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COMPLEX AFFINE TRANSVERSAL BUNDLES FOR SURFACES IN $\mathbf{C}^4$

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Abstract. Three different canonical constructions of transversal bundles for non-degenerate complex surfaces in $\mathbf{C}^4$ are presented and different induced connections are obtained. The bundles turn out to be always holomorphic, but only one of them is equiaffine.

1. Preliminaries. The purpose of this paper is to study some four dimensional submanifolds of eight dimensional space $\mathbf{R}^8$. The submanifolds are endowed with a complex structure which is compatible with the canonical complex structure in $\mathbf{R}^8$, so we treat them as complex surfaces holomorphically immersed in $\mathbf{C}^4$. Canonically determined transversal bundles and induced connections for real surfaces in $\mathbf{R}^4$ were obtained in [**2**] and [**4**]. An equiaffine structure for such surfaces was found by Nomizu and Vrancken ([**7**]) and further investigations showed that the construction leads to some natural geometric properties which are often generalizations of those well-known in the codimension-one case ([**3**]). In this paper we will make the assumptions suggested in B.Opozda's papers [**8**], [**9**] and obtain three complex transversal bundle satisfying the conditions proposed in [**2**], [**4**] and [**7**], respectively. Moreover we prove that these bundles are holomorphic. Only the bundle satisfying Nomizu–Vrancken's conditions leads to the equiaffine connection.

Let M be a two-dimensional complex submanifold of $\mathbf{C}^4$, that is, there exists an immersion $f\colon M \to \mathbf{C}^4$ which is holomorphic. It means that $f_*JX = Jf_*X$ where J denotes the complex structures on M and in $\mathbf{C}^4$. Each tangent space T_xM has a natural structure of a complex vector space with the multiplication by i given by J. Unless otherwise stated the vector fields, connections,

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bilinear and linear forms and functions used in the paper are of class $\mathcal{C}^{\infty}_{\mathbf{R}}$ (see [**8**], [**9**], compare also another approach in [**1**], [**3**]). Since this work is only of the local character and we can identify both the complex structures on M and $\mathbf{C}^4$, we can also identify M, as a complex manifold, with its image in $\mathbf{C}^4$.

Let σ denote a transversal complex plane bundle, that is $\mathbf{C}^4 = \sigma_x \oplus T_xM$ over $\mathbf{C}$. Let ξ_1, ξ_2 be transversal vector fields that span σ locally. It means that they are linearly independent over $\mathbf{C}$. If D denotes the standard affine connection on $\mathbf{C}^4$, then we have:

$$D_XY = \nabla_XY + h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2, \tag{1.1}$$

$$D_X\xi_j = -S_{\xi_j}(X) + \tau_j^1(X)\xi_1 + \tau_j^2(X)\xi_2, \tag{1.2}$$

for $j = 1,2$, where the connection ∇ is given by the condition $\nabla_XY \in TM$ and $S_{\xi_j}(X) \in TM$ as well. We call the connection ∇ the induced connection and the surface together with the connection – the induced affine structure.

It is straightforward to prove that ∇ is a torsion-free linear connection which is compatible with J (it is then called a complex connection). It means that $\nabla J = 0$, and equivalently that $\nabla_X JY = J\nabla_XY$ for arbitrary vector fields X, Y.

From now on we will use the simplified notation S_j instead of S_{ξ_j}. We can also see that both h^j are $\mathbf{C}$-valued, $\mathbf{C}$-bilinear 2-forms, S_j are (1,1)-$\mathbf{R}$-linear tensors and τ_k^j are $\mathbf{C}$-valued, $\mathbf{R}$-linear 1-forms (see [**6**], [**8**], [**9**]).

Given ξ_1 and ξ_2 we also define a complex-valued $\mathbf{C}$-linear skew-symmetric 2-form θ by $\theta(X,Y) = \mathrm{Det}[X, Y, \xi_1, \xi_2]$, where Det denotes the usual determinant in $\mathbf{C}^4$.

A transversal bundle σ is holomorphic if there is a holomorphic transversal frame of vector fields that span σ(locally). One can easily observe that this condition holds if and only if $S_j(JX) = JS_j(X)$ and $\tau_j^k(JX) = i\tau_j^k(X)$ for $j,k = 1,2$ (see [**8**], [**9**]). Objects satisfying such a property are called complex. We say that S_j, τ_j^k and h^j are induced by ξ_1 and ξ_2 and that ∇ is the connection induced by σ. The bundle σ is holomorphic if and only if the induced connection is also holomorphic.

