Viscosity approximation methods for the implicit midpoint rule of nonexpansive mappings in CAT(0) Spaces

Liang-cai Zhao[a], Shih-sen Chang[b], Lin Wang[c] , Gang Wang[c]

[a]College of Mathematics, Yibin University,
Yibin, Sichuan 644007, China

[b]Center for General Educatin, China Medical University,
Taichung, 40402, Taiwan

[c]College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
[1]

Abstract The purpose of this paper is to introduce the implicit midpoint rule of nonexpansive mappings in *CAT(0) spaces*. The strong convergence of this method is proved under certain assumptions imposed on the sequence of parameters. Moreover, it is shown that the limit of the sequence generated by the implicit midpoint rule solves an additional variational inequality. Applications to nonlinear Volterra integral equations and nonlinear variational inclusion problem are included. The results presented in the paper extend and improve some recent results announced in the current literature.

MSC: 47H09, 47J25.

Key Words: viscosity, implicit midpoint rule, nonexpansive mapping, projection, variational inequality, CAT(0) space.

1 Introduction

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, please refer to [1-8].

Based on the above fact, in 2015, Xu et al. [9] and Yao et al. [10] presented the following viscosity implicit midpoint rule for nonexpansive mappings in a Hilbert spaces:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad \forall n \geq 0, \tag{1.2}$$

where $\alpha_n \in (0,1)$ and f is a contraction. Under suitable conditions and by using a very complicated method, the authors proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T, which is also the unique solution of the following variational inequality

$$\langle (I-f)q, x-q\rangle \geq 0, \quad \forall x \in Fix(T). \tag{1.3}$$

On the other hand, the theory and applications of CAT(0) space have been studied extensively by many authors.

Recall that a metric space (X,d) is called a CAT(0) space, if it is geodetically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is known that any complete, simply connected Riemannian manifold

[1]Corresponding authors S. S. Chang (e-mail: changss2013@163.com)

having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces [11, 23], $R-$trees, Euclidean buildings[12],and many others. A complete CAT(0) space is often called a Hadamard space. A subset K of a CAT(0) space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$, where $[x, y]$ is the uniquely geodesic joining x and y. For a thorough discussion of CAT(0) spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger[11].

Motivated and inspired by the research going on in this direction, it is naturally to put forward the following

Open Question Can we establish the viscosity implicit midpoint rule for nonexpanssive mapping in CAT(0) and generalize the main results in [9, 10] to CAT(0) spaces?

The purpose of this paper is to give an affirmative answer to the above open question. In our paper we introduce and consider the following semi-implicit algorithm which is called the viscosity implicit midpoint rule in CAT(0):

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geq 0. \tag{1.4}$$

Under suitable conditions, some strong converge theorems to a fixed point of the nonexpansive mapping in CAT(0) space are proved. Moreover, it is shown that the limit of the sequence $\{x_n\}$ generated by (1.4) solves an additional variational inequality. As applications, we shall utilize the results presented in the paper to study the existence problems of solutions of nonlinear variational inclusion problem, and nonlinear Volterra integral equations. The results presented in the paper also extend and improve the main results in Xu [9], Yao et al. [10] and others.

2 Preliminaries

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y), \quad \text{and} \quad d(z, y) = (1-t)d(x, y).$$

The following lemmas play an important role in our paper.

Lemma 2.1 [13] Let X be a $CAT(0)$ space, $x, y, z \in X$ and $t \in [0, 1]$. Then

(i) $d(tx \oplus (1-t)y,\, z) \leq td(x, z) + (1-t)d(y, z)$.

(ii) $d^2(tx \oplus (1-t)y,\, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$.

Lemma 2.2 [14] Let X be a $CAT(0)$ space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then

$$d(\lambda p \oplus (1-\lambda)q,\, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p, r) + (1-\lambda)d(q, s).$$

Berg and Nikolaev [15] introduced the concept of quasilinearization as follows. Let us denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \qquad (a, b, c, d \in X).$$

It is easy to seen that $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{cd},\overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = -\langle\overrightarrow{ba},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ax},\overrightarrow{cd}\rangle + \langle\overrightarrow{xb},\overrightarrow{cd}\rangle = \langle\overrightarrow{ab},\overrightarrow{cd}\rangle$ for all $a,b,c,d \in X$. We say that X satisfies the Cauchy–Schwarz inequality if

