Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. ,
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some identities of degenerate Daehee numbers arising from certain differential equations

Dae San Kim[a], Taekyun Kim[b]

[a]Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
[b]Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Abstract

In this paper. we introduce the degenerate Daehee numbers and study a family ol differential equations associaled with the generating function of these numbers. From those differential equations, we will be able to obtain some new and interesting combinatorial identities involving the degenerate Daehee numbers and generalized harmonic numbers.

Keywords: degenerate Daehee numbers, differential equation, generalized harmonic numbers
2010 MSC: 05A19, 11B83, 34A34.

1. Introduction

The Daehee polynomials $D_n^{(r)}(x)$ of order r are given by the generating function

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^r = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.1}$$

For $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers of order r. In particular, if $r = 1$, then $D_n(x) = D_n^{(1)}(x)$ *and* $D_n = D_n^{(1)}$ are respectively called Daehee polynomials and Daehee numbers.

As a degenerate version of Daehee numbers D_n, we introduce what we call the degenerate Daehee numbers $D_{n,\lambda}$ defined by

$$\frac{\lambda\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}{\log(1+\lambda t)} = \sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}. \tag{1.2}$$

Email addresses: dskim@sogang.ac.kr (Dae San Kim), tkkim@kw.ac.kr (Taekyun Kim)

Received

We observe here that $D_{n,\lambda} \to D_n$ as $\lambda \to 0$. Also it is easy to see that

$$D_{n,\lambda} = \sum_{l=0}^{n} S_l(n,l)\lambda^{n-l} D_l = \sum_{l=0}^{n} \frac{(-1)^l l!}{l+1} S_1(n,l)\lambda^{n-l}. \tag{1.3}$$

Here $S_1(n,l)$ is the Stirling number of the first kind.

Many mathematicians have studied the arithmetic and combinatorial properties of degenerate versions of special numbers and polynomials, some of which are the degenerate Bernoulli polynomials (also called Korobov polynomials of the second kind), the degenerate Bernouili polynomials of the second kind (also called Korobov polynomials of the first kind), the degenerate Euler polynomials, the degenerate poly-Bernoulli polynomials. the degenerate poly-Bernoulli polynomials of the second, the degenerate falling factorial polynomials, and the degenerate Changhee polynomials (see [2, 1, 4, 8, 13, 15, 16, 23])

On the other hand, in [9, 10], Kim and Kim, and Kim developed some new methods for obtaining identities related to Bernoulli numbers of the second kind and Frobenius-Euler polynomials of higher order arising from certain non-linear differential equations. This idea of obtaining some interesting combinatorial identities by using differential equations satisfied by the generating function of special numbers or special polynomials turned out to be very fruitful (see [9, 10, 12, 14]).

The generalized harmonic numbers are defined as follows:

$$H_{N,0} = 1, \quad \text{for all } N, \tag{1.4}$$

$$H_{N,1} = H_N = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \tag{1.5}$$

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j}, \quad (2 \le j \le N). \tag{1.6}$$

These special numbers have appeared previously in the paper [9].

The purpose of this paper is to introduce the degenerate Daehee numbers and study a family of differential equations associated with the generating function of these numbers. From those differential equations, we will be able to obtain some new and interesting combinatorial identities involving the degenerate Daehee numbers and generalized harmonic numbers.

2. Differential equations arising from the generating function of degenerate Daehee numbers

Let

$$F(t) = F = \log\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right). \tag{2.1}$$

Then, by taking the derivative with respect to t of (2.1), we get

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t) \\ &= \left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^{-1} \frac{1}{1+\lambda t} \\ &= \frac{1}{1+\lambda t} e^{-\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \\ &= \frac{1}{1+\lambda} e^{-F}. \end{aligned} \tag{2.2}$$

From (2.2), we note that

$$\begin{aligned} F^{(2)} &= \frac{d}{dt}F^{(1)} \\ &= \frac{(-1)\lambda}{(1+\lambda t)^2}e^{-F} + \frac{1}{1+\lambda t}\left(-F^{(1)}\right)e^{-F} \\ &= \frac{(-1)\lambda}{(1+\lambda t)^2}e^{-F} + \frac{(-1)}{(1+\lambda t)^2}e^{-2F}. \end{aligned} \tag{2.3}$$

