



Quadratic ρ -functional inequalities in β -homogeneous normed spaces

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Abstract

In this paper, we solve the quadratic ρ -functional inequalities

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ & \leq \left\| \rho \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right) \right\|, \end{aligned} \quad (1)$$

where ρ is a fixed complex number with $|\rho| < 1$, and

$$\begin{aligned} & \left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right\| \\ & \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \end{aligned} \quad (2)$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ -functional inequalities (1) and (2) in β -homogeneous complex Banach spaces.

Keywords: Hyers-Ulam stability; β -homogeneous space; quadratic ρ -functional inequality.

2010 MSC: Primary 39B62, 39B72, 39B52, 39B82

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Received 2012-3-5

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F -norm if it satisfies the following conditions:

- (FN₁) $\|x\| = 0$ if and only if $x = 0$;
- (FN₂) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
- (FN₃) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
- (FN₄) $\|\lambda_n x\| \rightarrow 0$ provided $\lambda_n \rightarrow 0$;
- (FN₅) $\|\lambda x_n\| \rightarrow 0$ provided $x_n \rightarrow 0$.

Then $(X, \|\cdot\|)$ is called an F^* -space. An F -space is a complete F^* -space.

An F -norm is called β -homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [16]). A β -homogeneous F -space is called a β -homogeneous complex Banach space.

In Section 2, we solve the quadratic ρ -functional inequality (1) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (1) in β_2 -homogeneous complex Banach space.

In Section 3, we solve the quadratic ρ -functional inequality (2) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (2) in β_2 -homogeneous complex Banach space.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1 -homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2 -homogeneous complex Banach space with norm $\|\cdot\|$.

2. Quadratic ρ -functional inequality (1) in β -homogeneous complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the quadratic ρ -functional inequality (1) in complex normed spaces.

Lemma 2.1. *If a mapping $f : G \rightarrow Y$ satisfies*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ & \leq \left\| \rho \left(4f \left(\frac{x+y}{2} \right) + f(x-y) - 2f(x) - 2f(y) \right) \right\| \end{aligned} \quad (2.1)$$

for all $x, y \in G$, then $f : G \rightarrow Y$ is quadratic.

Proof. Assume that $f : G \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho| \|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in G$.

Thus

$$f \left(\frac{x}{2} \right) = \frac{1}{4} f(x) \quad (2.2)$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ & \leq \left\| \rho \left(4f \left(\frac{x+y}{2} \right) + f(x-y) - 2f(x) - 2f(y) \right) \right\| \\ & = |\rho| \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (2.1) in β -homogeneous complex Banach spaces.

Theorem 2.2. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ & \leq \left\| \rho \left(4f \left(\frac{x+y}{2} \right) + f(x-y) - 2f(x) - 2f(y) \right) \right\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (2.3)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \quad (2.4)$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta \|x\|^r \quad (2.5)$$

for all $x \in X$. So

$$\left\| f(x) - 4f \left(\frac{x}{2} \right) \right\| \leq \frac{2\theta}{2^{\beta_1 r}} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \end{aligned} \quad (2.6)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{k \rightarrow \infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} &\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\ &= \lim_{n \rightarrow \infty} \left\| 4^n \left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \left\| 4^n \rho \left(4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right) \right\| \\ &\quad + \lim_{n \rightarrow \infty} \frac{4^{\beta_2 n}}{2^{\beta_1 r n}} \theta (\|x\|^r + \|y\|^r) \\ &= \left\| \rho \left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y) \right) \right\| \end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned} &\left\| Q\left(\frac{x+y}{2}\right) + Q\left(\frac{x-y}{2}\right) - 2Q(x) - 2Q(y) \right\| \\ &\leq \left\| \rho \left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y) \right) \right\| \end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ &\leq \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| + \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \\ &\leq \frac{2\theta}{2^{\beta_1 r} - 4^{\beta_2}} \frac{4^{\beta_2 q}}{2^{\beta_1 q r}} \|x\|^r, \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q , as desired. $\square$

Theorem 2.3. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \quad (2.7)$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2\theta}{4^{\beta_2}} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \\ &\leq \frac{2\theta}{4^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 r}}{4^{\beta_2 j}} \|x\|^r \end{aligned} \quad (2.8)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Remark 2.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β -homogeneous real Banach space, then all the assertions in this section remain valid.

3. Quadratic ρ -functional inequality (2) in β -homogeneous complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the quadratic ρ -functional inequality (2) in β -homogeneous complex normed spaces.

Lemma 3.1. If a mapping $f : G \rightarrow Y$ satisfies

$$\begin{aligned} \left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right\| \\ \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \end{aligned} \quad (3.1)$$

for all $x, y \in G$, then $f : G \rightarrow Y$ is quadratic.

Proof. Assume that $f : G \rightarrow Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\|4f\left(\frac{x}{2}\right) - f(x)\| \leq 0$ and so

$$4f\left(\frac{x}{2}\right) = f(x) \quad (3.2)$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &= \left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right\| \\ &\leq |\rho| \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (3.1) in β -homogeneous complex Banach spaces.

Theorem 3.2. Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} & \left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right\| \\ & \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (3.3)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r} \theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \quad (3.4)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| = \left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta \|x\|^r \quad (3.5)$$

for all $x \in X$. So

$$\begin{aligned} \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \end{aligned} \quad (3.6)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{k \rightarrow \infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2 $\square$

Theorem 3.3. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r} \theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \quad (3.7)$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2^{\beta_1 r}}{4^{\beta_2}} \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \\ &\leq \sum_{j=l+1}^m \frac{2^{\beta_1 r j}}{4^{\beta_2 j}} \theta \|x\|^r \end{aligned} \quad (3.8)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Remark 3.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β -homogeneous real Banach space, then all the assertions in this section remain valid.

Acknowledgement

G. Lu was supported by Doctoral Science Foundation of Shengyang University of Technology (No.521101302) and the Project Sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry (No.2014-62009). Y. Jin was supported by National Natural Science Foundation of China (11361066) and the study of high-precision algorithm for high dimensional delay partial differential equations 2014-2017.

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