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle \le d(a,b)d(c,d).$$

for all $a,b,c,d \in X$. It is well known [15] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwarz inequality.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. The metric projection $P_C : X \to C$ is defined by

$$u = P_C(x) \;\Leftrightarrow\; d(u,x) = \inf\{d(y,x) : y \in C\}, \quad \forall x \in X.$$

Lemma 2.3 [16] Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C(x)$ if and only if u is a solution of the following variational inequality

$$\langle\overrightarrow{yu},\overrightarrow{ux}\rangle \ge 0, \quad \forall y \in C.$$

i.e., u satisfies the following inequality equation:

$$d^2(x,y) - d^2(y,u) - d^2(u,x) \ge 0,\ \forall y \in C.$$

Lemma 2.4 [17] Every bounded sequence in a complete $CAT(0)$ space always has a $\Delta-$convergent subsequence.

Lemma 2.5 [18] Let X be a complete $CAT(0)$ space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ $\Delta-$converges to x if and only if $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}),\overrightarrow{xy}\rangle \le 0$ for all $y \in X$.

Lemma 2.6 [19] Let X be a complete $CAT(0)$ space. Then for all $u,x,y \in X$, the following inequality holds

$$d^2(x,u) \le d^2(y,u) + 2\langle\overrightarrow{xy},\overrightarrow{xu}\rangle.$$

Lemma 2.7 [20] Let X be a complete $CAT(0)$ space. For any $t \in [0,1]$ and $u,v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x,y \in X$,

(i) $\langle\overrightarrow{u_tx},\overrightarrow{u_ty}\rangle \le t\langle\overrightarrow{ux},\overrightarrow{u_ty}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{u_ty}\rangle$,

(ii) $\langle\overrightarrow{u_tx},\overrightarrow{uy}\rangle \le t\langle\overrightarrow{ux},\overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{uy}\rangle$ and $\langle\overrightarrow{u_tx},\overrightarrow{vy}\rangle \le t\langle\overrightarrow{ux},\overrightarrow{vy}\rangle + (1-t)\langle\overrightarrow{vx},\overrightarrow{vy}\rangle$.

Lemma 2.8 [21] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n,\ \forall n \ge 0, \tag{2.1}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(2) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0$ or $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

Theorem 3.1 *Let C be a closed convex subset of a complete $CAT(0)$ space X, and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let f be a contraction on C with coefficient $k \in [0,1)$, and for the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geq 0. \tag{3.1}$$

where $\{\alpha_n\} \in (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ converges strongly to $\tilde{x} = P_{Fix(T)} f(\tilde{x})$, which is a fixed point of T and it is also a solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad \forall\, x \in Fix(T).$$

i.e., $\tilde{x}$ satisfies the following inequality equation:

$$d^2(f(\tilde{x}), x) - d^2(\tilde{x}, x) - d^2(f(\tilde{x}), \tilde{x}) \geq 0, \ \forall x \in Fix(T).$$

Proof We divided the proof into four steps.

Step 1. We prove that $\{x_n\}$ is bounded. To see this we take $p \in Fix(T)$ to deduce that

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(\alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\
&\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\
&\leq \alpha_n (d(f(x_n), f(p)) + d(f(p), p)) + (1-\alpha_n) d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\
&\leq \alpha_n k d(x_n, p) + \alpha_n d(f(p), p) + (1-\alpha_n) d\left(\frac{x_n \oplus x_{n+1}}{2}, p\right) \\
&\leq \alpha_n k d(x_n, p) + \alpha_n d(f(p), p) + \frac{1-\alpha_n}{2}(d(x_n, p) + d(x_{n+1}, p)).
\end{aligned}$$

It then follows that

$$\frac{1+\alpha_n}{2} d(x_{n+1}, p) \leq \frac{1+(2k-1)\alpha_n}{2} d(x_n, p) + \alpha_n d(f(p), p),$$

and, moreover

$$\begin{aligned}
d(x_{n+1}, p) &\leq \frac{1+(2k-1)\alpha_n}{1+\alpha_n} d(x_n, p) + \frac{2\alpha_n}{1+\alpha_n} d(f(p), p) \\
&= \left(1 - \frac{2(1-k)\alpha_n}{1+\alpha_n}\right) d(x_n, p) + \frac{2(1-k)\alpha_n}{1+\alpha_n} \frac{1}{1-k} d(f(p), p) \\
&\leq \max\left\{ d(x_n, p), \frac{1}{1-k} d(f(p), p) \right\}.
\end{aligned}$$

By induction we readily obtain

$$d(x_n,p) \le \max\{d(x_0,p), \frac{1}{1-k}d(f(p),p)\}.$$

for all $n \ge 0$. Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{T(\frac{x_n \oplus x_{n+1}}{2})\}$.