Further, by taking the derivative with respect to t of (2.3), we obtain

$$\begin{aligned} F^{(3)} &= \frac{d}{dt}F^{(2)} \\ &= \frac{(-1)^2 2\lambda^2}{(1+\lambda t)^3}e^{-F} + \frac{(-1)\lambda}{(1+\lambda t)^2}\left(-F^{(1)}\right)e^{-F} \\ &\quad + \frac{(-1)^2 2\lambda}{(1+\lambda t)^3}e^{-2F} + \frac{(-1)}{(1+\lambda t)^2}\left(-2F^{(1)}\right)e^{-2F} \\ &= \frac{(-1)^2}{(1+\lambda t)^3}\left(2\lambda^2 e^{-F} + 3\lambda e^{-2F} + 2e^{-3F}\right). \end{aligned} \tag{2.4}$$

Continuing this process, we are led to put

$$F^{(N)} = \frac{(-1)^{N-1}}{(1+\lambda t)^N}\sum_{k=1}^{N} a_k\left(N \mid \lambda\right)e^{-kF}, \tag{2.5}$$

for $N = 1, 2, 3, \ldots$.

On the one hand, from (2.5), we have

$$\begin{aligned} F^{(N+1)} &= \frac{d}{dt}F^{(N)} \\ &= \frac{(-1)^N N\lambda}{(1+\lambda t)^{N+1}}\sum_{k=1}^{N} a_k\left(N \mid \lambda\right)e^{-kF} \\ &\quad + \frac{(-1)^{N-1}}{(1+\lambda t)^N}\sum_{k=1}^{N} a_k\left(N \mid \lambda\right)\left(-kF^{(1)}\right)e^{-kF} \\ &= \frac{(-1)^N N\lambda}{(1+\lambda t)^{N+1}}\sum_{k=1}^{N} a_k\left(N \mid \lambda\right)e^{-kF} \\ &\quad + \frac{(-1)^N}{(1+\lambda t)^{N+1}}\sum_{k=1}^{N} a_k\left(N \mid \lambda\right)ke^{-(k+1)F} \\ &= \frac{(-1)^N}{(1+\lambda t)^{N+1}}\sum_{k=1}^{N} \lambda N a_k\left(N \mid \lambda\right)e^{-kF} \\ &\quad + \frac{(-1)^N}{(1+\lambda t)^{N+1}}\sum_{k=2}^{N+1} (k-1)\, a_{k-1}\left(N \mid \lambda\right)e^{-kF} \\ &= \frac{(-1)^N}{(1+\lambda t)^{N+1}}\left\{\lambda N a_1\left(N \mid \lambda\right)e^{-F} + \sum_{k=2}^{N}\left(\lambda N a_k\left(N \mid \lambda\right) + (k-1)\, a_{k-1}\left(N \mid \lambda\right)\right)e^{-kF}\right. \\ &\quad \left. + N a_N\left(N \mid \lambda\right)e^{-(N+1)F}\right\}. \end{aligned} \tag{2.6}$$

On the other hand, by replacing N by $N+1$ in (2.5), we get

$$F^{(N+1)} = \frac{(-1)^N}{(1+\lambda t)^{N+1}} \sum_{k=1}^{N+1} a_k (N+1 \mid \lambda) e^{-kF}. \tag{2.7}$$

Now, by comparing (2.6) and (2.7), we have

$$a_1 (N+1 \mid \lambda) = \lambda N a_1 (N \mid \lambda), \tag{2.8}$$
$$a_{N+1} (N+1 \mid \lambda) = N a_N (N \mid \lambda), \tag{2.9}$$
$$a_k (N+1 \mid \lambda) = \lambda N a_k (N \mid \lambda) + (k-1) a_{k-1} (N \mid \lambda), \tag{2.10}$$

$(2 \leq k \leq N)$.

