subseteq g(X)$,

$(e'2)$ either f is $(g, \prec\succ)$-continuous or f and g are continuous or $(Y, d, \preceq)$ has TCC property.

Then f and g have a coincidence point.

Proof. This result follows from Theorem 2 on setting $\varphi(t) = \alpha t$ with $\alpha \in [0, 1)$ besides removing the following assumptions:

(i) $(Y, \preceq)$ is sequentially chainable,

(ii) f is g-admissible.

Condition (i) is used to prove that $\{gx_n\}$ is a Cauchy sequence. In this case, using the analogous technique as utilized in Theorem 2, we obtain

$$d(gx_{n+1}, gx_{n+2}) = d(fx_n, fx_{n+1}) \leq \alpha d(gx_n, gx_{n+1})\ \forall\ n \in \mathbb{N}_0$$

so that

$$d(gx_n, gx_{n+1}) \leq \alpha^n d(gx_0, gx_1)\ \forall\ n \in \mathbb{N}_0.$$

By classical technique, it can be easily shown that $\{gx_n\}$ is a Cauchy sequence. Thus, there is no need to use the condition (i) as we do not need to apply the contractivity condition on $d(gx_{m_k}, gx_{n_k})$.

Further, as $\varphi(0) = 0$, owing to Corollary 2, we can remove condition (ii).

Remark 8. Notice that Theorem 3 improves Theorem 1 in the following respects:
• In the context of hypotheses (e), the completeness of X is not necessary. Alternately, it can be replaced by the completeness of any subspace Y satisfying $f(X) \subseteq Y$.
• In the context of hypotheses (e'), the completeness of the range subspaces ($f(X)$ or $g(X)$) are not necessary. Alternately, it can be replaced by the completeness of any subspace Y satisfying $f(X) \subseteq Y \subseteq g(X)$.
• The involved metrical terms namely: completeness, continuity, g-continuity and compatibility in Theorem 1 are not necessary as they can be alternately replaced by their respective "$\prec\succ$-analogues".

Using the similar arguments to Corollary 1, we have the following consequence of Theorem 3.

Corollary 4. Let $(X, d, \preceq)$ be a $\prec\succ$-complete ordered metric space and f and g two self-mappings on X. Suppose that the following conditions hold:

(a) $f(X) \subseteq g(X)$,
(b) f is g-comparable,
(c) there exists $x_0 \in X$ such that $g(x_0) \prec\succ f(x_0)$,
(d) there exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(gx, gy)) \ \ \forall\, x, y \in X \text{ with } g(x) \prec\succ g(y),$$

(e) $(e1)$ f and g are $\prec\succ$-compatible,
$(e2)$ g is $\prec\succ$-continuous,
$(e3)$ either f is $\prec\succ$-continuous or $(X, d, \preceq)$ has g-TCC property,

or alternately

(e') $(e'1)$ there exists a $\prec\succ$-closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$,
$(e'2)$ either f is $(g, \prec\succ)$-continuous or f and g are continuous or $(Y, d, \preceq)$ has TCC property.

Then f and g have a coincidence point.

Remark 9. If g is onto in Corollary 1 (also in Corollary 4), then we can drop assumption (a) as in this case it trivially holds. Also, we can remove assumption $(e'1)$ as it trivially holds for $Y = g(X) = X$ using Proposition 3. Whenever, f is onto, owing to assumption (a), g must be onto and hence again same conclusion is immediate.

On using Remarks 1-5, we obtain the more natural versions of foregoing results in the form of the following consequence.

Corollary 5. Theorem 2 (also Theorem 3 and Corollaries 1-4) remains true if the usual metrical terms namely: completeness, closedness, compatibility (or commutativity/weak commutativity), continuity and g-continuity are used instead of their respective "$\prec\succ$-analogous".

