On Numerical Solutions of Time-Fraction Generalized Hirota-Satsuma Coupled KdV Equation

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Abstract: In this study, we obtained the approximate soliton solution of the fractional generalized Hirota-Satsuma coupled Korteweg-de Vries equation (GHS-cKdV) within the homotopy analysis method (HAM). Numerical results were successfully compared with other solutions obtained by the differential transform method (DTM) and the homotopy perturbation method (HPM). The numerical results indicated that the only few terms are sufficient to get the correct solutions. Also, the results given by tables and figures.

Keywords: HAM, Fractional partial differential equation (FPD), HS-cKdV equation , time fractional GHS-cKdV equation.

1. Introduction

During the past few decades, the fractional equations have been occured from structures in different fields of science and engineering [1]-[2]. The fractional KdV equation and the fractional HSKdV are the most important of these type equations [3]-[4]. Firstly, Gardner and Hirota found analytical solution for cKdV, which describes interactions of two long waves with different dispersion relations [5]-[6], later Hirota and Satsuma introduced a cKdV equation

$$\eta_\vartheta - a(\eta_{\varrho\varrho\varrho} + 6\eta\eta_\varrho) = 2b\varphi\varphi_\varrho, \tag{1.1}$$

$$\varphi_\vartheta + \varphi_{\varrho\varrho\varrho} + 3\eta\varphi_\varrho = 0, \tag{1.2}$$

where a and b arbitrary constants [7]. $2b\varphi\varphi_\varrho$ acts as a force term on the KdV wave system with the linear dispersion relation $\omega = a\boldsymbol{\kappa}^3$.

From (1.1) and (1.2) equation, consider the GHS-cKdV equation

$$\begin{aligned} \eta_\vartheta &= \frac{1}{2}\eta_{\varrho\varrho\varrho} - 3\eta\eta_\varrho + 3(\lambda\varpi)_\varrho, \\ \lambda_\vartheta &= -\lambda_{\varrho\varrho\varrho} + 3\eta\lambda_\varrho, \\ \varpi_\vartheta &= -\varpi_{\varrho\varrho\varrho} + 3\eta\varpi_\varrho \end{aligned} \tag{1.3}$$

[4].

This study, we consider the solution of GHS-cKdV of time-fractional order

$$\begin{aligned} {}^{\mathbf{c}}D_{\vartheta}^{\alpha}\eta &= \frac{1}{2}\eta_{\varrho\varrho\varrho} - 3\eta\eta_{\varrho} + 3(\lambda\varpi)_{\varrho}, \\ {}^{\mathbf{c}}D_{\vartheta}^{\alpha}\lambda &= -\lambda_{\varrho\varrho\varrho} + 3\eta\lambda_{\varrho}, \qquad\qquad 0 < \alpha < 1 \\ {}^{\mathbf{c}}D_{\vartheta}^{\alpha}\varpi &= -\varpi_{\varrho\varrho\varrho} + 3\eta\varpi_{\varrho}. \end{aligned} \tag{1.4}$$

If $\boldsymbol{\xi} = -\beta$ at that time

$$\eta(\varrho,\vartheta) = \frac{\beta - 2\boldsymbol{\kappa}^2}{3} + 2\boldsymbol{\kappa}^2 \tanh^2(\boldsymbol{\kappa}(\varrho - \boldsymbol{\xi}\vartheta)),$$

$$\begin{aligned} \lambda(\varrho,\vartheta) &= -\frac{4\boldsymbol{\kappa}^2\boldsymbol{\xi}_0(\beta+\boldsymbol{\kappa}^2)}{3\boldsymbol{\xi}_1^2} + \frac{4\boldsymbol{\kappa}^2(\beta+\boldsymbol{\kappa}^2)}{3\boldsymbol{\xi}_1}\tanh(\boldsymbol{\kappa}(\varrho - \boldsymbol{\xi}\vartheta)), \\ \varpi(\varrho,\vartheta) &= \boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 \tanh(\boldsymbol{\kappa}(\varrho - \boldsymbol{\xi}\vartheta)) \end{aligned} \tag{1.5}$$

are system of (1.4) when $\alpha = 1$ [8]-[9]. In this, $\boldsymbol{\kappa}, \boldsymbol{\xi}_0, \boldsymbol{\xi}_1 \neq 0$ and β are arbitrary constants [8].

In this study, we will used to the HAM that was developed by Liao in [10]. This method contains the auxiliary parameter $\hbar$ which povides us with a simple way to adjust and control the convergence region of solution series for large or small values of ϱ and ϑ [11]-[17].

2. Basic Definitions

Definition 2.1: The R-L fractional integral oparetor of order $\alpha \geq 0$, of a function $f \in \boldsymbol{\xi}_{\mu}$, $\mu \geq -1$, is described as

$$D_{\varsigma}^{-\alpha}\eta(\varrho,\varsigma) = \frac{1}{\Gamma(\alpha)}\int_0^{\varsigma}(\varsigma-\sigma)^{\alpha-1}\eta(\varrho,\sigma)d\sigma,\ \alpha > 0,\ \varsigma > 0 \tag{2.1}$$

[18]-[19]. If $\alpha \geq 0,\ m-1 < \alpha \leq m,\ m \in \mathbb{N}$ and $A^m[a,b]$, then

$${}^{\mathbf{c}}D^{\alpha}D^{-\alpha}f(\varrho) = f(\varrho), \tag{2.2}$$

$${}_aD_{\varrho}^{-\alpha}\ {}_a^{\mathbf{c}}D_{\varrho}^{\alpha}f(\varrho) = f(\varrho) - \sum_{\mathbf{k}=0}^{m-1}\frac{D^{\mathbf{k}}f(a)}{\mathbf{k}!}(\varrho - a)^{\mathbf{k}} \tag{2.3}$$

[21].

Definition 2.2: The C-D is define as

$${}^{\mathbf{c}}D_{\varsigma}^{\alpha}f(\varrho) = \frac{1}{\Gamma(m-\alpha)}\int_0^{\varsigma}(\varsigma-\sigma)^{m-\alpha-1}f^{(m)}(\sigma)d\sigma, m-1 < \alpha \leq m, m \in \mathbb{N} \tag{2.4}$$

[22]-[24].