Step 2. We show that $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$. Observe that

$$\begin{aligned}
d(x_{n+1},x_n) &= d\Big(\alpha_n f(x_n) \oplus (1-\alpha_n)T\Big(\frac{x_n \oplus x_{n+1}}{2}\Big), \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1})T\Big(\frac{x_{n-1} \oplus x_n}{2}\Big)\Big) \\
&\le d\Big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(\frac{x_n \oplus x_{n+1}}{2}), \alpha_n f(x_n) \oplus (1-\alpha_n)T(\frac{x_{n-1} \oplus x_n}{2})\Big) \\
&\quad + d\Big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(\frac{x_{n-1} \oplus x_n}{2}), \alpha_n f(x_{n-1}) \oplus (1-\alpha_n)T(\frac{x_{n-1} \oplus x_n}{2})\Big) \\
&\quad + d\Big(\alpha_n f(x_{n-1}) \oplus (1-\alpha_n)T(\frac{x_{n-1} \oplus x_n}{2}), \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1})T(\frac{x_{n-1} \oplus x_n}{2})\Big) \\
&\le (1-\alpha_n)d\Big(T(\frac{x_n \oplus x_{n+1}}{2}), T(\frac{x_{n-1} \oplus x_n}{2})\Big) + \alpha_n d(f(x_n), f(x_{n-1})) \\
&\quad + |\alpha_n - \alpha_{n-1}| d\Big(f(x_{n-1}), T(\frac{x_{n-1} \oplus x_n}{2})\Big) \\
&\le (1-\alpha_n)d\Big(\frac{x_n \oplus x_{n+1}}{2}, \frac{x_{n-1} \oplus x_n}{2}\Big) + \alpha_n k d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|M \\
&\le \frac{(1-\alpha_n)}{2}\big[d(x_{n+1},x_n) + d(x_n,x_{n-1})\big] + \alpha_n k d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}|M.
\end{aligned}$$

Here $M > 0$ is a constant such that

$$M \ge \sup\Big\{d\Big(f(x_{n-1}), T(\frac{x_{n-1} \oplus x_n}{2})\Big), \, n \ge 0\Big\}.$$

It turns out that

$$\frac{1+\alpha_n}{2} d(x_{n+1},x_n) \le \frac{1+(2k-1)\alpha_n}{2} d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}|.$$

Consequently, we arrive at

$$\begin{aligned}
d(x_{n+1},x_n) &\le \frac{1+2k\alpha_n - \alpha_n}{1+\alpha_n} d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}| \\
&= \frac{1+\alpha_n + 2k\alpha_n - 2\alpha_n}{1+\alpha_n} d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}| \\
&= \Big(1 - \frac{2(1-k)\alpha_n}{1+\alpha_n}\Big) d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}|.
\end{aligned} \tag{3.2}$$

Since $\{\alpha_n\} \in (0,1)$,then $1+\alpha_n < 2$, $\frac{1}{1+\alpha_n} > \frac{1}{2}$, $(1 - \frac{2(1-k)\alpha_n}{1+\alpha_n}) < (1-(1-k)\alpha_n)$. we have

$$d(x_{n+1},x_n) \le (1-(1-k)\alpha_n) d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}|. \tag{3.3}$$

By virtue of the conditions (ii) and (iii), we can apply Lemma 2.8 to (3.3) to obtain $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$.

Step 3. We show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. In fact, we have

$$\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d\Big(x_{n+1}, T(\frac{x_n \oplus x_{n+1}}{2})\Big) + d\Big(T(\frac{x_n \oplus x_{n+1}}{2}), Tx_n\Big)\\
&\le d(x_n, x_{n+1}) + \alpha_n d\Big(f(x_n), T(\frac{x_n \oplus x_{n+1}}{2})\Big) + d\Big(\frac{x_n \oplus x_{n+1}}{2}, x_n\Big)\\
&\le d(x_n, x_{n+1}) + \alpha_n d\Big(f(x_n), T(\frac{x_n \oplus x_{n+1}}{2})\Big) + \frac{1}{2} d(x_n, x_{n+1})\\
&\le \frac{3}{2} d(x_n, x_{n+1}) + \alpha_n M \to 0 \ \ (\text{as } n \to \infty).
\end{aligned}$$