In the following lines, we formulate results ensuring the uniqueness of coincidence point, point of coincidence and common fixed point corresponding to Theorems 2 and 3. For a pair of self-mappings f and g defined on a nonempty set X and a subset $E \subseteq X$, we denote the following sets:

$$\mathrm{C}(f, g) \ = \ \{x \in X : gx = fx\}, \ \textit{i.e.}, \text{ the set of all coincidence points of } f \text{ and } g.$$

$$\overline{\mathrm{C}}(f,g) = \{\overline{x} \in X : \overline{x} = gx = fx,\ x \in X\},\ \textit{i.e.}, \text{ the set of all points of coincidence of } f \text{ and } g,$$

$$\mathrm{C}(a,b,\prec\succ,E) = \text{the class of all } \prec\succ\text{-chains between } a \text{ and } b \text{ in } E.$$

Theorem 4. In addition to the hypotheses of Theorem 2 (also Theorem 3), suppose that the following condition holds:
(u_1) $\mathrm{C}(fx, fy, \prec\succ, gX)$ is nonempty, for each $x, y \in X$.
Then f and g have a unique point of coincidence.
Proof. We prove the result for Theorem 2 and analogously, similar arguments can be used for Theorem 3. In view of Theorem 2, $\overline{\mathrm{C}}(f,g) \neq \emptyset$. Take $\overline{x}, \overline{y} \in \overline{\mathrm{C}}(f,g)$, then $\exists\ x, y \in X$ such that

$$\overline{x} = g(x) = f(x) \text{ and } \overline{y} = g(y) = f(y). \tag{22}$$

Now, we show that $\overline{x} = \overline{y}$. As $f(x), f(y) \in f(X) \subseteq g(X)$, by (u_0), there exists a $\prec\succ$-chain $\{gz_1, gz_2, ..., gz_k\}$ between $f(x)$ and $f(y)$ in $g(X)$, where $z_1, z_2, ..., z_k \in X$. Owing to (22), without loss of generality, we may choose $z_1 = x$ and $z_k = y$. We have

$$g(z_i) \prec\succ g(z_{i+1}) \text{ for each } i\ (1 \leq i \leq k-1). \tag{23}$$

Define the constant sequences $z_n^1 = x$ and $z_n^k = y$, then using (22), we have $g(z_{n+1}^1) = f(z_n^1) = \overline{x}$ and $g(z_{n+1}^k) = f(z_n^k) = \overline{y}\ \forall\ n \in \mathbb{N}_0$. Put $z_0^2 = z_2,\ z_0^3 = z_3, ...,\ z_0^{k-1} = z_{k-1}$. Since $f(X) \subseteq g(X)$, on the lines similar to that of Theorem 2, we can define sequences $\{z_n^2\},\ \{z_n^3\}, ...,\ \{z_n^{k-1}\}$ in X such that $g(z_{n+1}^2) = f(z_n^2),\ g(z_{n+1}^3) = f(z_n^3), ...,\ g(z_{n+1}^{k-1}) = f(z_n^{k-1})\ \forall\ n \in \mathbb{N}_0$. Hence, we have

$$g(z_{n+1}^i) = f(z_n^i)\ \ \forall\ n \in \mathbb{N}_0 \text{ and for each } i\ (1 \leq i \leq k). \tag{24}$$

Now we claim that

$$g(z_n^i) \prec\succ g(z_n^{i+1})\ \ \forall\ n \in \mathbb{N}_0 \text{ and for each } i\ (1 \leq i \leq k-1). \tag{25}$$

We prove this fact by the method of mathematical induction. In view of(23), (25) holds for $n = 0$. Suppose that (25) holds for $n = r > 0$, *i.e.*,

$$g(z_r^i) \prec\succ g(z_r^{i+1})\ \text{ for each } i\ (1 \leq i \leq k-1).$$

On using g-comparability of f, we obtain

$$f(z_r^i) \prec\succ f(z_r^{i+1})\ \text{ for each } i\ (1 \leq i \leq k-1),$$

which on using (24), gives rise

$$g(z_{r+1}^i) \prec\succ g(z_{r+1}^{i+1})\ \text{ for each } i\ (1 \leq i \leq k-1).$$

It follows that (25) holds for $n = r+1$. Thus, by induction, (25) holds for all $n \in \mathbb{N}_0$. Now for each $n \in \mathbb{N}_0$ and for each i $(1 \leq i \leq k-1)$, define $t_n^i := d(gz_n^i, gz_n^{i+1})$. We claim that

$$\lim_{n\to\infty} t_n^i = 0\ \text{ for each } i\ (1 \leq i \leq k-1). \tag{26}$$