3. The HAM Solution of The GHS-cKdV

According to the system (1.4). We start the application of the HAM for solving this system using the initial conditions. In this system,

$$\begin{aligned} \eta(\varrho,0) &= \frac{\beta-2\boldsymbol{\kappa}^2}{3}+2\boldsymbol{\kappa}^2\tanh^2(\boldsymbol{\kappa}\varrho), \\ \lambda(\varrho,0) &= -\frac{4\boldsymbol{\kappa}^2\boldsymbol{\xi}_0(\beta+\boldsymbol{\kappa}^2)}{3\boldsymbol{\xi}_1^2}+\frac{4\boldsymbol{\kappa}^2(\beta+\boldsymbol{\kappa}^2)\tanh(\boldsymbol{\kappa}\varrho)}{3\boldsymbol{\xi}_1}, \\ \varpi(\varrho,0) &= \boldsymbol{\xi}_0+\boldsymbol{\xi}_1\tanh(\boldsymbol{\kappa}\varrho) \end{aligned} \tag{3.1}$$

are the initial conditions of $\eta(\varrho,\vartheta)$, $\lambda(\varrho,\vartheta)$ and $\varpi(\varrho,\vartheta)$ [8]-[9].

We choose

$$L[\psi_i(\varrho,\vartheta;p)] = {}^cD_\vartheta^\alpha[\psi_i(\varrho,\vartheta;p)],\ i=1,2,3 \tag{3.2}$$

with property $L[\mathbf{c}_i]=0$ where $\mathbf{c}_i$ is constants. We define N_i by

$$\begin{aligned} N_1[\psi_1(\varrho,\vartheta,p)] &= {}^{\mathbf{c}}D_\vartheta^\alpha\psi_1(\varrho,\vartheta,p)-\frac{1}{2}\frac{\partial^3\psi_1(\varrho,\vartheta,p)}{\partial\varrho^3}+3\psi_1(\varrho,\vartheta,p)\frac{\partial\psi_1(\varrho,\vartheta,p)}{\partial\varrho} \\ &\quad -3\psi_2(\varrho,\vartheta,p)\frac{\partial\psi_3(\varrho,\vartheta,p)}{\partial\varrho}-3\frac{\partial\psi_2(\varrho,\vartheta,p)}{\partial\varrho}\psi_3(\varrho,\vartheta,p), \\ N_2[\psi_2(\varrho,\vartheta,p)] &= {}^{\mathbf{c}}D_\vartheta^\alpha\psi_2(\varrho,\vartheta,p)+\frac{\partial^3\psi_2(\varrho,\vartheta,p)}{\partial\varrho^3}-3\psi_1(\varrho,\vartheta,p)\frac{\partial\psi_2(\varrho,\vartheta,p)}{\partial\varrho}, \\ N_3[\psi_3(\varrho,\vartheta,p)] &= {}^{\mathbf{c}}D_\vartheta^\alpha\psi_3(\varrho,\vartheta,p)+\frac{\partial^3\psi_3(\varrho,\vartheta,p)}{\partial\varrho^3}-3\psi_1(\varrho,\vartheta,p)\frac{\partial\psi_3(\varrho,\vartheta,p)}{\partial\varrho}. \end{aligned} \tag{3.3}$$

So, we install the zeroth-order deformation equations

$$\begin{aligned} (1-p)L[\psi_1(\varrho,\vartheta,p)-\eta_0(\varrho,\vartheta)] &= p\hbar H(\vartheta)N[\psi_1(\varrho,\vartheta,p)], \\ (1-p)L[\psi_2(\varrho,\vartheta,p)-\lambda_0(\varrho,\vartheta)] &= p\hbar H(\vartheta)N[\psi_2(\varrho,\vartheta,p)], \\ (1-p)L[\psi_3(\varrho,\vartheta,p)-\varpi_0(\varrho,\vartheta)] &= p\hbar H(\vartheta)N[\psi_3(\varrho,\vartheta,p)]. \end{aligned} \tag{3.4}$$

In (3.4), for $p=0$ and $p=1$ respectively

$$\begin{aligned} \psi_1(\varrho,\vartheta,0) &= \eta_0(\varrho,\vartheta),\ \ \psi_1(\varrho,\vartheta,1)=\eta(\varrho,\vartheta) \\ \psi_2(\varrho,\vartheta,0) &= \lambda_0(\varrho,\vartheta),\ \ \psi_2(\varrho,\vartheta,1)=\lambda(\varrho,\vartheta) \\ \psi_3(\varrho,\vartheta,0) &= \varpi_0(\varrho,\vartheta),\ \ \psi_3(\varrho,\vartheta,1)=\varpi(\varrho,\vartheta). \end{aligned} \tag{3.5}$$

So we get mth-order DEqs;

$$\begin{aligned} L[\eta_m(\varrho,\vartheta)-\chi_m\eta_{m-1}(\varrho,\vartheta)] &= \hbar H(\vartheta)R_m(\eta_{m-1}(\varrho,\vartheta)), \\ L[\lambda_m(\varrho,\vartheta)-\chi_m\lambda_{m-1}(\varrho,\vartheta)] &= \hbar H(\vartheta)R_m(\lambda_{m-1}(\varrho,\vartheta)), \\ L[\varpi_m(\varrho,\vartheta)-\chi_m\varpi_{m-1}(\varrho,\vartheta)] &= \hbar H(\vartheta)R_m(\varpi_{m-1}(\varrho,\vartheta)) \end{aligned} \tag{3.6}$$