Step 4. Now we prove

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \le 0.$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\Delta-$converges to $\tilde{x}$ and

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle = \limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_j}\tilde{x}} \rangle. \tag{3.4}$$

Since $\{x_{n_j}\}$ $\Delta-$converges to $\tilde{x}$, by Lemma 2.5, we have

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_j}\tilde{x}} \rangle \le 0.$$

This together with (3.4) shows that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle = \limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_j}\tilde{x}} \rangle \le 0.$$

Step 5. Finally, we prove that $x_n \to \tilde{x} \in Fix(T)$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set $z_n = \alpha_n \tilde{x} \oplus (1-\alpha_n) T(\frac{x_n \oplus x_{n+1}}{2})$. It follows from Lemma 2.6 and Lemma 2.7 that

$$\begin{aligned}
d^2(x_{n+1}, \tilde{x}) &\le d^2(z_n, \tilde{x}) + 2\langle \overrightarrow{x_{n+1}z_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\\
&\le (1-\alpha_n)^2 d^2\Big(T(\frac{x_n \oplus x_{n+1}}{2}), \tilde{x}\Big) + 2\Big[\alpha_n \langle \overrightarrow{f(x_n)z_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\\
&\quad + (1-\alpha_n)\Big\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\, z_n}, \overrightarrow{x_{n+1}\tilde{x}} \Big\rangle\Big]\\
&\le (1-\alpha_n)^2 d^2\Big(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\Big) + 2\Big[\alpha_n \alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\\
&\quad + \alpha_n(1-\alpha_n)\Big\langle \overrightarrow{f(x_n) T(\frac{x_n \oplus x_{n+1}}{2})}, \overrightarrow{x_{n+1}\tilde{x}} \Big\rangle\\
&\quad + \alpha_n(1-\alpha_n)\Big\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\, \tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \Big\rangle\\
&\quad + (1-\alpha_n)^2\Big\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\, T(\frac{x_n \oplus x_{n+1}}{2})}, \overrightarrow{x_{n+1}\tilde{x}} \Big\rangle\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2\Big[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{f(x_n)T(\frac{x_n \oplus x_{n+1}}{2})}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle \\
&\quad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle\Big] \\
&\leq (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq (1-\alpha_n)^2 \left[\frac{1}{2} d^2(x_n, \tilde{x}) + \frac{1}{2} d^2(x_{n+1}, \tilde{x}) - \frac{1}{4} d^2(x_n, x_{n+1})\right] \\
&\quad + 2\alpha_n \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq \frac{(1-\alpha_n)^2}{2}[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] \\
&\quad + 2\alpha_n k d(x_n, \tilde{x}) d(x_{n+1}, \tilde{x}) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq \frac{(1-\alpha_n)^2}{2}[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] \\
&\quad + \alpha_n k[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq \left(\frac{(1-\alpha_n)^2}{2} + \alpha_n k\right)[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq \frac{1-2(1-k)\alpha_n}{2}[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + \alpha_n^2 M_1 + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.
\end{aligned}
$$

Here $M_1 > 0$ is a constant such that

$$M_1 \geq \sup\{d^2(x_n, \tilde{x}),\, n \geq 0\}.$$

It follows that

$$
\begin{aligned}
d^2(x_{n+1}, \tilde{x}) &\leq \frac{1-2(1-k)\alpha_n}{1+2(1-k)\alpha_n} d^2(x_n, \tilde{x}) + \frac{2\alpha_n^2}{1+2(1-k)\alpha_n} M_1 \\
&\quad + \frac{4\alpha_n}{1+2(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq \left(1 - \frac{2(1-k)\alpha_n}{1+(1-k)\alpha_n}\right) d^2(x_n, \tilde{x}) + \frac{2\alpha_n^2}{1+(1-k)\alpha_n} M_1 \\
&\quad + \frac{4\alpha_n}{1+(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.
\end{aligned}
$$