In the paper we often use holomorphic vector fields and recall (see [**5**], [**6**]) that a vector field Y is holomorphic if and only if $\nabla_{JX}Y = J\nabla_XY$ holds for an arbitrary vector field X and a complex connection ∇. A connection ∇ is called holomorphic if for holomorphic vector fields X and Y, ∇_XY is also a holomorphic vector field. For tangent vector fields X, Y, Z – not necessarily holomorphic – we can obtain the equations of Gauss (1.3), Codazzi ((1.4), (1.5), (1.6) and (1.7)) and Ricci ((1.8), (1.9), (1.10) and (1.11)), like in real geometry ([**7**]):

(1.3)
$$R(X,Y)Z = h^1(Y,Z)S_1X + h^2(Y,Z)S_2X - h^1(X,Z)S_1Y - h^2(X,Z)S_2Y,$$

(1.4)
$(\nabla_X h^1)(Y,Z) + \tau_1^1(X)h^1(Y,Z) + \tau_1^2(X)h^2(Y,Z)$ is symmetric in X, Y and Z,

(1.5)
$(\nabla_X h^2)(Y,Z) + \tau_2^1(X)h^1(Y,Z) + \tau_2^2(X)h^2(Y,Z)$ is symmetric in X, Y and Z,

(1.6)
$$(\nabla_X S_1)Y - (\nabla_Y S_1)X = -\tau_1^1(Y)S_1X + \tau_1^1(X)S_1Y - \tau_2^1(Y)S_2X + \tau_2^1(X)S_2Y,$$

(1.7)
$$(\nabla_X S_2)Y - (\nabla_Y S_2)X = -\tau_1^2(Y)S_1X + \tau_1^2(X)S_1Y - \tau_2^2(Y)S_2X + \tau_2^2(X)S_2Y,$$

$$h^1(X,S_1Y) - h^1(Y,S_1X) = d\tau_1^1(X,Y) + \tau_2^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_2^1(X), \tag{1.8}$$

$$\begin{aligned} h^2(X,S_1Y) - h^2(Y,S_1X) = d\tau_2^1(X,Y) &+ \tau_1^1(Y)\tau_2^1(X) - \tau_2^1(Y)\tau_1^1(X) \\ &+ \tau_2^1(Y)\tau_2^2(X) - \tau_2^2(Y)\tau_2^1(X), \end{aligned} \tag{1.9}$$

$$h^2(X,S_2Y) - h^2(Y,S_2X) = d\tau_2^2(X,Y) + \tau_1^2(Y)\tau_2^1(X) - \tau_2^1(Y)\tau_1^2(X), \tag{1.10}$$

$$\begin{aligned} h^1(X,S_2Y) - h^1(Y,S_2X) = d\tau_1^2(X,Y) &+ \tau_1^1(X)\tau_1^2(Y) - \tau_1^2(X)\tau_1^1(Y) \\ &+ \tau_1^2(X)\tau_2^2(Y) - \tau_2^2(X)\tau_1^2(Y). \end{aligned} \tag{1.11}$$

For another transversal plane bundle $\tilde{\sigma}$ and its local basis $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ we have

$$\xi_1 = \phi\tilde{\xi}_1 + \psi\tilde{\xi}_2 + Z_1 \tag{1.12}$$

$$\xi_2 = \alpha\tilde{\xi}_1 + \beta\tilde{\xi}_2 + Z_2, \tag{1.13}$$

where $\alpha, \beta, \psi, \phi$ are local complex-valued $\mathcal{C}^\infty_{\mathbf{R}}$ functions on M such that $\phi\beta - \alpha\psi \neq 0$ and Z_1, Z_2 are tangent vector fields.

Let $\tilde{\nabla}$, $\tilde{S}_j$, $\tilde{\tau}_j^k$ and $\tilde{h}^j$ be the objects induced by $\tilde{\xi}_1$ and $\tilde{\xi}_2$. Using (1.1), (1.2), (1.12) and (1.13) we get:

$$\tilde{\nabla}_X Y = \nabla_X Y + h^1(X,Y)Z_1 + h^2(X,Y)Z_2, \tag{1.14}$$

$$\tilde{h}^1(X,Y) = \phi h^1(X,Y) + \alpha h^2(X,Y), \tag{1.15}$$

$$\tilde{h}^2(X,Y) = \psi h^1(X,Y) + \beta h^2(X,Y). \tag{1.16}$$

2. The affine metrics. Let $u = \{X_1, X_2\}$ be a local frame of class $\mathcal{C}^\infty_{\mathbf{R}}$ on a neighbourhood U of a point $p \in M$. We do not assume that X_1, X_2 are holomorphic but only linearly independent over $\mathbf{C}$. Define a symmetric

R-bilinear **C**-valued form G_u:

$$G_u(Y,Z) = \frac{1}{2}\,\mathrm{Det}[X_1, X_2, D_Y X_1, D_Z X_2] + \frac{1}{2}\,\mathrm{Det}[X_1, X_2, D_Z X_1, D_Y X_2]. \tag{2.1}$$