Here,

$$
\begin{aligned}
R_m(\eta_{m-1}(\varrho,\vartheta)) &= {}^{\mathbf{c}}D_{\vartheta}^{\alpha}\eta_{m-1}(\varrho,\vartheta) - \frac{1}{2}\frac{\partial^3\eta_{m-1}(\varrho,\vartheta)}{\partial\varrho^3} + 3\sum_{i=0}^{m-1}\eta_i(\varrho,\vartheta)\frac{\partial\eta_{m-1-i}(\varrho,\vartheta)}{\partial\varrho} \\
&\quad -3\sum_{i=0}^{m-1}\left(\lambda_i(\varrho,\vartheta)\frac{\partial\varpi_{m-1-i}(\varrho,\vartheta)}{\partial\varrho} + \varpi_i(\varrho,\vartheta)\frac{\partial\lambda_{m-1-i}(\varrho,\vartheta)}{\partial\varrho}\right) \\
R_m(\lambda_{m-1}(\varrho,\vartheta)) &= {}^{\mathbf{c}}D_{\vartheta}^{\alpha}\lambda_{m-1}(\varrho,\vartheta) + \frac{\partial^3\lambda_{m-1}(\varrho,\vartheta)}{\partial\varrho^3} - 3\sum_{i=0}^{m-1}\eta_i(\varrho,\vartheta)\frac{\partial\lambda_{m-1-i}(\varrho,\vartheta)}{\partial\varrho}, \qquad (3.7) \\
R_m(\varpi_{m-1}(\varrho,\vartheta)) &= {}^{\mathbf{c}}D_{\vartheta}^{\alpha}\varpi_{m-1}(\varrho,\vartheta) + \frac{\partial^3\varpi_{m-1}(\varrho,\vartheta)}{\partial\varrho^3} - 3\sum_{i=0}^{m-1}\eta_i(\varrho,\vartheta)\frac{\partial\varpi_{m-1-i}(\varrho,\vartheta)}{\partial\varrho}.
\end{aligned}
$$

Now the solution of the mth-order DEqs. (3.6) for $m \geq 1$ become

$$
\begin{aligned}
\eta_m(\varrho,\vartheta) &= \chi_m\eta_{m-1}(\varrho,\vartheta) + \hbar H(\vartheta)L^{-1}[R_m(\eta_{m-1}(\varrho,\vartheta))], \\
\lambda_m(\varrho,\vartheta) &= \chi_m\lambda_{m-1}(\varrho,\vartheta) + \hbar H(\vartheta)L^{-1}[R_m(\lambda_{m-1}(\varrho,\vartheta))], \qquad (3.8) \\
\varpi_m(\varrho,\vartheta) &= \chi_m\varpi_{m-1}(\varrho,\vartheta) + \hbar H(\vartheta)L^{-1}[R_m(\varpi_{m-1}(\varrho,\vartheta))].
\end{aligned}
$$

If we use (2.4) in Eqs. (3.8), then we get

$$
\begin{aligned}
\eta_m(\varrho,\vartheta) &= D_{\vartheta}^{-\alpha}\,{}^{\mathbf{c}}D_{\vartheta}^{\alpha}\eta(\varrho,\vartheta) = \eta(\varrho,\vartheta) - \sum_{\mathbf{k}=0}^{n-1}\frac{\partial^{\mathbf{k}}\eta(\varrho,0)}{\partial\vartheta^{\mathbf{k}}}\frac{\vartheta^{\mathbf{k}}}{\mathbf{k}!},\ \vartheta>0,\ n-1<\alpha<n, \\
\lambda_m(\varrho,\vartheta) &= D_{\vartheta}^{-\alpha}\,{}^{\mathbf{c}}D_{\vartheta}^{\alpha}\lambda(\varrho,\vartheta) = \lambda(\varrho,\vartheta) - \sum_{\mathbf{k}=0}^{n-1}\frac{\partial^{\mathbf{k}}\lambda(\varrho,0)}{\partial\vartheta^{\mathbf{k}}}\frac{\vartheta^{\mathbf{k}}}{\mathbf{k}!},\ \vartheta>0,\ n-1<\alpha<n, \qquad (3.9) \\
\varpi_m(\varrho,\vartheta) &= D_{\vartheta}^{-\alpha}\,{}^{\mathbf{c}}D_{\vartheta}^{\alpha}\varpi(\varrho,\vartheta) = \varpi(\varrho,\vartheta) - \sum_{\mathbf{k}=0}^{n-1}\frac{\partial^{\mathbf{k}}\varpi(\varrho,0)}{\partial\vartheta^{\mathbf{k}}}\frac{\vartheta^{\mathbf{k}}}{\mathbf{k}!},\ \vartheta>0,\ n-1<\alpha<n.
\end{aligned}
$$

Instead of (3.9) and $R_m(\eta_{m-1})$, $R_m(\lambda_{m-1})$,$R_m(\varpi_{m-1})$,

$$
\begin{aligned}
\eta_m(\varrho,\vartheta) &= (\chi_m+\hbar)\eta_{m-1}(\varrho,\vartheta)-\hbar\eta_{m-1}(\varrho,0)+\hbar\frac{1}{\Gamma(\alpha)}\int\limits_0^{\vartheta}(\vartheta-\tau)^{\alpha-1}(-\frac{1}{2}\frac{\partial^3\eta_{m-1}(\varrho,\tau)}{\partial\varrho^3} \\
&\quad +3\sum_{i=0}^{m-1}\eta_i(\varrho,\tau)\frac{\partial\eta_{m-1-i}(\varrho,\tau)}{\partial\varrho}-3\sum_{i=0}^{m-1}(\lambda_i(\varrho,\tau)\frac{\partial\varpi_{m-1-i}(\varrho,\tau)}{\partial\varrho} \\
&\quad +\varpi_i(\varrho,\tau)\frac{\partial\lambda_{m-1-i}(\varrho,\tau)}{\partial\varrho}))d\tau, \qquad (3.10) \\
\lambda_m(\varrho,\vartheta) &= (\chi_m+\hbar)\lambda_{m-1}(\varrho,\vartheta)-\hbar\lambda_{m-1}(\varrho,0)+\hbar\frac{1}{\Gamma(\alpha)}\int\limits_0^{\vartheta}(\vartheta-\tau)^{\alpha-1}(\frac{\partial^3\lambda_{m-1}(\varrho,\tau)}{\partial\varrho^3} \\
&\quad -3\sum_{i=0}^{m-1}\eta_i(\varrho,\tau)\frac{\partial\lambda_{m-1-i}(\varrho,\tau)}{\partial\varrho})d\tau, \\
\varpi_m(\varrho,\vartheta) &= (\chi_m+\hbar)\varpi_{m-1}(\varrho,\vartheta)-\hbar\varpi_{m-1}(\varrho,0)+\hbar\frac{1}{\Gamma(\alpha)}\int\limits_0^{\vartheta}(\vartheta-\tau)^{\alpha-1}(\frac{\partial^3\varpi_{m-1}(\varrho,\tau)}{\partial\varrho^3} \\
&\quad -3\sum_{i=0}^{m-1}\eta_i(\varrho,\tau)\frac{\partial\varpi_{m-1-i}(\varrho,\tau)}{\partial\varrho})d\tau,
\end{aligned}
$$

writable and $H(\vartheta)$ can be choosen in the form $H(\vartheta)=1$.

