INTEGRAL INEQUALITIES OF EXTENDED SIMPSON'S TYPE FOR (α, m)-ε-CONVEX FUNCTIONS

JUN ZHANG, ZHI-LI PEI, GAO-CHAO XU, XIAO-HUI ZHOU, AND FENG QI

Dedicated to Y. J. Cho on the occasion of his 65th birthday

1. Introduction

The following definition is well known in the literature:

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex function if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.2 ([5])**.** Let X be a real linear space and $D \subseteq X$ be a convex set and $f : D \to \mathbb{R}$ be a mapping on D, for any constant $\varepsilon \ge 0$, then $f(x)$ is said to be a ε-convex on D, if satisfies the following inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon \tag{1.2}$$

holds for all $x, y \in D$ and $t \in [0,1]$.

In [11] the concept of m-convex functions below was innovated.

Definition 1.3 ([11])**.** For $f : [0,b] \to \mathbb{R}$, and $b > 0$ and $m \in (0,1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

Definition 1.4 ([7])**.** For $f : [0,b] \to \mathbb{R}$, and $b > 0$ and $(\alpha, m) \in (0,1]^2$, if

$$f(\lambda x + m(1-\lambda)y) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y) \tag{1.3}$$

is valid for all $x, y \in [0,b]$ and $\lambda \in [0,1]$, then we say that $f(x)$ is a (α, m)-convex function on $[0,b]$.

2010 *Mathematics Subject Classification.* Primary 26A51, 26D15; Secondary 41A55.
Key words and phrases. extended Simpson's type integral inequality; (α, m)-ε-convex function.
This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

Theorem 1.1 ([1, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.4}$$

Theorem 1.2 ([8, Theorem 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ and $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.6}$$

In [2], the following Hermite-Hadamard type inequality for m-convex functions was proved.

Theorem 1.3 ([2]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L_1([a, b])$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}. \tag{1.7}$$

In this paper, we will introduce a new concept "(α, m)-ε-convex function" and establish some integral inequalities of Simpson's type for (α, m)-ε-convex functions.

2. Definition and Lemmas

Now we give a definition of the called (α, m)-ε-convex functions.

Definition 2.1. For $f : [0, b^*] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, for any constant $\varepsilon \ge 0$, then f is said to be a (α, m)-ε-convex on I, if

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) + \varepsilon \tag{2.1}$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Remark 1. (1) If f is (α, m)-ε-convex on $[0, b^*]$ and $\alpha = 1$, then we say that f is an m-ε-convex on $[0, b^*]$.

(2) If f is (α, m)-ε-convex on $[0, b^*]$ and $\alpha = m = 1$, then it is ε-convex on $[0, b^*]$.

To establish some new extended Simpson's type inequalities for (α, m)-ε-convex functions, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$ and $\lambda \ge 0$, and $n \in \mathbb{N}_+$, then*

$$\frac{1}{n(\lambda+2)}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_{2k}) + \lambda\sum_{k=1}^{n} f(x_{2k-1}) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

$$=\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1\left(\frac{\lambda}{\lambda+2}-t\right)f'\big(tx_{2k-2}+(1-t)x_{2k-1}\big)\,\mathrm{d}t\right.$$
$$\left.+\sum_{k=1}^{n}\int_0^1\left(\frac{2}{\lambda+2}-t\right)f'\big(tx_{2k-1}+(1-t)x_{2k}\big)\,\mathrm{d}t\right]. \tag{2.2}$$

In particular, we have

(1) *if* $\lambda=4$, *then*

$$\frac{1}{6n}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+4\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$
$$=\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1\left(\frac{2}{3}-t\right)f'\big(tx_{2k-2}+(1-t)x_{2k-1}\big)\,\mathrm{d}t\right.$$
$$\left.+\sum_{k=1}^{n}\int_0^1\left(\frac{1}{3}-t\right)f'\big(tx_{2k-1}+(1-t)x_{2k}\big)\,\mathrm{d}t\right]; \tag{2.3}$$

(2) *if* $\lambda=0$, *then*

$$\frac{1}{2n}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$
$$=\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1(-t)f'\big(tx_{2k-2}+(1-t)x_{2k-1}\big)\,\mathrm{d}t\right.$$
$$\left.+\sum_{k=1}^{n}\int_0^1(1-t)f'\big(tx_{2k-1}+(1-t)x_{2k}\big)\,\mathrm{d}t\right], \tag{2.4}$$

where $x_k=a+\frac{k(b-a)}{2n}(k=0,1,\cdots,2n)$.