On fixing i, the two cases arise. Firstly, suppose that $t_{n_0}^i = d(gz_{n_0}^i, gz_{n_0}^{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then by g-admissibility of f, we obtain $d(fz_{n_0}^i, fz_{n_0}^{i+1}) = 0$. Consequently on using (25), we get $t_{n_0+1}^i = d(gz_{n_0+1}^i, gz_{n_0+1}^{i+1}) = d(fz_{n_0}^i, fz_{n_0}^{i+1}) = 0$. Thus by induction, we get $t_n^i = 0 \; \forall \; n \geq n_0$, yielding thereby $\lim\limits_{n\to\infty} t_n^i = 0$. On the other hand, suppose that $t_n > 0 \; \forall \; n \in \mathbb{N}_0$. Then, on using (24), (25) and assumption (e), we have

$$\begin{aligned} t_{n+1}^i &= d(gz_{n+1}^i, gz_{n+1}^{i+1}) \\ &= d(fz_n^i, fz_n^{i+1}) \\ &\leq \varphi(d(gz_n^i, z_n^{i+1})) \\ &= \varphi(t_n^i) \end{aligned}$$

so that

$$t_{n+1}^i \leq \varphi(t_n^i).$$

Hence on applying Lemma 1, we obtain $\lim\limits_{n\to\infty} t_n^i = 0$. Thus, in both the cases, (26) holds for each i $(1 \leq i \leq k-1)$. On using triangular inequality and (26), we obtain

$$d(\overline{x}, \overline{y}) \leq t_n^1 + t_n^2 + \cdots + t_n^{k-1} \to 0 \;\; as \;\; n \to \infty$$

so that

$$\overline{x} = \overline{y}.$$

Corollary 6. Theorem 4 remains true if we replace the condition (u_1) by one of the following conditions (besides retaining rest of the hypotheses):
(u_1^1) $(fX, \preceq)$ is totally ordered,
(u_1^2) $(X, \preceq)$ is (f, g)-directed.
Proof. Suppose that (u_1^1) holds, then for each pair $x, y \in X$, we have

$$f(x) \prec\succ f(y),$$

which implies that $\{fx, fy\}$ is a $\prec\succ$-chain between $f(x)$ and $f(y)$ in $g(X)$ so that $\mathrm{C}(fx, fy, \prec\succ, gX)$ is nonempty, for each $x, y \in X$, *i.e.*, (u_1) holds and hence Theorem 4 is applicable.
Next, assume that (u_1^2) holds, then for each pair $x, y \in X$, $\exists \; z \in X$ such that

$$f(x) \prec\succ g(z) \prec\succ f(y),$$

which implies that $\{fx, gz, fy\}$ is a $\prec\succ$-chain between $f(x)$ and $f(y)$ in $g(X)$ so that $\mathrm{C}(fx, fy, \prec\succ, gX)$ is nonempty, for each $x, y \in X$, *i.e.*, (u_1) holds and hence Theorem 4 is applicable.

Theorem 5. In addition to the hypotheses of Theorem 4, suppose that the following condition holds:
(u_2) one of f and g is one-one.
Then f and g have a unique coincidence point.
Proof. Take $x, y \in \mathrm{C}(f, g)$, then in view of Theorem 4, we have

$$g(x) = f(x) = f(y) = g(y).$$

As f or g is one-one, we have

$$x = y.$$

Theorem 6. In addition to the hypotheses embodied in condition (e') of Theorem 4, suppose that the following condition holds:
$(e'3)$ f and g are weakly compatible.
Then f and g have a unique common fixed point.
Proof. Owing to Remark 5 as well as assumption $(e'3)$, the mappings f and g are weakly

compatible. Let x be a coincidence point of f and g. Write $g(x) = f(x) = \overline{x}$, then in view of Lemma 4, $\overline{x}$ is also a coincidence point of f and g. It follows from Theorem 4 with $y = \overline{x}$ that $g(x) = g(\overline{x})$, *i.e.*, $\overline{x} = g(\overline{x})$, which yields that

$$\overline{x} = g(\overline{x}) = f(\overline{x}).$$

Hence, $\overline{x}$ is a common fixed point of f and g. To prove uniqueness, assume that x^* is another common fixed point of f and g. Then again from Theorem 4, we have

$$x^* = g(x^*) = g(\overline{x}) = \overline{x}.$$

This completes proof.