Arrangement of Eq. (3.10) gives m-th order deformation equations

$$
\begin{aligned}
\eta_m(\varrho,\vartheta) &= \chi_m\eta_{m-1}(\varrho,\vartheta)+\hbar D_\vartheta^{-\alpha}[R_m(\eta_{m-1}(\varrho,\vartheta))], \\
\lambda_m(\varrho,\vartheta) &= \chi_m\lambda_{m-1}(\varrho,\vartheta)+\hbar D_\vartheta^{-\alpha}[R_m(\lambda_{m-1}(\varrho,\vartheta))], \qquad (3.11) \\
\varpi_m(\varrho,\vartheta) &= \chi_m\varpi_{m-1}(\varrho,\vartheta)+\hbar D_\vartheta^{-\alpha}[R_m(\varpi_{m-1}(\varrho,\vartheta))].
\end{aligned}
$$

Therefore, we are obtained components as

$$
\begin{aligned}
\eta_0(\varrho,\vartheta) &= \frac{\beta-2\boldsymbol{\kappa}^2}{3}+2\boldsymbol{\kappa}^2\tanh^2(\boldsymbol{\kappa}\varrho), \\
\eta_1(\varrho,\vartheta) &= -4\hbar\boldsymbol{\kappa}^3\beta\sec\hbar^2(\boldsymbol{\kappa}\varrho)\tanh(\boldsymbol{\kappa}\varrho)\frac{\vartheta^\alpha}{\Gamma(\alpha+1)}, \\
\eta_2(\varrho,\vartheta) &= -4\hbar(1+\hbar)\boldsymbol{\kappa}^3\beta\sec\hbar^2(\boldsymbol{\kappa}\varrho)\tanh(\boldsymbol{\kappa}\varrho)\frac{\vartheta^\alpha}{\Gamma(\alpha+1)} \\
&\quad -4^{1-\alpha}\hbar^2\boldsymbol{\kappa}^4\sqrt{\pi}\beta^2\sec\hbar^4(\boldsymbol{\kappa}\varrho)(-2+\cosh(2\boldsymbol{\kappa}\varrho))\frac{\vartheta^{2\alpha}}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}, \\
&\quad \vdots,
\end{aligned}
$$

$$\begin{aligned}
\lambda_0(\varrho,\vartheta) &= -\frac{4\boldsymbol{\kappa}^2\boldsymbol{\xi}_0(\beta+\boldsymbol{\kappa}^2)}{3\boldsymbol{\xi}_1^2}+\frac{4\boldsymbol{\kappa}^2(\beta+\boldsymbol{\kappa}^2)\tanh(\boldsymbol{\kappa}\varrho)}{3\boldsymbol{\xi}_1},\\
\lambda_1(\varrho,\vartheta) &= -\frac{4\hbar\boldsymbol{\kappa}^3\beta(\beta+\boldsymbol{\kappa}^2)\sec\hbar^2(\boldsymbol{\kappa}\varrho)}{3\boldsymbol{\xi}_1^2}\frac{\vartheta^\alpha}{\Gamma(\alpha+1)},\\
\lambda_2(\varrho,\vartheta) &= -\frac{4\hbar(1+\hbar)\boldsymbol{\kappa}^3\beta(\beta+\boldsymbol{\kappa}^2)\sec\hbar^2(\boldsymbol{\kappa}\varrho)}{3\boldsymbol{\xi}_1^2}\frac{\vartheta^\alpha}{\Gamma(\alpha+1)}\\
&\quad -8\hbar^2\boldsymbol{\kappa}^4\beta^2(\boldsymbol{\kappa}^2+\beta)(4^{-\alpha}\sqrt{\pi}\vartheta^{2\alpha}\Gamma(\alpha))\sec\hbar^2(\boldsymbol{\kappa}\varrho)\tanh(\boldsymbol{\kappa}\varrho)\\
&\quad \frac{1}{3\Gamma(\alpha)\Gamma(\alpha+1)\boldsymbol{\xi}_1\Gamma\left(\alpha+\frac{1}{2}\right)},\\
&\quad \vdots,
\end{aligned}$$

$$\begin{aligned}
\varpi_0(\varrho,\vartheta) &= \boldsymbol{\xi}_0+\boldsymbol{\xi}_1\tanh(\boldsymbol{\kappa}\varrho),\\
\varpi_1(\varrho,\vartheta) &= -\hbar\boldsymbol{\kappa}\beta\sec\hbar^2(\boldsymbol{\kappa}\varrho)\boldsymbol{\xi}_1\frac{\vartheta^\alpha}{\Gamma(\alpha+1)},\\
\varpi_2(\varrho,\vartheta) &= -\hbar(1+\hbar)\boldsymbol{\kappa}\beta\sec\hbar^2(\boldsymbol{\kappa}\varrho)\boldsymbol{\xi}_1\frac{\vartheta^\alpha}{\Gamma(\alpha+1)}\\
&\quad -2\hbar^2\boldsymbol{\kappa}^2\beta^2 4^{-\alpha}\sqrt{\pi}\sec\hbar^2(\boldsymbol{\kappa}\varrho)\boldsymbol{\xi}_1\tanh(\boldsymbol{\kappa}\varrho)\frac{\vartheta^{2\alpha}\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)},\\
&\quad \vdots
\end{aligned}$$

and so on.