Proof. By integration by parts, the result is followed. The proof is completed. □

By taking $n=1$ in Lemma 2.1, we have the following identities.

Lemma 2.2. *Let* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I°, $a,b\in I^\circ$ *with* $a<b$. *If* $f'\in L_1([a,b])$ *and* $\lambda\ge 0$, *then*

$$\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$
$$=\frac{b-a}{4}\left[\int_0^1\left(\frac{\lambda}{\lambda+2}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right.$$
$$\left.+\int_0^1\left(\frac{2}{\lambda+2}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}t\right].$$

If letting $\lambda=0$ in Lemma 2.2, we can obtain

Lemma 2.3 ([1, p. 91, Lemma 2.1])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on I°. If $f' \in L_1([a, b])$ for $a, b \in I$ with $a < b$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}t. \tag{2.5}$$

3. Some new integral inequalities of Simpson's type

In this section, the integral inequalities of Simpson's type related to (α, m)-ε-convex function are discussed.

Theorem 3.1. *Let $f : \mathbb{R}_0 = [0, +\infty) \to \mathbb{R}$ be differentiable function on $\mathbb{R}_0$, $a, b \in \mathbb{R}_0$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is (α, m)-ε-convex on $\left[0, \frac{b}{m}\right]$ for constant $\varepsilon \ge 0$, $\lambda \ge 0$, $(\alpha, m) \in (0, 1]^2$ and $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ \le&\frac{b-a}{4}\left[C(\lambda)\right]^{1-\frac{1}{q}}\left\{\left[A(\alpha,\lambda)|f'(a)|^q+m(C(\lambda)-A(\alpha,\lambda))\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon C(\lambda)\right]^{\frac{1}{q}}\right.\\ &\left.+\left[B(\alpha,\lambda)\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(C(\lambda)-B(\alpha,\lambda))\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon C(\lambda)\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{3.1}$$

where

$$A(\alpha,\lambda) = \frac{[2\alpha(\lambda+2)+4](\lambda+2)^\alpha+\lambda^2(2\lambda^\alpha-(\lambda+2)^\alpha)}{(\alpha+1)(\alpha+2)(\lambda+2)^{\alpha+2}}, \tag{3.2}$$

$$B(\alpha,\lambda) = \frac{(\alpha\lambda+\lambda-2)(\lambda+2)^{\alpha+1}+2^{\alpha+3}}{(\alpha+1)(\alpha+2)(\lambda+2)^{\alpha+2}}, C(\lambda) = \frac{\lambda^2+4}{2(\lambda+2)^2}. \tag{3.3}$$

Proof. Since $|f'|^q$ is (α, m)-ε-convex function on $\left[0, \frac{b}{m}\right]$, from the Lemma 2.2 and Hölder's integral inequality, we have that

$$\begin{aligned}&\left|\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ \le&\frac{b-a}{4}\left[\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right.\\ &\left.+\int_0^1\left|t-\frac{2}{\lambda+2}\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}t\right]\\ \le&\frac{b-a}{4}\left\{\left(\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\left(t^\alpha|f'(a)|^q\right.\right.\right.\\ &\left.\left.+m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right)\mathrm{d}t\right]^{\frac{1}{q}}+\left(\int_0^1\left|t-\frac{2}{\lambda+2}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\\ &\left.\times\left[\int_0^1\left|t-\frac{2}{\lambda+2}\right|\left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right)\mathrm{d}t\right]^{\frac{1}{q}}\right\}\end{aligned}$$

$$=\frac{b-a}{4}\big[C(\lambda)\big]^{1-\frac{1}{q}}\left\{\left[A(\alpha,\lambda)|f'(a)|^q+m(C(\lambda)-A(\alpha,\lambda))\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon C(\lambda)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[B(\alpha,\lambda)\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(C(\lambda)-B(\alpha,\lambda))\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon C(\lambda)\right]^{\frac{1}{q}}\right\}.$$

The proof of Theorem 3.1 is thus completed. ☐

Corollary 3.1.1. *Under the assumptions of Theorem 3.1, if $\lambda = 0$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{8}\left\{\left[\frac{2}{\alpha+2}|f'(a)|^q+\frac{\alpha m}{\alpha+2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{2}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha m(\alpha+2)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right\}. \tag{3.4}$$

