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Strong convergence of hybrid Bregman projection algorithm for split feasibility and fixed point problems in Banach spaces

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Abstract

In this paper, we consider and study split feasibility and fixed point problems involved in Bregman quasi-strictly pseudo-contractive mapping in Banach spaces. It is proven that the sequences generated by the proposed iterative algorithm converge strongly to the common solution of split feasibility and fixed point problems.

Keywords: Split feasibility problem, fixed point problem, Bregman quasi-strictly pseudo-contractive mapping, Bregman projection, strong convergence.
2010 MSC: 47J25, 47H10, 65J15, 90C25.

1. Introduction

Throughout this paper, we assume that C and Q be nonempty closed convex sets in p-uniformly convex and uniformly smooth real Banach spaces E_1 and E_2, respectively. Let A be a bounded linear operator from E_1 to E_2 with its adjoint A^*. Let T be a nonlinear mapping from C to itself. We use $Fix(T)$ to denote the set of all fixed points of the mapping T, that is, $Fix(T) = \{x \in C : Tx = x\}$.

This paper is concerned on studying the following split feasibility and fixed point problems:

$$\text{Find } x^* \in C \cap Fix(T) \text{ such that } Ax^* \in Q. \tag{1.1}$$

Let $\Gamma = \{x^* : x^* \in C \cap Fix(T) \text{ such that } Ax^* \in Q\}$ be the set of all solutions of (1.1). In the sequel, we assume $\Gamma \neq \emptyset$. A special case of (1.1) is the following split feasibility problem (in short, SFP):

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.2}$$

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Let $\Gamma_0 = \{x^* : x^* \in C$ such that $Ax^* \in Q\}$ be the set of all solutions of (1.2). Then, we have that Γ_0 is a closed and convex subset of E_1.

The theory of the SFP was first introduced and learned by Censor [7] in finite-dimensional space for modeling inverse problems which arise from phase retrievals and in image reconstruction has been discussed in the last two decades and much intensively in the last ten years. A large number of algorithms related to the SFP have been studied; see, for example, [5, 8, 4] and the references therein. Recently, it has been found that the SFP can be used in intensity modulated radiation therapy, please see [6, 8, 9] and the references therein. The algorithm suggested by Censor in [7] involves the computation of the inverse A^{-1}, so, it can not be widely used. A seemingly more popular algorithm is the CQ algorithm [5]:

$$x_{n+1} = P_C\left(I - \gamma A^*(I - P_Q)A\right)x_n, \quad n \geq 0, \tag{1.3}$$

where $x_0 \in \mathcal{H}_1$ (a Hilbert space) and $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^*A. Recently, the SFP was studied in a more general framework, for example, Banach spaces. More specifically, Schöpfer *et al*. [13] proposed the following algorithm in p-uniformly convex and uniformly smooth real Banach spaces:

$$x_{n+1} = \Pi_C J^*\left(Jx_n - \gamma A^* J(I - P_Q)Ax_n\right), \tag{1.4}$$

where Π_C denotes the Bregman projection and J the duality mapping, they established weak convergence of algorithm (1.4) under some mild conditions. Obviously the above algorithm (1.4) convers the CQ algorithm (1.3) as a special case.

It is worth pointing out that only weak convergence result is established in [13]. However, the strong convergence is more acceptable than the weak convergence in some practical applications. Wang [16] considered the following iterative algorithm for multiple-sets split feasibility problem in p-uniformly convex and uniformly smooth real Banach spaces:

$$\begin{cases} y_n = T_n x_n, \\ D_n = \{v \in E_1 : \Delta_p(y_n, v) \leq \Delta_p(x_n, v)\}, \\ E_n = \{v \in E_1 : \langle x_n - v, J_p\overline{x} - J_p x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{D_n \bigcap E_n}\overline{x}. \end{cases} \tag{1.5}$$

Using the idea in the work of Nakajo [11], Wang proved the strong convergence of iterative algorithm (1.5). Subsequently, Takahashi [14] proposed the following hybrid projection algorithm for the SFP in uniformly convex and uniformly smooth real Banach spaces:

$$\begin{cases} z_n = x_n - rJ_{E_1}^{-1}A^*J_{E_2}\left(Ax_n - P_Q Ax_n\right), \\ C_n = \{v \in E_1 : \langle z_n - v, J_{E_1}(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{v \in E_1 : \langle x_n - v, J_{E_1}(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_1. \end{cases} \tag{1.6}$$

Basing mainly on the hybrid method, he proved the strong convergence of iterative algorithm (1.6).