As a result, the m th- order approximations of $\eta(\varrho,\vartheta),\lambda(\varrho,\vartheta)$ and $\varpi(\varrho,\vartheta)$ are given by

$$\eta(\varrho,\vartheta)=\sum_{m=0}^{\infty}\eta_m(\varrho,\vartheta),\ \lambda(\varrho,\vartheta)=\sum_{m=0}^{\infty}\lambda_m(\varrho,\vartheta),\ \ \varpi(\varrho,\vartheta)=\sum_{m=0}^{\infty}\varpi_m(\varrho,\vartheta). \tag{3.12}$$

Theorem (Convergence Theorem)

On condition that $\eta(\varrho,\vartheta)=\eta_0(\varrho,\vartheta)+\sum\limits_{m=1}^{\infty}\eta_m(\varrho,\vartheta)$, $\lambda(\varrho,\vartheta)=\lambda_0(\varrho,\vartheta)+\sum\limits_{m=1}^{\infty}\lambda_m(\varrho,\vartheta)$ and $\varpi(\varrho,\vartheta)=\varpi_0(\varrho,\vartheta)+\sum\limits_{m=1}^{\infty}\varpi_m(\varrho,\vartheta)$ converges, where $\eta_m(\varrho,\vartheta),\lambda_m(\varrho,\vartheta)$ and $\varpi_m(\varrho,\vartheta)$ are governed by (3.11) under the definitions (3.7) and (3.8), it must be solutions of time-fraction GHS-cKdV equation (1.4).

Proof

If the series

$$\sum_{m=0}^{\infty}\eta_m(\varrho,\vartheta),\sum_{m=0}^{\infty}\lambda_m(\varrho,\vartheta)\text{ and }\sum_{m=0}^{\infty}\varpi_m(\varrho,\vartheta)$$

convergence, then we can write

$$\begin{aligned}
S_1(\varrho,\vartheta) &= \sum_{m=0}^{\infty}\eta_m(\varrho,\vartheta),\\
S_2(\varrho,\vartheta) &= \sum_{m=0}^{\infty}\lambda_m(\varrho,\vartheta),\\
S_3(\varrho,\vartheta) &= \sum_{m=0}^{\infty}\varpi_m(\varrho,\vartheta),
\end{aligned}$$

and it holds

$$\begin{aligned}
\lim_{n\to\infty}\eta_n(\varrho,\vartheta) &= 0,\\
\lim_{n\to\infty}\lambda_n(\varrho,\vartheta) &= 0,\\
\lim_{n\to\infty}\varpi_n(\varrho,\vartheta) &= 0.
\end{aligned}\tag{3.13}$$

Using the Definition (3.7), we get

$$\begin{aligned}
\hbar\sum_{m=1}^{\infty}R_m(\eta_{m-1}(\varrho,\vartheta)) &= \sum_{m=1}^{\infty}L[\eta_m(\varrho,\vartheta)-\chi_m\eta_{m-1}(\varrho,\vartheta)]\\
&= \lim_{n\to\infty}\sum_{m=1}^{\infty}L[\eta_m(\varrho,\vartheta)-\chi_m\eta_{m-1}(\varrho,\vartheta)]\\
&= L[\lim_{n\to\infty}\sum_{m=1}^{\infty}(\eta_m(\varrho,\vartheta)-\chi_m\eta_{m-1}(\varrho,\vartheta))]\\
&= L[\lim_{n\to\infty}\eta_n(\varrho,\vartheta)]=0,
\end{aligned}\tag{3.14}$$

Similarly,

$$\hbar\sum_{m=1}^{\infty}R_m(\lambda_{m-1}(\varrho,\vartheta))=L[\lim_{n\to\infty}\lambda_n(\varrho,\vartheta)]=0,\tag{3.15}$$

$$\hbar\sum_{m=1}^{\infty}R_m(\varpi_{m-1}(\varrho,\vartheta))=L[\lim_{n\to\infty}\varpi_n(\varrho,\vartheta)]=0\tag{3.16}$$

From $\hbar\neq 0$, $\sum_{m=1}^{\infty}R_m(\overrightarrow{\eta}_m(\varrho,\vartheta))=0$, $\sum_{m=1}^{\infty}R_m(\overrightarrow{\lambda}_m(\varrho,\vartheta))=0$ and $\sum_{m=1}^{\infty}R_m(\overrightarrow{\varpi}_m(\varrho,\vartheta))=0$.
From (3.7), it holds