Theorem 3.2. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is (α,m)-ε-convex on $\left[0,\frac{b}{m}\right]$ for constant $\varepsilon\ge 0$, and $(\alpha,m)\in(0,1]^2$ and $q\ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left[\frac{2(2^\alpha\alpha+1)|f'(a)|^q+m(2^\alpha(\alpha^2+\alpha+2)-2)|f'(b/m)|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)}+\frac{\varepsilon}{2}\right]^{\frac{1}{q}}. \tag{3.5}$$

Proof. Since $|f'|^q$ is (α,m)-ε-convex function on $\left[0,\frac{b}{m}\right]$, by the Lemma 2.3 and Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{2}\int_0^1|1-2t|\big|f'\big(ta+(1-t)b\big)\big|^q\mathrm{d}t$$
$$\le\frac{b-a}{2}\left[\int_0^1|1-2t|\,\mathrm{d}t\right]^{1-\frac{1}{q}}\left[\int_0^1|1-2t|\big[t^\alpha|f'(a)|^q+m(1-t^\alpha)|f'(b/m)|^q+\varepsilon\big]\mathrm{d}t\right]^{\frac{1}{q}}$$
$$=\frac{b-a}{4}\left[\frac{2(2^\alpha\alpha+1)|f'(a)|^q+m(2^\alpha(\alpha^2+\alpha+2)-2)|f'(b/m)|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)}+\frac{\varepsilon}{2}\right]^{\frac{1}{q}}.$$

Theorem 3.1 is proved. ☐

Theorem 3.3. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is (α,m)-ε-convex on $\left[0,\frac{b}{m}\right]$ for constant $\varepsilon\ge 0$, $\lambda\ge 0$, $(\alpha,m)\in(0,1]^2$ and $q>1$, then*

$$\left|\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$

$$\le \frac{b-a}{4}\left(\frac{\lambda^{\frac{2q-1}{q-1}}+2^{\frac{2q-1}{q-1}}}{(\lambda+2)^{\frac{2q-1}{q-1}}}\right)^{1-\frac{1}{q}}\left\{\left[\frac{1}{\alpha+1}|f'(a)|^q+\frac{\alpha m}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha m}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right\}. \tag{3.6}$$

Proof. From the Lemma 2.2 and the Hölder's integral inequality, by the (α,m)-ε-convexity of function $|f'|^q$, we have

$$\left|\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le \frac{b-a}{4}\left\{\left(\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|^{\frac{q}{q-1}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\int_0^1\left(t^\alpha|f'(a)|^q\right.\right.\right.$$
$$\left.\left.+m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right)\mathrm{d}t\right]^{\frac{1}{q}}+\left(\int_0^1\left|t-\frac{2}{\lambda+2}\right|^{q/(q-1)}\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\left.\times\left[\int_0^1\left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right)\mathrm{d}t\right]^{\frac{1}{q}}\right\}$$
$$=\frac{b-a}{4}\left(\frac{\lambda^{\frac{2q-1}{q-1}}+2^{\frac{2q-1}{q-1}}}{(\lambda+2)^{\frac{2q-1}{q-1}}}\right)^{1-\frac{1}{q}}\left\{\left[\frac{1}{\alpha+1}|f'(a)|^q+\frac{\alpha m}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha m}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right\}.$$

Theorem 3.3 is proved. □

Corollary 3.3.1. *Under the assumptions of Theorem 3.3, if $\lambda=0$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left\{\left[\frac{1}{\alpha+1}|f'(a)|^q\right.\right.$$
$$\left.\left.+\frac{\alpha m}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}+\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha m}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right]^{\frac{1}{q}}\right\}.$$

Theorem 3.4. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is (α,m)-ε-convex on $\left[0,\frac{b}{m}\right]$ for constant $\varepsilon\ge0$, and $(\alpha,m)\in(0,1]^2$ and $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\frac{|f'(a)|^q+\alpha m|f'(b/m)|^q}{\alpha+1}+\varepsilon\right]^{\frac{1}{q}}. \tag{3.7}$$

Proof. Since $|f'|^q$ is (α,m)-ε-convex function on $\left[0,\frac{b}{m}\right]$, from the Lemma 2.3 and Hölder's integral inequality, we have that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$

$$
\begin{aligned}
&\le \frac{b-a}{2}\int_0^1 \big|1-2t\big|\big|f'\big(ta+(1-t)b\big)\big|^q \mathrm{d}t\\
&\le \frac{b-a}{2}\left[\int_0^1 \big|1-2t\big|^{q/(q-1)}\,\mathrm{d}t\right]^{1-\frac{1}{q}}\left[\int_0^1 \big[t^\alpha|f'(a)|^q+m(1-t^\alpha)|f'(b/m)|^q+\varepsilon\big]\mathrm{d}t\right]^{\frac{1}{q}}\\
&= \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\frac{|f'(a)|^q+\alpha m|f'(b/m)|^q}{\alpha+1}+\varepsilon\right]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 3.4 is proved. □