From (2.2) and (2.5), we note

$$F^{(1)} = \frac{1}{1+\lambda t} e^{-F} = a_1 (1 \mid \lambda) \frac{1}{1+\lambda t} e^{-F}. \tag{2.11}$$

Thus, by (2.11), we obtain

$$a_1 (1 \mid \lambda) = 1. \tag{2.12}$$

In addition, from (2.3) and (2.5), we observe

$$\begin{aligned} F^{(2)} &= \frac{(-1)\lambda}{(1+\lambda t)^2} e^{-F} + \frac{(-1)}{(1+\lambda t)^2} e^{-2F} \\ &= \frac{(-1)}{(1+\lambda t)^2} \left(a_1 (2 \mid \lambda) e^{-F} + a_2 (2 \mid \lambda) e^{-2F} \right). \end{aligned} \tag{2.13}$$

Hence, from (2.13), we have

$$a_1 (2 \mid \lambda) = \lambda, \quad a_2 (2 \mid \lambda) = 1. \tag{2.14}$$

Now, we are ready to determine $a_k (N+1 \mid \lambda)$'s appearing in (2.8), (2.9) and (2.10). From (2.8), we get

$$\begin{aligned} a_1 (N+1 \mid \lambda) &= \lambda N a_1 (N \mid \lambda) \\ &= \lambda N \lambda (N-1) a_1 (N-1 \mid \lambda) \\ &\vdots \\ &= (\lambda N) \lambda (N-1) \cdots \lambda 2 a_1 (2 \mid \lambda) \\ &= \lambda^N N!. \end{aligned} \tag{2.15}$$

By (2.9), we have

$$\begin{aligned} a_{N+1} (N+1 \mid \lambda) &= N a_N (N \mid \lambda) \\ &= N (N-1) a_{N-1} (N-1 \mid \lambda) \\ &\vdots \\ &= N (N-1) \cdots 2 a_2 (2 \mid \lambda) \\ &= N!. \end{aligned} \tag{2.16}$$

We remark here that the $N \times N$ matrix with the (i,j) entry given by $a_i (j \mid \lambda) (1 \leq i, j \leq n)$ is given by

$$\begin{bmatrix} 1 & \lambda & \lambda^2 2! & \cdots & \lambda^{N-1} (N-1)! \\ 0 & 2! & & & \\ 0 & 0 & 3! & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & (N-1)! \end{bmatrix}.$$

We now turn our attention to $a_k(N+1 \mid \lambda)$, for $2 \le k \le N$. For $k=2$ in (2.10), we have

$$\begin{aligned}
&a_2(N+1 \mid \lambda) && (2.17)\\
&= \lambda N a_2(N \mid \lambda) + a_1(N \mid \lambda)\\
&= \lambda N a_2(N \mid \lambda) + \lambda^{N-1}(N-1)!\\
&= \lambda N\left(\lambda(N-1)a_2(N-1 \mid \lambda) + \lambda^{N-2}(N-2)!\right) + \lambda^{N-1}(N-1)! && (2.18)\\
&= \lambda^2 N(N-1)a_2(N-1 \mid \lambda) + \lambda^{N-1}N!\left(\frac{1}{N-1}+\frac{1}{N}\right)\\
&= \lambda^2 N(N-1)\left(\lambda(N-2)a_2(N-2 \mid \lambda) + \lambda^{N-3}(N-3)!\right) + \lambda^{N-1}N!\left(\frac{1}{N-1}+\frac{1}{N}\right)\\
&= \lambda^3 N(N-1)(N-2)a_2(N-2 \mid \lambda) + \lambda^{N-1}N!\left(\frac{1}{N-2}+\frac{1}{N-1}+\frac{1}{N}\right)\\
&\vdots\\
&= \lambda^{N-1}N(N-1)\cdots 2a_2(2 \mid \lambda) + \lambda^{N-1}N!\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right)\\
&= \lambda^{N-1}N!H_{N,1}.
\end{aligned}$$