$$\begin{aligned}
\sum_{m=1}^{\infty}R_m(\eta_{m-1}(\varrho,\vartheta)) &= \sum_{m=1}^{\infty}[D_\vartheta^\alpha\eta_{m-1}(\varrho,\vartheta)-\frac{1}{2}(\eta_{m-1}(\varrho,\vartheta))_{\varrho\varrho\varrho}+3\sum_{i=0}^{m-1}(\eta_i(\varrho,\vartheta)(\eta_{m-1-i}(\varrho,\vartheta))_\varrho\\
&\quad -\lambda_i(\varrho,\vartheta)\varpi_{m-1-i}(\varrho,\vartheta)-\varpi_i(\varrho,\vartheta)\lambda_{m-1-i}(\varrho,\vartheta))]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} D_{\vartheta}^{\alpha}\eta_{m-1}(\varrho,\vartheta) - \frac{1}{2}\sum_{m=1}^{\infty}(\eta_{m-1}(\varrho,\vartheta))_{\varrho\varrho\varrho} + 3\sum_{m=1}^{\infty}[\sum_{i=0}^{m-1}(\eta_i(\varrho,\vartheta)(\eta_{m-1-i}(\varrho,\vartheta)_{\varrho}) \\
&\quad -\lambda_i(\varrho,\vartheta)(\varpi_{m-1-i}(\varrho,\vartheta))_{\varrho} - \varpi_i(\varrho,\vartheta)(\lambda_{m-1-i}(\varrho,\vartheta))_{\varrho}] \\
&= \sum_{m=0}^{\infty} D_{\vartheta}^{\alpha}\eta_{m}(\varrho,\vartheta) - \frac{1}{2}\sum_{m=0}^{\infty}(\eta_{m}(\varrho,\vartheta))_{\varrho\varrho\varrho} + 3\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty}[(\eta_i(\varrho,\vartheta)(\eta_{m-1-i}(\varrho,\vartheta)_{\varrho}) \\
&\quad -\lambda_i(\varrho,\vartheta)(\varpi_{m-1-i}(\varrho,\vartheta))_{\varrho} - \varpi_i(\varrho,\vartheta)(\lambda_{m-1-i}(\varrho,\vartheta))_{\varrho}] \\
&= D_{\vartheta}^{\alpha}\sum_{m=0}^{\infty}\eta_{m}(\varrho,\vartheta) - \frac{1}{2}(\sum_{m=0}^{\infty}\eta_{m}(\varrho,\vartheta))_{\varrho\varrho\varrho} + 3(\sum_{i=0}^{\infty}\eta_i(\varrho,\vartheta)\sum_{m=i+1}^{\infty}(\eta_{m-1-i}(\varrho,\vartheta))_{\varrho} \\
&\quad -\sum_{i=0}^{\infty}\lambda_i(\varrho,\vartheta)\sum_{m=i+1}^{\infty}(\varpi_{m-1-i}(\varrho,\vartheta))_{\varrho} - \sum_{i=0}^{\infty}\varpi_i(\varrho,\vartheta)\sum_{m=i+1}^{\infty}(\lambda_{m-1-i}(\varrho,\vartheta))_{\varrho}) \\
&= D_{\vartheta}^{\alpha}S_1(\varrho,\vartheta) - \frac{1}{2}(S_1(\varrho,\vartheta))_{\varrho\varrho\varrho} + 3(S_1(\varrho,\vartheta)(S_1(\varrho,\vartheta))_{\varrho} - S_2(\varrho,\vartheta)(S_3(\varrho,\vartheta))_{\varrho} \qquad (3.17) \\
&\quad -S_3(\varrho,\vartheta)(S_2(\varrho,\vartheta))_{\varrho} \\
&= 0.
\end{aligned}
$$

From the condition $\eta(\varrho,0)=0$ and $\eta_m(\varrho,0)=0$,

$$
S_1(\varrho,0) = \sum_{m=0}^{\infty}\eta_m(\varrho,0) = \eta_0(\varrho,\vartheta) + \sum_{m=1}^{\infty}\eta_{m-1}(\varrho,0) = f(\varrho,\vartheta). \tag{3.18}
$$

In the same like (3.17)

$$
\sum_{m=1}^{\infty} R_m(\lambda_{m-1}(\varrho,\vartheta)) = 0, \sum_{m=1}^{\infty} R_m(\varpi_{m-1}(\varrho,\vartheta)) = 0 \tag{3.19}
$$

From the conditions $\lambda(\varrho,0)=0, \lambda_m(\varrho,0)=0$ and $\varpi(\varrho,0)=0, \varpi_m(\varrho,0)=0$

$$
S_2(\varrho,0) = \sum_{m=0}^{\infty}\lambda_m(\varrho,0) = \lambda_0(\varrho,0) + \sum_{m=1}^{\infty}\lambda_{m-1}(\varrho,0) = g(\varrho,\vartheta).
$$

$$
S_3(\varrho,0) = \sum_{m=0}^{\infty}\varpi_m(\varrho,0) = \varpi_0(\varrho,0) + \sum_{m=1}^{\infty}\varpi_{m-1}(\varrho,0) = \hbar(\varrho,\vartheta).
$$

Consequently, $S(\varrho,\vartheta)$ must be the exact solution of Eqs. (1.4) and (3.1).

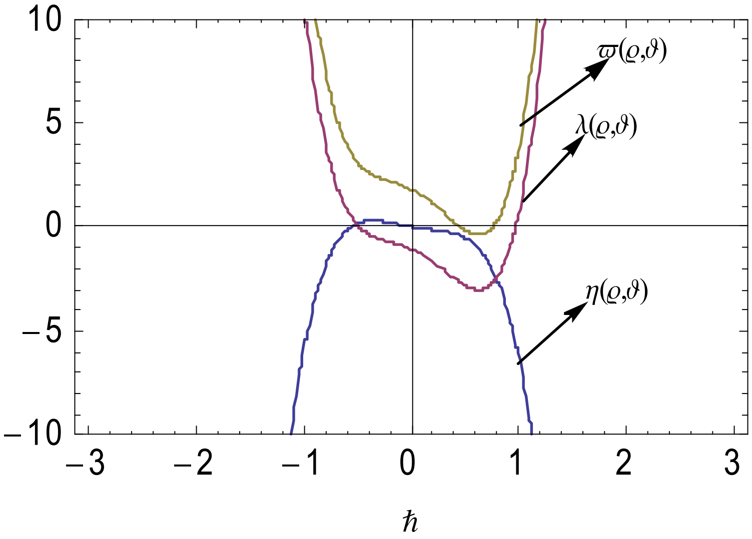

Figure 1. The $\hbar$ curves of 5 th-order approximate solutions obtained by the HAM for $\alpha = 0.5$ and $\boldsymbol{\kappa} = 1, \beta = 0.5, \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1.5, \varrho = 0.2, \vartheta = 0.6$.

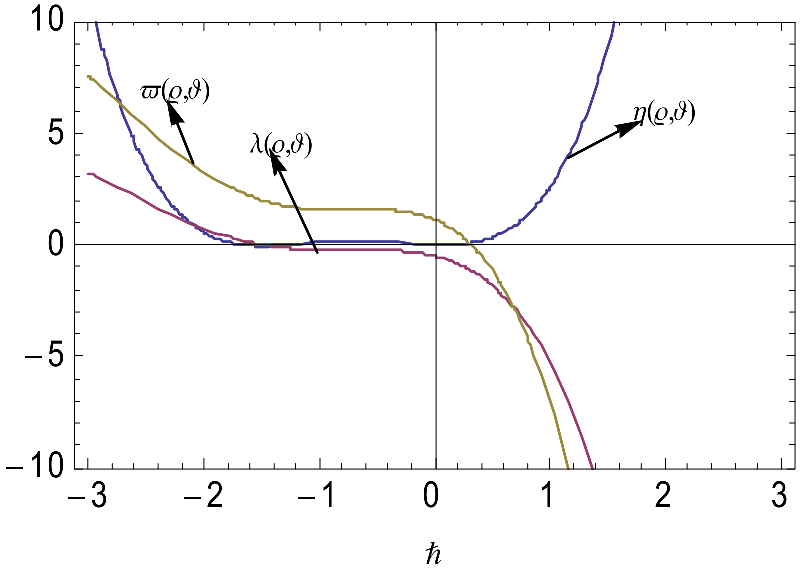

Figure 2. The $\hbar$ curves of 5 th-order approximate solutions obtained by the HAM for $\alpha = 0.75$ and $\boldsymbol{\kappa} = 0.5,\ \beta = 1.5,\ \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1,\ \varrho = 0.2, \vartheta = 0.6$.