3. Fixed Point Theorems

On setting $g = I$, the identity mapping on X, in foregoing results, we get the following corresponding fixed point theorems.

Theorem 7. Let $(X, d, \preceq)$ be an ordered metric space and f a self-mapping on X. Let Y be a $\prec\succ$-complete subspace of X such that $f(X) \subseteq Y$ and $(Y, \preceq)$ is sequentially chainable. Suppose that the following conditions hold:

(i) f is comparable,
(ii) either f is $\prec\succ$-continuous or $(Y, d, \preceq)$ has TCC property,
(iii) there exists $x_0 \in X$ such that $x_0 \prec\succ f(x_0)$,
(iv) there exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(x, y)) \ \ \forall\ x, y \in X \text{ with } x \prec\succ y.$$

Then f has a fixed point.

Corollary 7. Let $(X, d, \preceq)$ be a $\prec\succ$-complete ordered metric space such that $(X, \preceq)$ is sequentially chainable and f a self-mapping on X. Suppose that the following conditions hold:

(i) f is comparable,
(ii) either f is $\prec\succ$-continuous or $(X, d, \preceq)$ has TCC property,
(iii) there exists $x_0 \in X$ such that $x_0 \prec\succ f(x_0)$,
(iv) there exists $\varphi \in \Omega$ such that

$$d(fx, fy) \leq \varphi(d(x, y)) \ \ \forall\ x, y \in X \text{ with } x \prec\succ y.$$

Then f has a fixed point.

Theorem 8. Let $(X, d, \preceq)$ be an ordered metric space and f a self-mapping on X. Let Y be a $\prec\succ$-complete subspace of X such that $f(X) \subseteq Y$. Suppose that the following conditions hold:

(i) f is comparable,
(ii) either f is $\prec\succ$-continuous or $(Y, d, \preceq)$ has TCC property,
(iii) there exists $x_0 \in X$ such that $x_0 \prec\succ f(x_0)$,
(iv) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \ \ \forall\ x, y \in X \text{ with } x \prec\succ y.$$

Then f has a fixed point.

Corollary 8. Let $(X, d, \preceq)$ be a $\prec\succ$-complete ordered metric space and f a self-mapping on X. Suppose that the following conditions hold:
(i) f is comparable,
(ii) either f is $\prec\succ$-continuous or $(X, d, \preceq)$ has TCC property,
(iii) there exists $x_0 \in X$ such that $x_0 \prec\succ f(x_0)$,
(iv) there exists $\alpha \in [0, 1)$ such that

$$d(fx, fy) \leq \alpha d(x, y) \;\; \forall\, x, y \in X \text{ with } x \prec\succ y.$$

Then f has a fixed point.

Theorem 9. In addition to the hypotheses of Theorem 7 (also Theorem 8), suppose that the following conditions holds:
(u) $\mathrm{C}(fx, fy, \prec\succ)$ is nonempty for each $x, y \in X$.
Then f has a unique fixed point.

Corollary 9. Theorem 9 remains true if we replace the condition (u) by one of the following conditions:
(u^1) $(fX, \preceq)$ is totally ordered,
(u^2) $(X, \preceq)$ is f-directed.

4. Examples

In this section, we furnish some examples, which illustrate our newly proved results.