On the other hand, in 1967, Bregman [3] used the so-called Bregman distance function to design and analyze feasibility and optimization algorithms. After that, many authors found that the so-called Bregman distance function could be applied in different ways in order to construct iterative algorithms for solving not only feasibility and optimization problems, but also variational inequality problems, equilibria problems, fixed points problems and so on (see, *e.g*, [2, 10, 12, 18, 19, 20] and the references therein). The fixed point theory with respect to Bregman distance has been studied in the last decade and a lot of good results were published intensively in the last five years. Many authors concentrated their energies on constructing the fixed point of Bregman nonlinear operators by utilizing the Bregman distance and the Bregman projection, see [15, 17] and the references therein. In 2015, Wang [17] studied a new hybrid Bregman projection iterative algorithm for Bregman quasi-strictly pseudo-contractive mapping and proved strong convergence result in reflexive Banach spaces. In particularly, he proposed the following iterative method:

$$\begin{cases} x_0 \in C \textit{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ C_{n+1} = \{v \in C_n : D_f(x_n, Tx_n) \leq \frac{1}{1-\kappa}\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - v\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{1.7}$$

where $\kappa \in [0, 1)$. Then the sequence $\{x_n\}$ converges strongly to $p = P_{Fix(T)}x_0$, $P_{Fix(T)}$ is the Bregman projection of E onto $Fix(T)$.

In this paper, motivated and inspired by the above research work going on in this field, we propose a new hybrid projection method for solving split feasibility and fixed point problems (1.1) involved in Bregman quasi-strictly pseudo-contractive mapping in p-uniformly convex and uniformly smooth real Banach spaces. Our modification is mainly based on the schemes (1.5), (1.6) and (1.7). Furthermore, we will prove the strong convergence theorem for the proposed algorithm.

2. Preliminaries

Let $1 < q \leq 2 \leq p$ with $1/p + 1/q = 1$. Let E be a real Banach space.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S(E), \|x-y\| \geq \epsilon\right\},$$

for any x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. E is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$; p-uniformly convex if there exists $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| - \|x-y\|) - 1 : x \in S(E), \|y\| = t\right\}.$$

E is called uniformly smooth if $\lim_{t\to 0} \rho_E(t)/t = 0$. By setting $1 < q \leq 2 \leq p$, a Banach space E is called q-uniformly smooth if there exists $C_q > 0$ such that $\rho_E(t) \leq C_q t^q$ for all $t > 0$. We assume that E is p-uniformly convex and uniformly smooth, which implies that its dual space, E^*, is q-uniformly smooth and uniformly convex. In this situation, it is known that the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = \left(J_{E^*}^q\right)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^*.

The q-uniformly smooth spaces have the following conclusion.

Lemma 2.1. *[16] If E is a q-uniformly smooth space, then there is a constant $C_q > 0$ such that*

$$\|x-y\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q,$$

for all $x, y \in E$, where $C_q > 0$ is the q-uniformly smoothness constant of E and J_E^q is the duality mapping from E into 2^{E^*} defined by

$$J_E^q(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^q,\ \|x^*\| = \|x\|^{q-1}\}, \quad \forall x, y \in E.$$

Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the Bregman distance with respect to f is defined by

$$\Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x\rangle, \quad \forall x, y \in E.$$

It is note worthy that the duality mapping J_p is the derivative of the function $f_p = \frac{1}{p}\|x\|^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned}\Delta_p(x,y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}\left(\|y\|^p - \|x\|^p\right) + \langle J_E^p x, x-y\rangle \\ &= \frac{1}{q}\left(\|x\|^p - \|y\|^p\right) - \langle J_E^p x - J_E^p y, x\rangle.\end{aligned}$$

From the definition of $\Delta_p(\cdot,\cdot)$, we get

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) + \left\langle z-y, J_E^p x - J_E^p z\right\rangle, \tag{2.1}$$

and

$$\Delta_p(x,y) + \Delta_p(y,x) = \left\langle x-y, J_E^p x - J_E^p y\right\rangle, \tag{2.2}$$

for any $x,y,z \in E$. All in all, the Bregman distance is not a metric because of the lack of symmetry. For the p-uniformly convex space, the metric and Bregman distance has the following relation

$$\tau\|x-y\|^p \le \Delta_p(x,y) \le \left\langle x-y, J_E^p x - J_E^p y\right\rangle, \tag{2.3}$$

where $\tau > 0$. Obviously, if $\{x_n\}$ and $\{y_n\}$ are both bounded sequences of a p-uniformly convex and uniformly smooth space E, then $x_n - y_n \to 0$ as $n \to \infty$ implies that $\Delta_p(x_n, y_n) \to 0$ as $n \to \infty$.