Theorem 3.5. *For $n\in\mathbb{N}_+$ and $n\ge 2$, let $f:\mathbb{R}_0\to\mathbb{R}$ be differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|$ is (α,m)-ε-convex on $\big[0,\frac{b}{m}\big]$ for constant $\varepsilon\ge 0$, $\lambda\ge 0$ and $(\alpha,m)\in(0,1]^2$, then*

$$
\begin{aligned}
&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+\lambda\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le \frac{b-a}{4n^2}\sum_{k=1}^{n}\Big[A(\alpha,\lambda)\big|f'\big(x_{2k-2}\big)\big|+m(C(\lambda)-A(\alpha,\lambda))\big|f'\big(x_{2k-1}/m\big)\big|\\
&\qquad + B(\alpha,\lambda)\big|f'\big(x_{2k-1}\big)\big|+m(C(\lambda)-B(\alpha,\lambda))\big|f'\big(x_{2k}/m\big)\big|+2\varepsilon C(\lambda)\Big], \qquad (3.8)
\end{aligned}
$$

where $x_k=a+\frac{k(b-a)}{2n}(k=0,1,\cdots,2n)$ and $A(\alpha,\lambda)$, $B(\alpha,\lambda)$ and $C(\alpha)$ are defined by (3.2) and (3.3) respectively.

Proof. Since $|f'|$ is (α,m)-ε-convex function on $\big[0,\frac{b}{m}\big]$, from the Lemma 2.1 and Hölder's integral inequality, we have that

$$
\begin{aligned}
&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+\lambda\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le \frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\big|f'\big(tx_{2k-2}+(1-t)x_{2k-1}\big)\big|\mathrm{d}t\right.\\
&\qquad \left.+\sum_{k=1}^{n}\int_0^1\left|t-\frac{2}{\lambda+2}\right|\big|f'\big(tx_{2k-1}+(1-t)x_{2k}\big)\big|\mathrm{d}t\right]\\
&\le \frac{b-a}{4n^2}\left\{\sum_{k=1}^{n}\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\Big[t^\alpha\big|f'\big(x_{2k-2}\big)\big|+m(1-t^\alpha)\big|f'\big(x_{2k-1}/m\big)\big|+\varepsilon\Big]\mathrm{d}t\right.\\
&\qquad \left.+\sum_{k=1}^{n}\int_0^1\left|t-\frac{2}{\lambda+2}\right|\Big[t^\alpha\big|f'\big(x_{2k-1}\big)\big|+m(1-t^\alpha)\big|f'\big(x_{2k}/m\big)\big|+\varepsilon\Big]\mathrm{d}t\right\}\\
&= \frac{b-a}{4n^2}\sum_{k=1}^{n}\Big[A(\alpha,\lambda)\big|f'\big(x_{2k-2}\big)\big|+m(C(\lambda)-A(\alpha,\lambda))\big|f'\big(x_{2k-1}/m\big)\big|\\
&\qquad + B(\alpha,\lambda)\big|f'\big(x_{2k-1}\big)\big|+m(C(\lambda)-B(\alpha,\lambda))\big|f'\big(x_{2k}/m\big)\big|+2\varepsilon C(\lambda)\Big]. \qquad (3.9)
\end{aligned}
$$

The proof of Theorem 3.5 is thus completed. □

Corollary 3.5.1. *Under the assumptions of Theorem 3.3, then*

$$\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+\lambda\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4n}\left\{\big[(1-m)(A(\alpha,\lambda)+B(\alpha,\lambda))+2mC(\lambda)\big]\sup_{a\le x\le b/m}|f'(x)|+2\varepsilon C(\lambda)\right\}.$$

Theorem 3.6. *For $n\in\mathbb{N}_+$ and $n\ge 2$, let $f:\mathbb{R}_0\to\mathbb{R}$ be differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|$ is m-ε-convex on $\big[0,\frac{b}{m^2}\big]$ for constant $\varepsilon\ge 0$ and $m\in(0,1]$, then*

$$\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+\lambda\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{8n^2}\left[\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}|f'(a)|\right.$$
$$+m\frac{3n(\lambda^2+4)(\lambda+2)-(\lambda^3+12\lambda+16)}{6(\lambda+2)^3}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)$$
$$\left.+m^2\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}\left|f'\left(\frac{b}{m^2}\right)\right|+\frac{2\varepsilon(\lambda^2+4)}{(\lambda+2)^2}\right]. \tag{3.10}$$