Since $1+(1-k)\alpha_n < 2-k$, $\frac{1}{1+(1-k)\alpha_n} > \frac{1}{2-k}$. we have

$$\begin{aligned} d^2(x_{n+1},\tilde{x}) &\le \left(1-\frac{2(1-k)\alpha_n}{2-k}\right)d^2(x_n,\tilde{x}) + \frac{2\alpha_n^2}{1+(1-k)\alpha_n}M_1 \\ &\quad + \frac{4\alpha_n}{1+(1-k)\alpha_n}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\le \left(1-\frac{2(1-k)\alpha_n}{2-k}\right)d^2(x_n,\tilde{x}) + 2\alpha_n^2 M_1 + \frac{4\alpha_n}{1+(1-k)\alpha_n}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle. \end{aligned}$$

Take $\gamma_n = \frac{2(1-k)\alpha_n}{2-k}$, $\delta_n = 2\alpha_n^2 M_1 + \frac{4\alpha_n}{1+(1-k)\alpha_n}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle$. It follows from conditions (i), (ii) and (3.4) that $\{\gamma_n\} \subset (0,1)$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty}\frac{2-k}{1-k}\left(\alpha_n M_1 + \frac{2}{1+(1-k)\alpha_n}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\right) \le 0.$$

From Lemma 2.8 we have that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof.

Remark 3.2 Since every Hilbert space is a complete CAT(0) space, Theorem 3.1 is an improvement and generalization of the main results in Xu et al. [9] and Yao et al. [10].

The following result can be obtained from Theorem 3.1 immediately.

Theorem 3.3 *Let C be a closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let f be a contraction on C with coefficient $k \in [0,1)$, and for the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \ge 0. \tag{3.5}$$

where $\{\alpha_n\} \in (0,1)$ satisfies the conditions: (i), (ii) and (iii) in Theorem 3.1. Then the sequence $\{x_n\}$ defined by (3.5) converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{Fix(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \tilde{x} - f(\tilde{x}),\ x - \tilde{x}\rangle \ge 0, \quad \forall\, x \in Fix(T).$$

4 Applications

4.1 Application to nonlinear variation inclusion problem

Let H be a real Hilbert space, $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then, the *resolvent mapping* $J_\lambda^M : H \to H$ *associated with* M, is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x),\ \forall x \in H, \tag{4.1}$$

for some $\lambda > 0$, where I stands identity operator on H.

We note that for all $\lambda > 0$ the resolvent operator J_λ^M is a single-valued nonexpansive mapping.

The "so-called" *monotone variational inclusion problem (in short, MVIP)* is to find $x^* \in H$ such that

$$0 \in B_1(x^*). \tag{4.2}$$

From the definition of resolvent mapping J_λ^M, it is easy to know that (MVIP) (4.2) is equivalent to find $x^* \in H$ such that

$$x^* \in Fix(J_\lambda^M) \;\; for\ some\ \lambda > 0. \tag{4.3}$$

For any given function $x_0 \in H$, define a sequence by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_\lambda^M \left(\frac{x_n \oplus x_{n+1}}{2} \right), \quad n \geq 0. \tag{4.4}$$

From Theorem 3.3 we have the following

Theorem 4.1 Let M, J_λ^M be the same as above. Let $f : H \to H$ be a contraction. Let $\{x_n\}$ be the sequence defined by (4.4). If the sequence $\{\alpha_n\} \in (0,1)$ satisfies the conditions: (i), (ii) and (iii) in Theorem 3.1 and $Fix(J_\lambda^M) \neq \emptyset$, then $\{x_n\}$ converges strongly to the solution of monotone variational inclusion (4.2), which is also a solution of the following variational inequality:

$$\langle \tilde{x} - f(\tilde{x}), x - \tilde{x} \rangle \geq 0, \quad \forall\, x \in Fix(J_\lambda^M).$$

4.2 Application to nonlinear Volterra integral equations

Let us consider the following nonlinear Volterra integral equation

$$x(t) = g(t) + \int_0^t F(t,s,x(s))ds, \quad t \in [0,1], \tag{4.5}$$

where g is a continuous function on $[0,1]$ and $F : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition.

$$|F(t,s,x) - F(t,s,y)| \leq |x-y|, \quad t,s \in [0,1] \;\; x,y \in \mathbb{R},$$

then equation (4.5) has at least one solution in $L^2[0,1]$ (see, for example, [22]).

Define a mapping $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tx)(t) = g(t) + \int_0^t F(t,s,x(s))ds, \quad t \in [0,1]. \tag{4.6}$$

It is easy to see that T is a nonexpansive mapping. This means that to find the solution of integral equation (4.5) is reduced to find a fixed point of the nonexpansive mapping T in $L^2[0,1]$.