Projections are an important tool for our work in this paper. We can define metric projection P_C as follows

$$P_C x = \operatorname{argmin}_{y\in C}\|x-y\|, \quad x \in E,$$

metric projection P_C can be characterized by the following variational inequality

$$\left\langle J_E^p(x - P_C x), z - P_C x\right\rangle \le 0, \quad z \in C. \tag{2.4}$$

Likewise, one can define the Bregman projection

$$\Pi_C x = \operatorname{argmin}_{y\in C}\Delta_p(x,y), \quad x \in E,$$

is the unique minimizer of the Bregman distance, The Bregman projection can also be characterized by the following variational inequality

$$\left\langle J_E^p x - J_E^p \Pi_C x, z - \Pi_C x\right\rangle \le 0, \quad z \in C, \tag{2.5}$$

from which one has

$$\Delta_p(\Pi_C x, z) \le \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \quad z \in C. \tag{2.6}$$

The metric projection and the Bregman projection with respect to f_2 are coincident in a Hilbert space, but in a more general framework, they are totally different. What is important is that the metric projection can not share property (2.6) as the Bregman projection in Banach spaces.

Following [1], we study the function $V_p : E^* \times E \to [0,\infty)$ associated with f_p, which is defined by

$$V_p(\overline{x}, x) = \frac{1}{q}\|\overline{x}\|^q - \langle \overline{x}, x\rangle + \frac{1}{p}\|x\|^p, \quad x \in E,\ \overline{x} \in E^*.$$

Then V_p is nonnegative and

$$V_p(\overline{x}, x) = \Delta_p(J_{E^*}^q \overline{x}, x), \quad x \in E,\ \overline{x} \in E^*.$$

Moreover, by the subdifferential inequality, we have

$$V_p(\overline{x}, x) + \langle \overline{y}, J^q_{E^*}\overline{x} - x\rangle \leq V_p(\overline{x} + \overline{y}, x), \quad x \in E,\ \overline{x}, \overline{y} \in E^*.$$

In addition, V_p is convex in the first variable. Thus, for all $z \in E$,

$$\Delta_p\left(J^q_{E^*}\left(\sum_{i=1}^N t_i J^p_E(x_i)\right), z\right) \leq \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.7}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$. For more details about V_p, please see [1].

Very recently, Ugwunnadi *et al*. [15] introduced the concept of Bregman quasi-strictly pseudo-contractive mapping and proved the strong convergence by using hybrid Bregman projection iterative algorithm for a Bregman quasi-strictly pseudo-contractive mapping.

Definition 2.2. A mapping $T : C \to C$ is said to be Bregman quasi-strictly pseudo-contractive mapping if there exists a constant $\kappa \in [0,1)$ and $Fix(T) \neq \emptyset$ such that

$$\Delta_p(Tx, x^*) \leq \Delta_p(x, x^*) + \kappa\Delta_p(Tx, x), \quad \forall x \in C,\ x^* \in Fix(T).$$

Definition 2.3. A mapping $T : C \to C$ is said to be Bregman quasi-nonexpansive mapping if $Fix(T) \neq \emptyset$ such that

$$\Delta_p(Tx, x^*) \leq \Delta_p(x, x^*), \quad \forall x \in C,\ x^* \in Fix(T).$$

Definition 2.4. A mapping $T : C \to C$ is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$ as $n \to \infty$, then $Tx = y$.

We shall adopt the notation: $x_n \to x$ means that $\{x_n\}$ converges to x strongly. Now, we give some examples of a Bregman quasi-strictly pseudo-contractive mapping.

Example 2.5. [17] Let E be a smooth space, and define $f(x) = \|x\|^2$ for all $x \in E$. Let $x_0 \neq 0$ be any element of E, $T : E \to E$ be defined as follows:

$$T(x) = \begin{cases} (1/2 + 1/2^{n+1})x_0, & x = (1/2 + 1/2^n)x_0, \\ -x, & x \neq (1/2 + 1/2^n)x_0 \end{cases}$$

for all $n \geq 1$. Then T is a Bregman quasi-strictly pseudo-contractive mapping.

Example 2.6. [15] Let $E = \mathbb{R}$ and define $T, f : [-1, 0] \to \mathbb{R}$ by $f(x) = x$ and $T(x) = 2x$ for all $x \in [-1, 0]$. Then T is a Bregman quasi-strictly pseudo-contractive mapping.

3. Main results

In this section, we will introduce the following algorithm and prove strong convergence theorem for finding the common solution of split feasibility and fixed point problems.