$\hbar$	ϑ	DTM [9]	HPM [8]	HAM
-2.6	0.2	0.00121759	0.00142938	5.77383×10^{-4}
-2.2	0.4	0.00157146	0.00188767	5.10103×10^{-4}
-2.35	0.6	0.00193597	0.00231728	5.55×10^{-6}
-2.2	0.8	0.00230311	0.00271511	4.0728×10^{-7}

Table 1. Absolute erros obtained when $\alpha = 0.5$ and $\boldsymbol{\kappa} = 0.1,\ \beta = 1.5, \varrho = 0.5$ for 3 th-order approximation of $\eta(\varrho, \vartheta)$.

$\hbar$	ϑ	DTM [9]	HPM [8]	HAM
-1.75	0.2	2.66655×10^{-4}	2.66655×10^{-4}	2.4006×10^{-5}
-1.651	0.4	5.40902×10^{-4}	4.22079×10^{-4}	6.95901×10^{-6}
-1.54	0.6	6.21223×10^{-4}	5.06039×10^{-4}	4.2746×10^{-6}
-1.45	0.8	4.13316×10^{-4}	5.33157×10^{-4}	2.36771×10^{-5}

Table 2. Absolute erros obtained when $\alpha = 0.5$ and $\boldsymbol{\kappa} = 0.1,\ \beta = 1.5, \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1.5, \varrho = 0.5$ for 3 th-order approximation $\lambda(\varrho, \vartheta)$.

$\hbar$	ϑ	DTM [9]	HPM [8]	HAM
-2.222	0.2	0.0677777	0.02998	1.35667×10^{-4}
-0.336	0.4	0.0694248	0.0471694	1.28267×10^{-7}
-0.438	0.6	0.0604485	0.0565524	1.21137×10^{-4}
-0.54	0.8	0.0461901	0.059583	1.6585×10^{-4}

Table 3. Absolute erros obtained when $\alpha = 0.5$ and $\boldsymbol{\kappa} = 0.1,\ \beta = 1.5, \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1.5, \varrho = 0.5$ for 3 th-order approximation $\varpi(\varrho, \vartheta)$.

$\hbar$	ϑ	HAM	ES	AE
	0.0	0.	0.0101627	0.0101627
-0.9	0.1	0.0826045	0.0826252	2.07372×10^{-5}
-0.388	0.2	0.172619	0.172751	1.31932×10^{-4}
-0.416	0.3	0.259719	0.259898	1.79292×10^{-4}
-0.42	0.4	0.343514	0.343798	2.84257×10^{-4}
-0.417	0.5	0.424265	0.424234	3.03379×10^{-5}
-0.411	0.6	0.501284	0.50104	2.43098×10^{-4}
-0.404	0.7	0.574846	0.574099	7.46711×10^{-4}
-0.396	0.8	0.643616	0.64334	2.75958×10^{-4}
-0.388	0.9	0.70864	0.708736	9.58075×10^{-5}
-0.381	1.0	0.77238	0.770298	2.08196×10^{-3}

Table 4. HAM, ES and absolute errors (AE) results of $\eta(\varrho, \vartheta)$ when $\boldsymbol{\kappa} = 1,\ \beta = 0.5,\ \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1.5,\ \varrho = 0.25$ and $\alpha = 0.5$ for 5 th-order approximation.

$\hbar$	ϑ	HAM	ES	AE
	0.0	-0.0196301	-0.0196301	3.46945×10^{-18}
-0.079	0.1	-0.0193281	-0.0193284	3.09595×10^{-7}
-0.12	0.2	-0.0190208	-0.0190271	6.29367×10^{-6}
-0.153	0.3	-0.0187258	-0.0187263	4.90707×10^{-7}
-0.185	0.4	-0.0184285	-0.0184261	2.39303×10^{-6}
-0.218	0.5	-0.0181262	-0.0181267	5.32422×10^{-7}
-0.25	0.6	-0.0178328	-0.0178282	4.62445×10^{-6}
-0.285	0.7	-0.0175349	-0.0175306	4.26116×10^{-6}
-0.324	0.8	-0.0172347	-0.0172343	4.4655×10^{-7}
-0.37	0.9	-0.0169306	-0.0169392	8.63545×10^{-6}
-0.419	1.0	-0.0166457	-0.0166455	1.75463×10^{-7}

Table 5. Comparison of the HAM, ES and absolute errors (AE) results of $\lambda(\varrho, \vartheta)$ when $\boldsymbol{\kappa} = 0.1,\ \beta = 1.5,\ \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1,\ \varrho = 0.25$ and $\alpha = 0.5$ for 5 th-order approximation.

$\hbar$	ϑ	HAM	ES	AE
	0.0	0.512497	0.512497	0.
-0.079	0.1	0.514998	0.514996	2.23619×10^{-6}
-0.1187	0.2	0.517499	0.517493	5.76808×10^{-6}
-0.153	0.3	0.519989	0.519989	2.1848×10^{-7}
-0.1859	0.4	0.522491	0.522485	6.30129×10^{-6}
-0.218	0.5	0.52497	0.524979	9.60925×10^{-6}
-0.2516	0.6	0.527472	0.527472	4.41146×10^{-6}
-0.287	0.7	0.52997	0.529964	6.24215×10^{-6}
-0.325	0.8	0.532452	0.532454	2.527737×10^{-6}
-0.3685	0.9	0.534951	0.534943	8.18447×10^{-6}
-0.419	1.0	0.537423	0.53743	6.6887×10^{-6}

Table 6. HAM, ES and absolute errors (AE) results of $\varpi(\varrho, \vartheta)$ when $\boldsymbol{\kappa} = 0.1,\ \beta = 0.5,\ \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 0.5,\ \varrho = 0.25$ and $\alpha = 0.5$ for 5 th-order approximation.