Proof. By the (3.9) and the m-ε-convexity of the function $|f'|$, we have that

$$\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+\lambda\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4n^2}\sum_{k=1}^{n}\Big[A(\alpha,\lambda)\big|f'\big(x_{2k-2}\big)\big|+m(C(\lambda)-A(\alpha,\lambda))\big|f'\big(x_{2k-1}/m\big)\big|$$
$$+B(\alpha,\lambda)\big|f'\big(x_{2k-1}\big)\big|+m(C(\lambda)-B(\alpha,\lambda))\big|f'\big(x_{2k}/m\big)\big|+2\varepsilon C(\lambda)\Big]$$
$$\le\frac{b-a}{4n^2}\sum_{k=1}^{n}\left\{\frac{\lambda^3+12\lambda+16}{6(\lambda+2)^3}\left[\frac{2n-(2k-2)}{2n}|f'(a)|+\frac{m(2k-2)}{2n}\left|f'\left(\frac{b}{m}\right)\right|\right]\right.$$
$$+\frac{m(\lambda^3+3\lambda^2+4)}{3(\lambda+2)^3}\left[\frac{2n-(2k-1)}{2n}\left|f'\left(\frac{a}{m}\right)\right|+\frac{m(2k-1)}{2n}\left|f'\left(\frac{b}{m^2}\right)\right|\right]$$
$$+\frac{\lambda^3+3\lambda^2+4}{3(\lambda+2)^3}\left[\frac{2n-(2k-1)}{2n}|f'(a)|+\frac{m(2k-1)}{2n}\left|f'\left(\frac{b}{m}\right)\right|\right]$$
$$\left.+\frac{m(\lambda^3+12\lambda+16)}{6(\lambda+2)^3}\left[\frac{2n-2k}{2n}\left|f'\left(\frac{a}{m}\right)\right|+\frac{2km}{2n}\left|f'\left(\frac{b}{m^2}\right)\right|\right]+\frac{2\varepsilon(\lambda^2+4)}{2(\lambda+2)^2}\right\}$$
$$=\frac{b-a}{8n^2}\left[\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}|f'(a)|\right.$$
$$+m\frac{3n(\lambda^2+4)(\lambda+2)-(\lambda^3+12\lambda+16)}{6(\lambda+2)^3}\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)$$
$$\left.+m^2\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}\left|f'\left(\frac{b}{m^2}\right)\right|+\frac{2\varepsilon(\lambda^2+4)}{(\lambda+2)^2}\right].$$

The proof of Theorem 3.6 is thus completed. □

Corollary 3.6.1. *Under the assumptions of Theorem 3.3, we have*

(1) *if* $\lambda = 4$, *then*

$$\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+4\sum_{k=1}^{n}f\big(x_{2k-1}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{1536n^2}\left[(45n+16)\big|f'(a)\big|+m(45n-16)\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right.$$
$$\left.+\,m^2(45n+16)\left|f'\left(\frac{b}{m^2}\right)\right|+180\varepsilon\right]; \tag{3.11}$$

(2) *if* $\lambda = 0$, *then*

$$\left|\frac{1}{2n}\left[f(a)+2\sum_{k=1}^{n-1}f\big(x_{2k}\big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{48n^2}\left[(3n+2)\big|f'(a)\big|+m(3n-2)\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\right.$$
$$\left.+\,m^2(3n+2)\left|f'\left(\frac{b}{m^2}\right)\right|+12\varepsilon\right]. \tag{3.12}$$

Acknowledgements. This work was partially supported by the National Natural Science Foundation under Grant No. 61373067 and No. 61672301 of China and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant No. 2015MS0123, China.

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(Zhang) College of Computer Science and Technology, Jilin University, Changchun 130012, China; College of Computer Science and Technology, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

E-mail address: `Zhangjun728@163.com`

(Pei) College of Computer Science and Technology, Inner Mongolia University for the Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

E-mail address: `zhilipei@sina.com`

(Xu) College of Computer Science and Technology, Jilin University, Changchun 130012, China.

E-mail address: `xugc@jlu.edu.cn`

(Zhou) College of Computer Science and Technology, Jilin University, Changchun 130012, China.

E-mail address: `496072945@qq.com`

(Qi) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China; Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

E-mail address: `qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com`

URL: `http://qifeng618.wordpress.com`