Theorem 3.1. *Let C and Q be nonempty closed convex sets in p-uniformly convex and uniformly smooth real Banach spaces E_1 and E_2, respectively. Let A: $E_1 \to E_2$ be a bounded linear operator with its adjoint A^*: $E_2^* \to E_1^*$. Let T be a closed Bregman quasi-strictly pseudo-contractive mapping from C to itself. Let the sequence $\{x_n\}$ be iteratively generated by $x_1 = x_0 \in C$, $D_1 = C_1 = C$,*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C J^q_{E_1^*}\left(J^p_{E_1}x_n - \lambda_n A^* J^p_{E_2}\left(Ax_n - P_Q Ax_n\right)\right), \\ z_n = J^q_{E_1^*}\left(\alpha_n J^p_{E_1}y_n + (1-\alpha_n)J^p_{E_1}Tx_n\right), \\ D_{n+1} = \{w \in D_n : \Delta_p(y_n, w) \leq \Delta_p(x_n, w)\}, \\ C_{n+1} = \Big\{w \in C_n : \Delta_p(z_n, x_n) \leq \frac{\kappa}{1-\kappa}\left\langle w - x_n, J^p_{E_1}Tx_n - J^p_{E_1}x_n\right\rangle \\ \qquad\qquad + \left\langle w - x_n, J^p_{E_1}z_n - J^p_{E_1}x_n\right\rangle\Big\}, \\ x_{n+1} = \Pi_{D_{n+1}\bigcap C_{n+1}}x_0, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\kappa \in [0,1)$. *Assume that* $\{\alpha_n\} \subset [c,d] \subset (0,1)$ *and* $\{\lambda_n\} \subset [a,b] \subset \left(0, \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}\right)$. *Then the sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to* $\Pi_\Gamma x_0$.

Proof. Taking $x^* \in \Gamma$, $x' \in C$, by the definition of T, we have

$$\Delta_p(Tx', x^*) \leq \Delta_p(x', x^*) + \kappa\Delta_p(Tx', x').$$

From (2.1), we get

$$\Delta_p(Tx', x^*) = \Delta_p(Tx', x') + \Delta_p(x', x^*) + \left\langle x' - x^*, J_{E_1}^p Tx' - J_{E_1}^p x' \right\rangle,$$

which implies that

$$\Delta_p(Tx', x') \leq \frac{1}{1-\kappa}\left\langle x^* - x', J_{E_1}^p Tx' - J_{E_1}^p x' \right\rangle. \tag{3.2}$$

Let $\{x_n\}$ be a sequence in $Fix(T)$ such that $x_n \to z$ as $n \to \infty$. From (3.2), we obtain

$$\Delta_p(Tz, z) \leq \frac{1}{1-\kappa}\left\langle x_n - z, J_{E_1}^p Tz - J_{E_1}^p z \right\rangle,$$

setting $n \to \infty$ in the above inequality, we have $\Delta_p(Tz, z) \leq 0$, it follows from (2.3) that $Tz = z$. Therefore, $Fix(T)$ is closed.

Next, let $z_1, z_2 \in Fix(T)$, for given $t \in (0,1)$, putting $z = tz_1 + (1-t)z_2$. From (3.2), we obtain, respectively,

$$\Delta_p(Tz, z) \leq \frac{1}{1-\kappa}\left\langle z_1 - z, J_{E_1}^p Tz - J_{E_1}^p z \right\rangle, \tag{3.3}$$

and

$$\Delta_p(Tz, z) \leq \frac{1}{1-\kappa}\left\langle z_2 - z, J_{E_1}^p Tz - J_{E_1}^p z \right\rangle. \tag{3.4}$$

Multiplying (3.3) by t and (3.4) by $1-t$, we have

$$\Delta_p(Tz, z) \leq \frac{1}{1-\kappa}\left\langle z - z, J_{E_1}^p Tz - J_{E_1}^p z \right\rangle,$$

setting $n \to \infty$ in the above inequality, we have $\Delta_p(Tz, z) \leq 0$, it follows from (2.3) that $Tz = z$. Therefore, $Fix(T)$ is convex. Since Γ_0 is a closed convex subset of E_1, we obtain that Γ is closed convex.

Now, from (3.1), we know D_n is closed for each $n \geq 1$. Note that $\Delta_p(y_n, w) \leq \Delta_p(x_n, w)$ is equivalent to

$$\left\langle J_{E_1}^p x_n - J_{E_1}^p y_n, w \right\rangle \leq \frac{1}{q}\left(\|x_n\|^p - \|y_n\|^p\right),$$

so that D_n is a halfspace, therefore, we get D_n is convex immediately.

For $n = 1$, $C_1 = C$ is closed convex essentially. Assume that C_n is closed convex for $n > 1$. For $w \in C_{n+1}$, we obtain

$$\Delta_p(z_n, x_n) \leq \frac{\kappa}{1-\kappa}\left\langle w - x_n, J_{E_1}^p Tx_n - J_{E_1}^p x_n \right\rangle + \left\langle w - x_n, J_{E_1}^p z_n - J_{E_1}^p x_n \right\rangle,$$

since $\left\langle \cdot, J_{E_1}^p Tx_n - J_{E_1}^p x_n \right\rangle$ and $\left\langle \cdot, J_{E_1}^p z_n - J_{E_1}^p x_n \right\rangle$ are continuous and linear in E_1, we get C_n is closed convex.