Here and in below $H_{N,j}(0 \le j \le N)$ are as defined in (1.4), (1.5) and (1.6).
For $k=3$ in (2.10), we obtain

$$\begin{aligned}
&a_3(N+1 \mid \lambda) && (2.19)\\
&= \lambda N a_3(N \mid \lambda) + 2a_2(N \mid \lambda)\\
&= \lambda N a_3(N \mid \lambda) + 2!\lambda^{N-2}(N-1)!H_{N-1,1}\\
&= \lambda N\left(\lambda(N-1)a_3(N-1 \mid \lambda) + 2!\lambda^{N-3}(N-2)!H_{N-2,1}\right)\\
&\quad +2!\lambda^{N-2}(N-1)!H_{N-1,1}\\
&= \lambda^2 N(N-1)a_3(N-1 \mid \lambda) + 2!\lambda^{N-2}N!\left(\frac{H_{N-2,1}}{N-1}+\frac{H_{N-1,1}}{N}\right)\\
&= \lambda^2 N(N-1)\left(\lambda(N-2)a_3(N-2 \mid \lambda) + 2!\lambda^{N-4}(N-3)!H_{N-3,1}\right)\\
&\quad +2!\lambda^{N-2}N!\left(\frac{H_{N-2,1}}{N-1}+\frac{H_{N-1,1}}{N}\right)\\
&= \lambda^3 N(N-1)(N-2)a_3(N-2 \mid \lambda) + 2!\lambda^{N-2}N!\left(\frac{H_{N-3,1}}{N-2}+\frac{H_{N-2,1}}{N-1}+\frac{H_{N-1,1}}{N}\right)\\
&\vdots\\
&= \lambda^{N-2}N(N-1)\cdots 3a_3(3 \mid \lambda) + 2!\lambda^{N-2}N!\left(\frac{H_{2,1}}{3}+\frac{H_{3,1}}{4}+\cdots+\frac{H_{N-1,1}}{N}\right)\\
&= 2!\lambda^{N-2}N!\left(\frac{H_{1,1}}{2}+\frac{H_{2,1}}{3}+\cdots+\frac{H_{N-1,1}}{N}\right)\\
&= 2!\lambda^{N-2}N!H_{N,2}.
\end{aligned}$$

Proceeding similarly to $k=2$ and $k=3$ cases, we can show that

$$a_4(N+1 \mid \lambda) = 3!\lambda^{N-3}N!H_{N,3}. \tag{2.20}$$

Continuing in this fashion, we can find that

$$a_k(N+1 \mid \lambda) = (k-1)!\lambda^{N-k+1}N!H_{N,k-1}, \quad (2 \le k \le N). \tag{2.21}$$

Here we observe that (2.21) holds also for $k=1$ and $k=N+1$ (cf. (2.15), (2.16)).

Thus, from (2.21), we obtain the following theorem.

Theorem 2.1. *For $N=1,2,3,\dots,$ let us consider the following family of differential equations:*

$$F^{(N)}=\frac{(-1)^{N-1}}{(1+\lambda t)^{N}}\sum_{k=1}^{N}(k-1)!\lambda^{N-k}(N-1)!H_{N-1,k-1}e^{-kF}, \tag{2.22}$$

where

$$\begin{aligned}
&H_{N,0}=1, \quad \textit{for all } N,\\
&H_{N,1}=H_N=\frac{1}{N}+\frac{1}{N-1}+\cdots+\frac{1}{1},\\
&H_{N,j}=\frac{H_{N-1,j-1}}{N}+\frac{H_{N-2,j-1}}{N-1}+\cdots+\frac{H_{j-1,j-1}}{j}, \quad (2\le j\le N).
\end{aligned}$$

Then the above family of differential equations in (2.22) *have a solution*

$$F=F(t)=\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right).$$

3. Applications of differential equations

Here we will use Theorem 2.1 in order to derive some new and interesting identity involving the degenerate Daehee numbers and generalized harmonic numbers.

From (2.1), we get

$$\begin{aligned}
F(t)&=\frac{\lambda\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}{\log(1+\lambda t)}\frac{1}{\lambda}\log(1+\lambda t)\\
&=\left(\sum_{l=0}^{\infty}D_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\lambda^{m-1}t^m\right)\\
&=\sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n-l-1}}{n-l}\right)t^n.
\end{aligned} \tag{3.1}$$

On the one hand, from (3.1) we obtain

$$\begin{aligned}
&F^{(N)}\\
&=\left(\frac{d}{dt}\right)^{N}F(t)\\
&=\sum_{n=N}^{\infty}(n)_N\left(\sum_{l=0}^{n-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n-l-1}}{n-l}\right)t^{n-N}\\
&=\sum_{n=0}^{\infty}(n+N)_N\left(\sum_{l=0}^{n+N-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n+N-l-1}}{n+N-l}\right)t^{n}\\
&=\sum_{n=0}^{\infty}(n+N)!\left(\sum_{l=0}^{n+N-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n+N-l-1}}{n+N-l}\right)\frac{t^n}{n!},
\end{aligned} \tag{3.2}$$

where $(x)_N=x(x-1)\cdots(x-N+1)$, for $N\ge 1$, and $(x)_0=1$.