Example 1. Let $X = \mathbb{R}$. On X, consider usual metric d and partial order $\preceq$ defined by $x \preceq y \Leftrightarrow |x| \leq |y|$ and $xy \geq 0$. Then $(X, d, \preceq)$ is a $\prec\succ$-complete ordered metric space. Define $f, g : X \to X$ by $f(x) = \frac{x^2}{6}$ and $g(x) = -x^2 \;\forall\; x \in X$. Then f is g-comparable. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{t}{4} \;\forall\; t \in [0, \infty)$, then $\varphi \in \Omega$. Now, for $x, y \in X$ with $g(x) \preceq g(y)$, we have

$$d(fx, fy) = \left|\frac{x^2}{6} - \frac{y^2}{6}\right| = \frac{1}{6}\left|x^2 - y^2\right| = \frac{1}{6}d(gx, gy) < \frac{1}{4}d(gx, gy) = \varphi(d(gx, gy)).$$

Therefore, f, g and φ satisfy assumption (d) of Theorem 2. By a routine calculation, one can also verify all the conditions mentioned in (e) of Theorem 2. Thus, all the conditions of Theorem 2 are satisfied for $Y = X$ and hence, f and g have a coincidence point in X. Moreover, here (u_1) holds and therefore, in view of Theorem 4, f and g have a unique point of coincidence (namely: $\overline{x} = 0$). Furthermore, f and g have a unique common fixed point (namely: $x = 0$) (due to Theorem 6).

Example 2. Consider $X = \mathbb{R}$ equipped with usual metric and usual partial order. Define $f, g : X \to X$ by $f(x) = 9$ and $g(x) = x^2 - 7 \;\forall\; x \in X$. Then f is g-comparable. Let $\varphi \in \Omega$ be arbitrary. Now, for $x, y \in X$ with $g(x) \preceq g(y)$, we have

$$d(fx, fy) = |9 - 9| = 0 \leq \varphi(|x^2 - y^2|) = \varphi(d(gx, gy)).$$

Thus f, g and φ satisfy the assumption (d) of Theorem 2. Also, the mappings f and g are not $\prec\succ$-compatible and hence (e) does not hold. But the subspace $Y := g(X) = [-7, \infty)$ is $\prec\succ$-complete and f and g are continuous, *i.e.*, all the conditions mentioned in (e')

are satisfied. Hence by Theorem 2, f and g have a coincidence point in X. Further, in this example (u_1) holds and henceforth, in view of Theorem 4, f and g have a unique point of coincidence (namely: $\overline{x} = 9$). Notice that neither f nor g is one-one, *i.e.*, (u_2) does not hold and hence, we can not apply Theorem 5, which guarantees the uniqueness of coincidence point. Observe that there are two coincidence points (namely: x=4 and x=-4). Also, f and g are not weakly compatible , *i.e.*, $(e'3)$ does not hold and hence, we can not apply Theorem 6, which ensures the uniqueness of common fixed point. Notice that there is no common fixed point of f and g.

Example 3. Let $X = [-\frac{1}{5}, \frac{1}{5}]$. Then $(X, d, \preceq)$ is an $\prec\succ$-complete ordered metric space under the usual metric and the usual partial order. Define $f : X \to X$ by $f(x) = x^2$, then f is comparable but not monotone. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{2t}{5}\ \forall\ t \in [0, \infty)$, then $\varphi \in \Omega$. Also, for $x, y \in X$ with $x \preceq y$, we have

$$d(fx, fy) = |x^2 - y^2| = |x + y||x - y| \leq \frac{2}{5}d(x, y) = \varphi(d(x, y)).$$

i.e. f satisfies the contractivity condition (iv) of Corollary 7. Thus, all the conditions mentioned in Corollary 7 and Theorem 9 are satisfied and hence f has a unique fixed point in X (namely: $x = 0$).

5. An application to Integral Equation

In this section, using certain results (particularly, Corollary 7 and Theorem 9) proved in Section 3, we study the existence and uniqueness of solution of the following integral equation:

$$u(t) = \int_0^T M(t, \xi, u(\xi))d\xi \quad \forall\ t \in I, \tag{27}$$

where $T > 0$, $I = [0, T]$, $u : I \to \mathbb{R}$ is unknown function and $M : I \times I \times \mathbb{R} \to \mathbb{R}$ is known function.

We denote $\mathcal{C}(S)$ by the space of all real valued continuous functions on a nonempty set S.