Let $x^* \in \Gamma$ and let $v_n = Ax_n - P_Q Ax_n$. It follows from (2.4) that

$$\begin{aligned}\left\langle J_{E_2}^p v_n, Ax_n - Ax^* \right\rangle &= \|Ax_n - P_Q Ax_n\|^p + \left\langle J_{E_2}^p v_n, P_Q Ax_n - Ax^* \right\rangle \\ &\geq \|Ax_n - P_Q Ax_n\|^p,\end{aligned}$$

applying Lemma 2.1, we have

$$\begin{aligned}
\Delta_p(y_n, x^*) &\leq \Delta_p\left(J_{E_1^*}^q\left(J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p v_n\right), x^*\right)\\
&= \frac{1}{q}\left\|J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p v_n\right\|^q - \left\langle J_{E_1}^p x_n, x^*\right\rangle + \lambda_n\left\langle J_{E_2}^p v_n, Ax^*\right\rangle + \frac{1}{p}\|x^*\|^p\\
&\leq \frac{1}{q}\left\|J_{E_1}^p x_n\right\|^q - \lambda_n\left\langle Ax_n, J_{E_2}^p v_n\right\rangle + \frac{C_q\left(\lambda_n\|A\|\right)^q}{q}\left\|J_{E_2}^p v_n\right\|^q\\
&\quad - \left\langle J_{E_1}^p x_n, x^*\right\rangle + \lambda_n\left\langle J_{E_2}^p v_n, Ax^*\right\rangle + \frac{1}{p}\|x^*\|^p\\
&= \frac{1}{q}\|x_n\|^p - \left\langle J_{E_1}^p x_n, x^*\right\rangle + \frac{1}{p}\|x^*\|^p + \lambda_n\left\langle J_{E_2}^p v_n, Ax^* - Ax_n\right\rangle\\
&\quad + \frac{C_q\left(\lambda_n\|A\|\right)^q}{q}\left\|J_{E_2}^p v_n\right\|^q\\
&= \Delta_p(x_n, x^*) + \lambda_n\left\langle J_{E_2}^p v_n, Ax^* - Ax_n\right\rangle + \frac{C_q\left(\lambda_n\|A\|\right)^q}{q}\left\|J_{E_2}^p v_n\right\|^q\\
&\leq \Delta_p(x_n, x^*) - \left(\lambda_n - \frac{C_q\left(\lambda_n\|A\|\right)^q}{q}\right)\|v_n\|^p.
\end{aligned} \tag{3.5}$$

By the assumption of $\{\lambda_n\}$, we have

$$\Delta_p(y_n, x^*) \leq \Delta_p(x_n, x^*), \tag{3.6}$$

so that $\Gamma \subset D_{n+1}$ for all $n \geq 1$. Next, we show $\Gamma \subset C_{n+1}$. Note that $\Gamma \subset C_1 = C$. Suppose $\Gamma \subset C_n$ for $n \geq 1$, then for all $x^* \in \Gamma \subset C_n$, from (2.7), (3.1), (3.2) and (3.6), we obtain

$$\begin{aligned}
\Delta_p(z_n, x^*) &= \Delta_p\left(J_{E_1^*}^q\left(\alpha_n J_{E_1}^p y_n + (1-\alpha_n) J_{E_1}^p T x_n\right), x^*\right)\\
&\leq \alpha_n \Delta_p(y_n, x^*) + (1-\alpha_n)\Delta_p(Tx_n, x^*)\\
&\leq \alpha_n \Delta_p(y_n, x^*) + (1-\alpha_n)\left(\Delta_p(x_n, x^*) + \kappa\Delta_p(Tx_n, x_n)\right)\\
&\leq \Delta_p(x_n, x^*) + \kappa\Delta_p(Tx_n, x_n)\\
&\leq \Delta_p(x_n, x^*) + \frac{\kappa}{1-\kappa}\left\langle x^* - x_n, J_{E_1}^p T x_n - J_{E_1}^p x_n\right\rangle.
\end{aligned} \tag{3.7}$$

From (2.1), we get

$$\Delta_p(z_n, x^*) = \Delta_p(z_n, x_n) + \Delta_p(x_n, x^*) + \left\langle x_n - x^*, J_{E_1}^p z_n - J_{E_1}^p x_n\right\rangle. \tag{3.8}$$

By (3.7) and (3.8), we obtain

$$\Delta_p(z_n, x_n) \leq \frac{\kappa}{1-\kappa}\left\langle x^* - x_n, J_{E_1}^p T x_n - J_{E_1}^p x_n\right\rangle + \left\langle x^* - x_n, J_{E_1}^p z_n - J_{E_1}^p x_n\right\rangle.$$

This shows that $x^* \in C_{n+1}$, which implies $\Gamma \subset C_{n+1}$ for all $n \geq 1$. Thus, $D_{n+1} \bigcap C_{n+1}$ is nonempty. So, $\{x_n\}$ is well defined.