Now, we observe that

$$
\begin{aligned}
& e^{-kF} && (3.3)\\
&= \sum_{m_1=0}^{\infty} \frac{(-k)^{m_1}}{m_1!} F^{m_1} \\
&= \sum_{m_1=0}^{\infty} (-k)^{m_1} \frac{1}{m_1!} \left(\log \left(1 + \frac{1}{\lambda} \log (1+\lambda t) \right) \right)^{m_1} && (3.4)\\
&= \sum_{m_1=0}^{\infty} (-k)^{m_1} \sum_{m_2=m_1}^{\infty} S_1(m_2, m_1) \left(\frac{1}{\lambda} \right)^{m_2} \frac{(\log(1+\lambda t))^{m_2}}{m_2!} \\
&= \sum_{m_2=0}^{\infty} \left(\sum_{m_1=0}^{m_2} (-k)^{m_1} S_1(m_2, m_1) \lambda^{-m_2} \right) \\
&\quad \times \sum_{m_3=m_2}^{\infty} S_1(m_3, m_2) \lambda^{m_3} \frac{t^{m_3}}{m_3!} \\
&= \sum_{m_3=0}^{\infty} \left(\sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-k)^{m_1} S_1(m_2, m_1) S_1(m_3, m_2) \lambda^{m_3-m_2} \right) \frac{t^{m_3}}{m_3!}.
\end{aligned}
$$

In turn, (3.3) gives us

$$
\begin{aligned}
& \frac{1}{(1+\lambda t)^N} e^{-kF} && (3.5)\\
&= \left(\sum_{l=0}^{\infty} \binom{N+l-1}{l} (-1)^l \lambda^l t^l \right) \\
&\quad \times \sum_{m_3=0}^{\infty} \left(\sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-k)^{m_1} S_1(m_2, m_1) S_1(m_3, m_2) \lambda^{m_3-m_2} \right) \frac{t^{m_3}}{m_3!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-1)^{n+m_1+m_3} \binom{n}{m_3} (N+n-m_3-1)_{n-m_3} \right. \\
&\quad \left. \times k^{m_1} \lambda^{n-m_2} S_1(m_2, m_1) S_1(m_3, m_2) \right) \frac{t^n}{n!}.
\end{aligned}
$$

On the other hand, from (1.2) and (3.5) we have

$$
\begin{aligned}
F^{(N)} = (-1)^{N-1} (N-1)! \sum_{n=0}^{\infty} & \left(\sum_{k=1}^{N} \sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \right. && (3.6)\\
& \times (-1)^{n+m_1+m_3} \binom{n}{m_3} (N+n-m_3-1)_{n-m_3} (k-1)! k^{m_1} \\
& \left. \times \lambda^{N+n-k-m_2} S_1(m_2, m_1) S_1(m_3, m_2) H_{n-1,k-1} \right) \frac{t^n}{n!}.
\end{aligned}
$$

By equating (3.2) and (3.6), we finally get the following theorem.

Theorem 3.1. *For $N = 1, 2, 3, \ldots,$ and $n = 0, 1, 2, \ldots,$ we have*

$$\begin{aligned}
&(-1)^{N+n+1} \frac{(n+N)!}{(N-1)!} \sum_{l=0}^{N+n-1} \frac{D_{l,\lambda}}{l!} \frac{(-\lambda)^{n+N-l-1}}{n+N-l} \\
&= \sum_{k=1}^{N} \sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-1)^{m_1+m_3} \binom{n}{m_3} (N+n-m_3-1)_{n-m_3} \\
&\quad \times (k-1)! k^{m_1} \lambda^{N+n-k-m_2} S_1(m_2, m_1) S_1(m_3, m_2) H_{N-1,k-1},
\end{aligned}$$

where $H_{N,j}$'s are as in (1.4), (1.5) *and* (1.6).