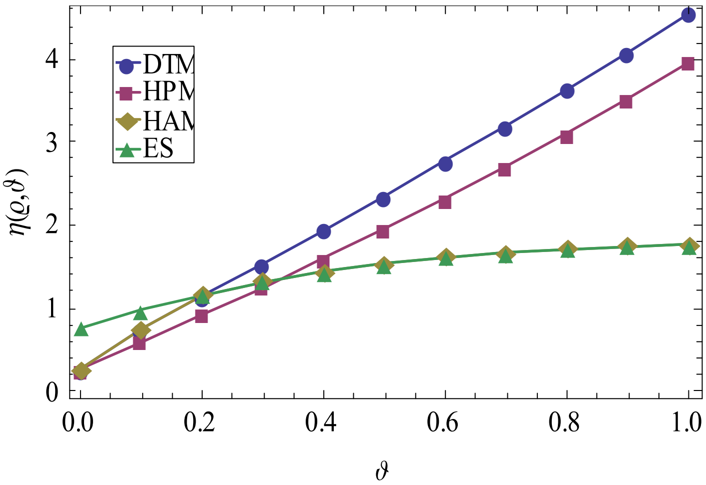

Figure 3. DTM, HPM, HAM and ES. for 3 th-order approximate when $\alpha = 0.75,\ \boldsymbol{\kappa} = 1,\ \beta = 1.5$ and $\varrho = 0.5$ of $\eta(\varrho, \vartheta)$.

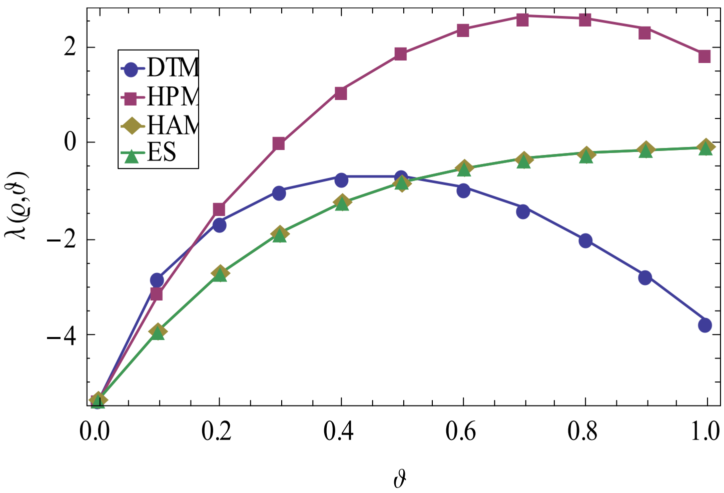

Figure 4. DTM, HPM, HAM and ES. for 3 th-order approximate when $\alpha = 0.75,\ \boldsymbol{\kappa} = 1.5,\ \beta = 1.5,\ \boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 1.5$ and $\varrho = 0.2$ of $\lambda(\varrho, \vartheta)$.

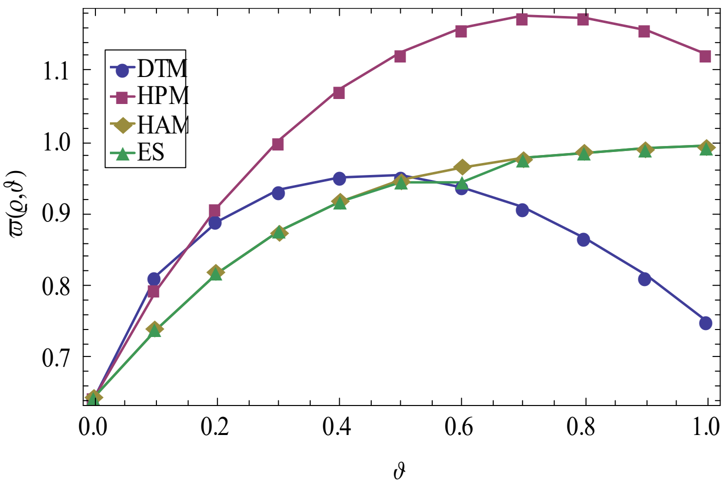

Figure 5. DTM, HPM, HAM and ES. for 3 th-order approximate when $\alpha = 0.75$, $\boldsymbol{\kappa} = 1.5$, $\beta = 1.5$, $\boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 = 0.5$ and $\varrho = 0.2$ of $\varpi(\varrho, \vartheta)$.

4.Conclusion

This artiche, we have achieved approximate solutions of a time-fraction GHS-cKdV equation via the HAM.

We have made convergence analysis and the range of $\hbar \neq 0$ was determined in Figure 1-2.

Absolute errors that were obtained by DTM [9] , HPM [8] and HAM have been seen in Table 1-3. As it has been seen from this tables, when $\hbar$ takes selecting the approximate, HAM gave better results than other two methods. In Tables 4-6, a comparison between HAM, ES and absolute error that was obtained by HAM for 5-terms of η, λ and ϖ was showed.

In Figures 3-5, a comparison between DTM, HPM, HAM and ES for $\alpha = 0.75$ and some values of $\boldsymbol{\kappa}, \beta, \boldsymbol{\xi}_0, \boldsymbol{\xi}_1$ and convergence control parameter $\hbar$ was presented. As it has been shown from the figures, DTM and HPM solutions walked off from ES but HAM solution approached the ES.

Consequently, in spite of solution which is found by HAM observed very closed to analytical solution, it is showed that solutions which is obtained with DTM and HPM is far to analytical solution. In conclusion, HAM provides us with a simple way to adjust and control the convergence regions and rates of approximation solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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