Definition 18. A function $\eta \in \mathcal{C}(I)$ is called a lower solution of (27), if

$$\eta(t) \leq \int_0^T M(t, \xi, \eta(\xi))d\xi \quad \forall\ t \in I.$$

Definition 19. A function $\eta \in \mathcal{C}(I)$ is called an upper solution of (27), if

$$\eta(t) \geq \int_0^T M(t, \xi, \eta(\xi))d\xi \quad \forall\ t \in I.$$

Let $\mathfrak{F}$ denotes the family of functions $\phi : [0, \infty] \to [0, \infty]$ satisfying the following conditions:

(i) ϕ is continuous and increasing,
(ii) $\phi(t) < t$ for each $t > 0$.

Typical examples of $\mathfrak{F}$ are $\phi(t) = \alpha.t,\ 0 \leq \alpha < 1,\ \phi(t) = \frac{t}{1+t}$ and $\phi(t) = \ln(1+t)$. Also, clearly $\mathfrak{F} \subset \Omega$.

Now, we prove the following result on the existence and uniqueness of the solution of the problem described by (27) in the presence of a lower solution (or an upper solution).

Theorem 10. In respect of the problem described by (27), suppose that the following assumptions hold:

(a) $M \in \mathcal{C}(I \times I \times \mathbb{R})$ and $M(t, \xi, x) \geq 0\ \forall\ t, \xi \in I, x \in \mathbb{R}$,

(b) there exists $\phi \in \mathfrak{F}$ such that for all $t, \xi \in I$ and for all $x, y \in \mathbb{R}$ with $x \leq y$,

$$0 \leq M(t, \xi, x) - M(t, \xi, y) \leq p(t, \xi)\phi(y - x),$$

where $p : I \times I \to [0, \infty)$ is a continuous function satisfying

$$\sup_{t \in I} \int_0^T p(t, \xi) d\xi \leq 1.$$

Then the existence of a lower solution (or an upper solution) of the problem (27) ensures the existence and uniqueness of the solution of this problem.

Proof. Define a function $\mathcal{A} : \mathcal{C}(I) \to \mathcal{C}(I)$ by

$$(\mathcal{A}u)(t) = \int_0^T M(t, \xi, u(\xi)) d\xi \quad \forall\ t \in I. \tag{28}$$

Clearly, if $u \in \mathcal{C}(I)$ is a fixed point of $\mathcal{A}$ then u is a solution of (27). On $\mathcal{C}(I)$, define a metric d given by:

$$d(u, v) = \sup_{t \in I} |u(t) - v(t)| \quad \forall\ u, v \in \mathcal{C}(I). \tag{29}$$

On $\mathcal{C}(I)$, define a partial order $\preceq$ given by:

$$u, v \in \mathcal{C}(I); u \preceq v \Longleftrightarrow u(t) \leq v(t)\ \ \forall\ t \in I. \tag{30}$$

Now, we check that all the conditions of Corollary 7 are satisfied. Clearly, $(\mathcal{C}(I), d, \preceq)$ is a $\prec\succ$-complete ordered metric space and $(\mathcal{C}(I), \preceq)$ is sequentially chainable.

(i) Take $u, v \in \mathcal{C}(I)$ such that $u \prec\succ v$, then by (30), we obtain

$$u(\xi) \leq v(\xi)\ \forall\ \xi \in I \text{ or } u(\xi) \geq v(\xi)\ \forall\ \xi \in I,$$

which, for each $t \in I$, using assumption (b) gives rise

$$M(t, \xi, u(\xi)) \geq M(t, \xi, v(\xi))\ \forall\ \xi \in I \text{ or } M(t, \xi, u(\xi)) \leq M(t, \xi, v(\xi))\ \forall\ \xi \in I. \tag{31}$$

On using (28), (31) and assumption (a), we get

$$\begin{aligned} (\mathcal{A}u)(t) &= \int_0^T M(t, \xi, u(\xi)) d\xi \\ &\geq \int_0^T M(t, \xi, v(\xi)) d\xi \\ &= (\mathcal{A}v)(t) \quad \forall\ t \in I \end{aligned}$$

or

$$\begin{aligned}(\mathcal{A}u)(t) &= \int_0^T M(t,\xi,u(\xi))d\xi \\ &\leq \int_0^T M(t,\xi,v(\xi))d\xi \\ &= (\mathcal{A}v)(t) \quad \forall\, t\in I,\end{aligned}$$

which owing to (30) imply that $\mathcal{A}(u) \prec\succ \mathcal{A}(v)$ so that $\mathcal{A}$ is comparable.