From (3.1) and (2.5), we have

$$\left\langle J_{E_1}^p x_0 - J_{E_1}^p x_n, z - x_n\right\rangle \leq 0, \quad z \in C_n.$$

Since $\Gamma \subset C_n$, we have

$$\left\langle J_{E_1}^p x_0 - J_{E_1}^p x_n, x^* - x_n\right\rangle \leq 0, \quad x^* \in \Gamma. \tag{3.9}$$

By (2.6) and for all $x^* \in \Gamma$, we have

$$\begin{aligned}
\Delta_p(x_n, x_0) &\leq \Delta_p(x^*, x_0) - \Delta_p(x^*, x_n)\\
&\leq \Delta_p(x^*, x_0),
\end{aligned}$$

this shows that $\{\Delta_p(x_n, x_0)\}$ is bounded. Hence, $\{x_n\}$ is bounded. By the construction of C_n, we get $x_m \in C_m \subset C_n$ and $x_n = \Pi_{C_n} x_0$ for any $m \geq n$. From (2.6), we obtain

$$\begin{aligned} \Delta_p(x_m, x_n) &= \Delta_p(x_m, \Pi_{C_n} x_0) \leq \Delta_p(x_m, x_0) - \Delta_p(\Pi_{C_n} x_0, x_0) \\ &= \Delta_p(x_m, x_0) - \Delta_p(x_n, x_0). \end{aligned} \tag{3.10}$$

Since $x_n = \Pi_{C_n} x_0$ and $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$, we have $\Delta_p(x_n, x_0) \leq \Delta_p(x_m, x_0)$ for all $m \geq n$. This implies that $\{\Delta_p(x_n, x_0)\}$ is nondecreasing and hence the limit $\lim_{n\to\infty} \Delta_p(x_n, x_0)$ exists. From (3.10), we obtain $\Delta_p(x_n, x_m) \to 0$ as $m, n \to \infty$. From (2.3), we have $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in $C \subset E_1$, so there exists $x \in E_1$ such that $x_n \to x$ as $n \to \infty$.

By using (2.1) and (2.5), we have

$$\begin{aligned} \Delta_p(x_0, \Pi_\Gamma x_0) &\geq \Delta_p(x_0, x_{n+1}) \\ &= \Delta_p(x_0, x_n) + \Delta_p(x_n, x_{n+1}) + \left\langle x_n - x_{n+1}, J_{E_1}^p x_0 - J_{E_1}^p x_n \right\rangle \\ &\geq \Delta_p(x_0, x_n) + \Delta_p(x_n, x_{n+1}) \\ &\geq \Delta_p(x_0, x_{n-1}) + \Delta_p(x_{n-1}, x_n) + \Delta_p(x_n, x_{n+1}) \\ &\vdots \\ &\geq \sum_{i=0}^{n} \Delta_p(x_i, x_{i+1}). \end{aligned}$$

Consequently, $\sum_{i=0}^{\infty} \Delta_p(x_i, x_{i+1}) < \infty$, which from (2.3) yields $\sum_{i=0}^{\infty} \|x_n - x_{n+1}\|^p < \infty$. This implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.11}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have,

$$\Delta_p(z_n, x_n) \leq \frac{\kappa}{1-\kappa} \left\langle x_{n+1} - x_n, J_{E_1}^p T x_n - J_{E_1}^p x_n \right\rangle + \left\langle x_{n+1} - x_n, J_{E_1}^p z_n - J_{E_1}^p x_n \right\rangle,$$

from (3.11) and (2.3), we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.12}$$

Since $x_{n+1} = \Pi_{D_{n+1}} x_0 \in D_{n+1}$, we get,

$$\Delta_p(y_n, x_{n+1}) \leq \Delta_p(x_n, x_{n+1}),$$

from (3.11), we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0,$$

so,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.13}$$

Since $J_{E_1}^p$ is norm-to-norm uniformly continuous, from (3.1), we get

$$\begin{aligned} \|J_{E_1}^p z_n - J_{E_1}^p x_n\| &= \|\alpha_n (J_{E_1}^p y_n - J_{E_1}^p x_n) + (1-\alpha_n)(J_{E_1}^p T x_n - J_{E_1}^p x_n)\| \\ &\geq (1-\alpha_n)\|J_{E_1}^p T x_n - J_{E_1}^p x_n\| - \alpha_n \|J_{E_1}^p y_n - J_{E_1}^p x_n\|, \end{aligned}$$