References

[1] L. Carlitz, *A degenerate Staudt-Clausen theorem*, Arch. Math. (Basel) **7** (1956), 28–33. MR 0074436 1

[2] ———, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51–88. MR 531621 1

[3] Y.-K. Cho, T. Kim, T. Mansour, and S.-H. Rim, *On a (r, s)-analogue of Changhee and Daehee numbers and polynomials*, Kyungpook Math. J. **55** (2015), no. 2, 225–232. MR 3367939

[4] D. V. Dolgy, D. S. Kim, T. Kim, and T. Mansour, *Degenerate poly-Bernoulli polynomials of the second kind*, J. Comput. Anal. Appl. **21** (2016), no. 5, 954–966. MR 3469957 1

[5] B. S. El-Desouky and A. Mustafa, *New results on higher-order Daehee and Bernoulli numbers and polynomials*, Adv. Difference Equ. (2016), 2016:32, 21. MR 3455249

[6] D. S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. (Ruse) **7** (2013), no. 117-120, 5969–5976. MR 3141903

[7] ———, *Identities arising from higher-order Daehee polynomial bases*, Open Math. **13** (2015), 196–208. MR 3314175

[8] ———, *A note on degenerate poly-Bernoulli numbers and polynomials*, Adv. Difference Equ. (2015), 2015:258, 8. MR 3385686 1

[9] ———, *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation*, Bull. Korean Math. Soc. **52** (2015), no. 6, 2001–2010. MR 3432732 1, 1

[10] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory **132** (2012), no. 12, 2854–2865. MR 2965196 1

[11] T. Kim and D. S. Kim, *A note on nonlinear Changhee differential equations*, Russ. J. Math. Phys. **23** (2016), no. 1, 88–92. MR 3482767

[12] T. Kim, D. S. Kim, K.-W. Hwang, and J. J. Seo, *Some identities of Laguerre polynomials arising from differential equations*, Adv. Difference Equ. (2016), 2016:159, 9. MR 3513919 1

[13] T. Kim, D. S. Kim, H. I. Kwon, D. V. Dolgy, and J.-J. Seo, *Degenerate falling factorial polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **26** (2016), no. 3, 481–499. 1

[14] T. Kim, D. S. Kim, T. Mansour, and J.-J. Seo, *Linear differential equations for families of polynomials*, J. Inequal. Appl. (2016), 2016:95, 8. MR 3476742 1

[15] N. M. Korobov, *On some properties of special polynomials*, Proceedings of the IV International Conference "Modern Problems of Number Theory and its Applications" (Russian) (Tula, 2001), vol. 1, 2001, pp. 40–49. MR 2035591 1

[16] H. I. Kwon, T. Kim, and J. J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc. **18** (2015), no. 3, 295–305. MR 3410551 1

[17] J. G. Lee, L.-C. Jang, J.-J. Seo, S.-K. Choi, and H. I. Kwon, *On Appell-type Changhee polynomials and numbers*, Adv. Difference Equ. (2016), 2016:160, 10. MR 3514494

[18] D. Lim and F. Qi, *On the Appell type λ-Changhee polynomials*, J. Nonlinear Sci. Appl. **9** (2016), no. 4, 1872–1876. MR 3452683

[19] E.-J. Moon, J.-W. Park, and S.-H. Rim, *A note on the generalized q-Daehee numbers of higher order*, Proc. Jangjeon Math. Soc. **17** (2014), no. 4, 557–565. MR 3243802

[20] H. Ozden, I. N. Cangul, and Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **18** (2009), no. 1, 41–48. MR 2479746

[21] J.-J. Seo, S. H. Rim, T. Kim, and S. H. Lee, *Sums products of generalized Daehee numbers*, Proc. Jangjeon Math. Soc. **17** (2014), no. 1, 1–9. MR 3184457

[22] Y. Simsek, *Apostol type Daehee numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **26** (2016), no. 3, 555–566.

[23] A. V. Ustinov, *Korobov polynomials and umbral analysis*, Chebyshevskiĭ Sb. **4** (2003), no. 4(8), 137–152. MR 2097912 1

[24] N. L. Wang and H. Li, *Some identities on the higher-order daehee and changhee numbers*, Pure Appl. Math. J. **4** (2015),