(ii) Take a sequence $\{u_n\} \subset \mathcal{C}(I)$ such that $u_n \updownarrow u \in \mathcal{C}(I)$. Then for each $t \in I$, $\{u_n(t)\}$ is a sequence in $\mathbb{R}$ converging to $u(t)$. Hence, $\{u_n(t)\}$ has a monotone subsequence $\{u_{n_k}(t)\}$. Therefore, for all $k \in \mathbb{N}_0$ and for all $t \in I$, we have

$$\begin{aligned}u_{n_k}(t) &\leq u(t) \quad \text{if } \{u_{n_k}(t)\} \text{ is increasing} \\ u_{n_k}(t) &\geq u(t) \quad \text{if } \{u_{n_k}(t)\} \text{ is decreasing,}\end{aligned}$$

which by using (30) implies that $u_{n_k} \prec\succ u \;\forall\; k \in \mathbb{N}_0$ so that $(\mathcal{C}(I), d, \preceq)$ has *TCC* property.

(iii) If $\eta \in \mathcal{C}(I)$ is a lower (resp. an upper) solution of (27), then using (28) and (30), we can verify that $\eta \preceq \mathcal{A}(\eta)$ (resp. $\eta \succeq \mathcal{A}(\eta)$). Hence, in both the cases, we have $\eta \prec\succ \mathcal{A}(\eta)$, for some lower or upper solution η.

(iv) Take $u, v \in \mathcal{C}(I)$ such that $u \preceq v$. On using (28), (29) and assumption (b), we obtain

$$\begin{aligned}d(\mathcal{A}u,\mathcal{A}v) &= \sup_{t\in I}|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)| = \sup_{t\in I}|\int_0^T M(t,\xi,u(\xi))d\xi - \int_0^T M(t,\xi,v(\xi))d\xi| \\ &= \sup_{t\in I}\int_0^T \big(M(t,\xi,u(\xi)) - M(t,\xi,v(\xi))\big)d\xi \\ &\leq \sup_{t\in I}\int_0^T p(t,\xi)\phi(v(\xi)-u(\xi))d\xi. \qquad (32)\end{aligned}$$

Given that ϕ is increasing on $[0,\infty)$ and $u \preceq v$, which implies that $\phi(v(\xi)-u(\xi)) \leq \phi(d(u,v))$ for all $\xi \in I$. Hence, (32) reduces to

$$d(\mathcal{A}u,\mathcal{A}v) \leq \phi(d(u,v))\sup_{t\in I}\int_0^T p(t,\xi)d\xi,$$

which again using assumption (b) gives rise

$$d(\mathcal{A}u,\mathcal{A}v) \leq \phi(d(u,v)) \;\; \forall\; u,v \in \mathcal{C}(I) \text{ such that } u \preceq v,$$

where $\phi \in \mathfrak{F} \subset \Omega$.

Thus, all the conditions of Corollary 7 are satisfied, which ensures that $\mathcal{A}$ has a fixed point.

Finally, choose arbitrary $u, v \in \mathcal{C}(I)$ and write $w := \max\{\mathcal{A}u, \mathcal{A}v\} \in \mathcal{C}(I)$. As $\mathcal{A}(u) \preceq w$ and $\mathcal{A}(v) \preceq w$, $\{\mathcal{A}u, w, \mathcal{A}v\}$ is a $\prec\succ$-chain between $\mathcal{A}(u)$ and $\mathcal{A}(v)$. Now, in view of Theorem 9, $\mathcal{A}$ has a unique fixed point, which is, indeed, a unique solution of the problem described by (27).

6. Conclusion

In an attempt to extend Theorem 1 from linear contractions to Boyd-Wong type nonlinear contractions, we were compelled to add two extra conditions in Theorem 2 (namely: sequential chainability and g-admissibility), which substantiate the utility of this extension. As per new work, readers may attempt to prove such results under various well known contractions such as: quasi contractions, Matkowski type contractions, weak nonlinear contractions, rational type contractions, Meir-Keeler type contractions, cyclic contractions, Geraghty-type contractions etc. besides using implicit relations.

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