this implies that

$$(1-\alpha_n)\|J_{E_1}^p T x_n - J_{E_1}^p x_n\| \leq \alpha_n \|J_{E_1}^p y_n - J_{E_1}^p x_n\| + \|J_{E_1}^p z_n - J_{E_1}^p x_n\|, \tag{3.14}$$

since $J^q_{E_1^*}$ is norm-to-norm uniformly continuous, setting $n \to \infty$ in (3.14), from (3.12), (3.13) and $\{\alpha_n\} \subset [c, d] \subset (0, 1)$, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0,$$

by the closedness of T, from $x_n \to x$, we obtain

$$Tx = x. \tag{3.15}$$

From (2.3) and (2.5), we have

$$\begin{aligned}
\Delta_p(x, \Pi_C x) &\le \left\langle x - \Pi_C, J^p_{E_1} x - J^p_{E_1} \Pi_C x \right\rangle \\
&= \left\langle x - x_n, J^p_{E_1} x - J^p_{E_1} \Pi_C x \right\rangle + \left\langle x_n - y_n, J^p_{E_1} x - J^p_{E_1} \Pi_C x \right\rangle \\
&\quad + \left\langle y_n - \Pi_C x, J^p_{E_1} x - J^p_{E_1} \Pi_C x \right\rangle \\
&\le \left\langle x - x_n, J^p_{E_1} x - J^p_{E_1} \Pi_C x \right\rangle + \left\langle x_n - y_n, J^p_{E_1} x - J^p_{E_1} \Pi_C x \right\rangle .
\end{aligned}$$

Setting $n \to \infty$ yields $\Delta_p(x, \Pi_C x) = 0$, we get $x \in C$.

From (3.5), we have

$$\left(\lambda_n - \frac{C_q (\lambda_n \|A\|)^q}{q}\right) \|v_n\|^p \le \Delta_p(x_n, x^*) - \Delta_p(y_n, x^*),$$

this together with $v_n = Ax_n - P_Q Ax_n$ and (3.13) implies that

$$\lim_{n\to\infty} \|Ax_n - P_Q Ax_n\| = 0. \tag{3.16}$$

By (2.4), we have

$$\begin{aligned}
\|Ax - P_Q Ax\|^p &= \left\langle J^p_{E_2} (Ax - P_Q Ax), Ax - P_Q Ax \right\rangle \\
&= \left\langle J^p_{E_2} (Ax - P_Q Ax), Ax - Ax_n \right\rangle + \left\langle J^p_{E_2} (Ax - P_Q Ax), Ax_n - P_Q Ax_n \right\rangle \\
&\quad + \left\langle J^p_{E_2} (Ax - P_Q Ax), P_Q Ax_n - P_Q Ax \right\rangle \\
&\le \left\langle J^p_{E_2} (Ax - P_Q Ax), Ax - Ax_n \right\rangle + \left\langle J^p_{E_2} (Ax - P_Q Ax), Ax_n - P_Q Ax_n \right\rangle .
\end{aligned}$$

From (3.16) and $Ax_n \to Ax$ as $n \to \infty$, setting $n \to \infty$ yields $\|Ax - P_Q Ax\|^p = 0$, we have $Ax \in Q$. Thus, we conclude that $x_n \to x \in \Gamma$.

Setting $n \to \infty$ in (3.9), we obtain

$$\left\langle J^p_{E_1} x_0 - J^p_{E_1} x, x^* - x \right\rangle \le 0, \quad x^* \in \Gamma.$$

By (2.5), we have $x = \Pi_\Gamma x_0$. □

Remark 3.2. Compared with the known results in the literature, our result is very different from those in the following aspects.

- The corresponding iterative algorithms in [14, Theorem 3.2], [16, Theorem 3.1], [17, Theorem 3.1] are extended for developing our algorithm which couples modified CQ method with Nakajo's iteration involved in Bregman quasi-strictly pseudo-contractive mapping in Theorem 3.1. Our iterative scheme in Theorem 3.1 can be viewed as a merger between corresponding iterative algorithms in [14, Theorem 3.2], [16, Theorem 3.1], [17, Theorem 3.1].