For any given function $x_0 \in L^2[0,1]$, define a sequence of functions $\{x_n\}$ in $L^2[0,1]$ by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T \left(\frac{x_n \oplus x_{n+1}}{2} \right), \quad n \geq 0. \tag{4.7}$$

From Theorem 3.3 we have the following

Theorem 4.2 Let $F,\ g,\ T, L^2[0,1]$ be the same as above. Let f be a contraction on $L^2[0,1]$ with coefficient $k \in [0,1)$. Let $\{x_n\}$ be the sequence defined by (4.7). If the sequence $\{\alpha_n\} \in (0,1)$ satisfies the conditions: (i), (ii) and (iii) in Theorem 3.1. Then $\{x_n\}$ converges strongly in $L^2[0,1]$ to the solution of integral equation (4.5) which is also a solution of the following variational inequality:

$$\langle \tilde{x} - f(\tilde{x}), x - \tilde{x} \rangle \geq 0, \quad \forall\ x \in Fix(T).$$

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to express their thanks to the Referee and the Editors for their helpful comments and advices.

This work was supported by Scientific Research Fund of SiChuan Provincial Education Department (No.14ZA0272), This work was also supported by the National Natural Science Foundation of China (Grant No.11361070), and the Natural Science Foundation of China Medical University, Taiwan.

References

[1] Moudafi. A, Viscosity approximation methods for fixed points problems, J.Math. Anal. Appl, 241, 46–55(2000).
[2] Attouch. H, Viscosity approximation methods for minimization problems, SIAM J. Optim, 6(3), 769–806(1996).
[3] Xu. HK, Viscosity approximation methods for nonexpansive mappings, J.Math. Anal. Appl, 298, 279–291(2004).
[4] Auzinger. W, Frank. R, Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case, Numer. Math, 56, 469–499(1989).
[5] Bader, G, Deuflhard, P: A semi-implicit mid-point rule for stiff systems of ordinary differential equations. Numer. Math. 41, 373–398 (1983)
[6] Schneider, C: Analysis of the linearly implicit mid-point rule for differential-algebra equations. Electron. Trans. Numer. Anal. 1, 1–10 (1993)
[7] Somalia, S: Implicit midpoint rule to the nonlinear degenerate boundary value problems. Int. J. Comput. Math. 79(3), 327–332 (2002)
[8] Van Veldhuxzen, M: Asymptotic expansions of the global error for the implicit midpoint rule (stiff case). Computing 33, 185–192 (1984)
[9] Xu, HK, Alghamdi, MA, Shahzad, N: The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces. Fixed Point Theory Appl. 2015(41), doi:10.1186/s13663–015–0282–9.
[10] Yao, YH, Shahzad, N, Liou, YC, Modified semi-implicit midpoint rule for nonexpansive mappings, Fixed Point Theory and Applications (2015) 2015:166.
[11] Bridson, MR, Haefliger, A: Metric Spaces of Non-positive Curvature. Grundlehren der Mathematischen Wissenschaften, vol. 319. Springer, Berlin (1999).
[12] Brown, KS (ed.): Buildings. Springer, New York (1989).

[13] Dhompongsa, S, Panyanak, B: On △-convergence theorems in CAT(0) spaces. Comput. Math. Appl. 56, 2572–2579 (2008).
[14] Banach, S: Metric Spaces of Nonpositive Curvature. Springer, New York (1999).
[15] Berg, ID, Nikolaev, IG: Quasilinearization and curvature of Alexandrov spaces. Geom. Dedic. 133, 195–218 (2008).
[16] Dehghan, H, Rooin, J: A characterization of metric projection in CAT(0) spaces. In: International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012), 10-12th May 2012, pp. 41-43. Payame Noor University, Tabriz (2012).
[17] Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. 68, 3689–3696 (2008).
[18] Kakavandi, BA: Weak topologies in complete CAT(0) metric spaces. Proc. Am. Math. Soc. 141, 1029–1039 (2013).
[19] Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. 8, 33–45 (2007).
[20] Wangkeeree, R, Preechasilp, P: Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces. J. Inequal. Appl. 2013, Article ID 93 (2013).
[21] Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. 66, 240–256 (2002).
[22] Zhang, SS, "Integral Equations", Chongqing press, Chongqing, 1984.
[23] Saejung, S, Halperns Iteration in CAT(0) Spaces, Fixed Point Theory and Applications Volume 2010, Article ID 471781, 13 pages, doi:10.1155/2010/471781.