- The construction of sets $\Big($such as $C_{n+1} = \Big\{w \in C_n : \Delta_p(z_n, x_n) \leq \frac{\kappa}{1-\kappa} \Big\langle w - x_n, J^p_{E_1} Tx_n - J^p_{E_1} x_n \Big\rangle + \Big\langle w - x_n, J^p_{E_1} z_n - J^p_{E_1} x_n \Big\rangle \Big\}\Big)$ in our iterative scheme is very different from the iterative algorithm in [14, Theorem 3.2] because our construction is mainly based on the definition of Bregman quasi-strictly pseudo-contractive mapping. Moreover, we attain strong convergence result in a broader framework, the p-uniformly convex and uniformly smooth Banach spaces.
- The technique of proving strong convergence in Theorem 3.1 is different from those in [14, Theorem 3.2], [17, Theorem 3.1] because our technique depends on Lemma 2.1 in Banach spaces.
- The problem of finding a common element of the set of solutions of split feasibility problem and the set of fixed points of a Bregman quasi-strictly pseudo-contractive mapping in our Theorem 3.1 is more general than the problem of finding a solution of split feasibility problem in [14, Theorem 3.2] and the problem of finding an element of the set of fixed points of a Bregman quasi-strictly pseudo-contractive mapping in [17, Theorem 3.1].

Since the class of Bregman quasi-nonexpansive mappings is Bregman quasi-strict pseudo-contractive, the following corollary is obtained by using Theorem 3.1.

Corollary 3.3. *Let C and Q be nonempty closed convex sets in p-uniformly convex and uniformly smooth real Banach spaces E_1 and E_2, respectively. Let A: $E_1 \to E_2$ be a bounded linear operator with its adjoint A^*: $E_2^* \to E_1^*$. Let T be a closed Bregman quasi-nonexpansive mapping from C to itself. Let sequence $\{x_n\}$ be iteratively generated by $x_1 = x_0 \in C$, $D_1 = C_1 = C$,*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C J^q_{E_1^*} \left(J^p_{E_1} x_n - \lambda_n A^* J^p_{E_2} (Ax_n - P_Q Ax_n) \right), \\ z_n = J^q_{E_1^*} \left(\alpha_n J^p_{E_1} y_n + (1-\alpha_n) J^p_{E_1} Tx_n \right), \\ D_{n+1} = \{w \in D_n : \Delta_p(y_n, w) \leq \Delta_p(x_n, w)\}, \\ C_{n+1} = \Big\{w \in C_n : \Delta_p(z_n, x_n) \leq \Big\langle w - x_n, J^p_{E_1} z_n - J^p_{E_1} x_n \Big\rangle \Big\}, \\ x_{n+1} = \Pi_{D_{n+1} \bigcap C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{3.17}$$

Assume that $\{\alpha_n\} \subset [c, d] \subset (0, 1)$ and $\{\lambda_n\} \subset [a, b] \subset \left(0, \left(\frac{q}{C_q \|A\|^q}\right)^{\frac{1}{q-1}}\right)$. Then the sequence $\{x_n\}$ defined by (3.17) converges strongly to $\Pi_\Gamma x_0$.

Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. Then we have the following corollary.

Corollary 3.4. *Let E_1 and E_2 be two L^p spaces with $2 \leq p < \infty$, $C \subset E_1$ and $Q \subset E_2$ be two nonempty closed convex sets. Let A: $E_1 \to E_2$ be a bounded linear operator with its adjoint A^*: $E_2^* \to E_1^*$. Let T be a closed Bregman quasi-strictly pseudo-contractive mapping from C to itself. Let sequence $\{x_n\}$ be iteratively generated by $x_1 = x_0 \in C$, $D_1 = C_1 = C$,*

$$\begin{cases} x_1 \in C, \\ y_n = \Pi_C J^q_{E_1^*} \left(J^p_{E_1} x_n - \lambda_n A^* J^p_{E_2} (Ax_n - P_Q Ax_n) \right), \\ z_n = J^q_{E_1^*} \left(\alpha_n J^p_{E_1} y_n + (1-\alpha_n) J^p_{E_1} Tx_n \right), \\ D_{n+1} = \{w \in D_n : \Delta_p(y_n, w) \leq \Delta_p(x_n, w)\}, \\ C_{n+1} = \Big\{w \in C_n : \Delta_p(z_n, x_n) \leq \frac{\kappa}{1-\kappa} \Big\langle w - x_n, J^p_{E_1} Tx_n - J^p_{E_1} x_n \Big\rangle \\ \qquad\qquad + \Big\langle w - x_n, J^p_{E_1} z_n - J^p_{E_1} x_n \Big\rangle \Big\}, \\ x_{n+1} = \Pi_{D_{n+1} \bigcap C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{3.18}$$

where $\kappa \in [0,1)$. *Assume that* $\{\alpha_n\} \subset [c,d] \subset (0,1)$ *and* $\{\lambda_n\} \subset [a,b] \subset \left(0, \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}\right)$. *Then the sequence* $\{x_n\}$ *defined by* (3.18) *converges strongly to* $\Pi_\Gamma x_0$.

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