Using a transversal plane bundle spanned by transversal vector fields $\{\xi_1, \xi_2\}$ we can write:

$$G_u(Y,Z) = \frac{1}{2}[X_1, X_2, \xi_1, \xi_2] \cdot \left(\begin{vmatrix} h^1(X_1,Y) & h^1(X_2,Z) \\ h^2(X_1,Y) & h^2(X_2,Z) \end{vmatrix} + \begin{vmatrix} h^1(X_1,Z) & h^1(X_2,Y) \\ h^2(X_1,Z) & h^2(X_2,Y) \end{vmatrix} \right). \tag{2.2}$$

Thus we can see that G_u is also **C**-linear, because each h^j is **C**-linear.

We can apply here Lemma 3.1 ([**7**]). It follows that for another local frame $v = \{Y_1, Y_2\}$ satisfying

$$Y_1 = aX_1 + bX_2,$$
$$Y_2 = cX_1 + dX_2$$

with $ad - bc \neq 0$ we have

$$G_v = (ad - bc)^2 G_u.$$

If we define

$$\det_u h = \begin{vmatrix} h(X_1,X_1) & h(X_1,X_2) \\ h(X_1,X_2) & h(X_2,X_2) \end{vmatrix} \tag{2.3}$$

where h is a symmetric **C**-bilinear form, we have

$$\det_v h = (ad - bc)^2 \det_u h.$$

Thus we can call a surface non-degenerate if the form G_u is non-degenerate (which does not depend on the choice of u).

From now on we will assume that the surface is non-degenerate. In a sufficiently small neighbourhood of each point we can define three unique branches of a **C**-valued, **C**-bilinear symmetric form

$$g_u(Y,Z) = \frac{G_u(Y,Z)}{(\det_u G_u)^{\frac{1}{3}}}.$$

We can easily see, like in the real case, that the set of three branches of g_u is independent of the choice of u. In this way we get locally three complex-valued metrics which we will denote by g. We will call each of them an affine metric on the surface M. Notice that if u is a holomorphic tangent frame, the function

$$M \ni x \longmapsto g_{u_x}(X,Y)$$

is holomorphic if the vector fields X and Y are holomorphic. It follows from the definition of G_u and g_u. Namely, given a holomorphic vector field X, the function $x \longmapsto X(x)$ is holomorphic, so is $x \longmapsto D_X Y(x)$ for holomorphic Y. G_u is defined using the complex determinant in $\mathbf{C}^4$, which is a holomorphic function applied to holomorphic vector fields Y, Z and holomorphic frame u. Therefore the mapping $x \longmapsto G_u(x)$ is also holomorphic. Now the definition of g_u requires only mappings that are also holopmorphic. Using this fact, we can always find holomorphic orthonormal frames or null frames (relative to g), starting from arbitrary holomorphic ones.

From now on we will choose an affine metric in a sufficiently small neighbourhood of a given point. For the sake of simplicity and convenience, we will work with null frames with respect to this affine metric g that is the frames $\{X_1, X_2\}$ satisfying $g(X_1, X_2) = 1$ and $g(X_j, X_j) = 0$ for $j = 1, 2$. In the following theorem we associate to each null tangent frame a unique transversal frame in a given transversal bundle (see [**7**]).

Theorem 2.1. *Let σ be a transversal plane bundle and $\{X_1, X_2\}$ be a null tangent frame. Then there exists a unique local transversal frame $\{\xi_1, \xi_2\}$ in σ such that:*

$$\mathrm{Det}[X_1, X_2, \xi_1, \xi_2] = -2, \tag{2.4}$$

and the second fundamental forms have the following matrices in the basis $\{X_1, X_2\}$:

$$h^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad h^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.5}$$

Proof. Let $u = \{X_1, X_2\}$ be a local null frame and $\{\xi_1, \xi_2\}$ be a local transversal frame that spans σ. Let $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ be another transversal frame. It can be expressed by:

$$\xi_1 = \phi\tilde{\xi}_1 + \psi\tilde{\xi}_2, \tag{2.6}$$

$$\xi_2 = \alpha\tilde{\xi}_1 + \beta\tilde{\xi}_2, \tag{2.7}$$

where $\phi\beta - \psi\alpha \neq 0$. We choose functions ϕ, ψ, β and α to so that the following conditions are satisfied (bars will denote the objects induced by $\{\tilde{\xi}_1, \tilde{\xi}_2\}$):

$$\tilde{h}^1(X_1, X_1) = 1, \tag{2.8}$$

$$\tilde{h}^1(X_2, X_2) = 0, \tag{2.9}$$

$$\tilde{h}^2(X_1, X_1) = 0, \tag{2.10}$$

$$[X_1, X_2, \tilde{\xi}_1, \tilde{\xi}_2] = -2. \tag{2.11}$$

Owing to (1.15) and (1.16) this system can be written in the following equivalent form:

$$1 = \phi h^1(X_1, X_1) + \alpha h^2(X_1, X_1), \tag{2.12}$$

$$0 = \phi h^1(X_2, X_2) + \alpha h^2(X_2, X_2), \tag{2.13}$$

$$0 = \psi h^1(X_1, X_1) + \beta h^2(X_1, X_1), \tag{2.14}$$

$$[X_1, X_2, \xi_1, \xi_2] = -2(\phi\beta - \psi\alpha). \tag{2.15}$$

We look at (2.12) and (2.13) as a system of linear equations with the determinant different from zero, because $g(X_1, X_2) = 1$. Thus we obtain

$$\phi = \frac{\begin{vmatrix} 1 & h^2(X_1, X_1) \\ 0 & h^2(X_2, X_2) \end{vmatrix}}{\begin{vmatrix} h^1(X_1, X_1) & h^2(X_1, X_1) \\ h^1(X_2, X_2) & h^2(X_2, X_2) \end{vmatrix}},$$

$$\alpha = \frac{\begin{vmatrix} h^1(X_1, X_1) & 1 \\ h^1(X_2, X_2) & 0 \end{vmatrix}}{\begin{vmatrix} h^1(X_1, X_1) & h^2(X_1, X_1) \\ h^1(X_2, X_2) & h^2(X_2, X_2) \end{vmatrix}}.$$

If we substitute α and ϕ obtained above to (2.15), then the equations (2.14) and (2.15) form another system of linear equations. Its determinant is equal to -2. We also see that the desired condition $\phi\beta - \psi\alpha \neq 0$ is true because $[X_1, X_2, \xi_1, \xi_2] \neq 0$. $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ are determined uniquely here, because if both frames $\{\xi_1, \xi_2\}$ and $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ satisfy the equations (2.12) up to (2.15), we get $\alpha = \psi = 0$ and $\beta = \phi = 1$. To complete the proof we use the fact that $\{X_1, X_2\}$ is a null frame. From $G_u(X_1, X_1) = 0$ we get $\tilde{h}^2(X_1, X_2) = 0$ by definition. $G_u(X_2, X_2) = 0$ gives $\tilde{h}^1(X_1, X_2)\tilde{h}^2(X_2, X_2) = 0$ and we also have

$$1 = \frac{G_u(X_1, X_2)}{(\det_u G_u)^{\frac{1}{3}}} = \tilde{h}^2(X_2, X_2)^{\frac{1}{3}}.$$

Thus

$$\tilde{h}^2(X_1, X_2) = 0,$$
$$\tilde{h}^2(X_2, X_2) = 1,$$
$$\tilde{h}^1(X_1, X_2) = 0$$

which completes the proof. □

From now on we will call the frame $\{\xi_1, \xi_2\}$ determined by the last theorem the transversal frame associated to the null tangent frame $\{X_1, X_2\}$. Notice that if we interchange the fields X_1 and X_2 in the frame, then the fields ξ_1, ξ_2 in the associated frame will also interchange.

The following lemma is straightforward and we omit its proof.

Lemma 2.2. *Let* $\{X_1, X_2\}$ *and* $\{\tilde{X}_1, \tilde{X}_2\}$ *be two tangent null frames. Let* $\{\xi_1, \xi_2\}$*and* $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ *be the associated transversal frames, respectively. Then there exists a* **C**-*valued non-zero function* a *such that*

$$\tilde{X}_1 = aX_1, \tag{2.16}$$

$$\tilde{X}_2 = a^{-1}X_2, \tag{2.17}$$

$$\tilde{\xi}_1 = a^2\xi_1, \tag{2.18}$$

$$\tilde{\xi}_2 = a^{-2}\xi_2, \tag{2.19}$$

after changing the order of X_1 *and* X_2*, as well as* ξ_1 *and* ξ_2*, if necessary. Moreover, if the fields* X_1 *and* X_2 *are holomorphic, then the function* a *is also holomorphic.*

Assume that an affine metric g is fixed (locally). Now we define locally a complex valued metric on an arbitrary transversal plane bundle σ. Let $u =\{X_1, X_2\}$ be a null frame and $\{\xi_1, \xi_2\}$ – the associated transversal frame in σ. We define a metric $g_u^\perp$ on σ by:

$$\begin{aligned} g_u^\perp(\xi_1, \xi_1) &= 0, \\ g_u^\perp(\xi_1, \xi_2) &= -2, \\ g_u^\perp(\xi_2, \xi_2) &= 0 \end{aligned} \tag{2.20}$$

and extend it to a **C**-bilinear, symmetric form.

Lemma 2.3. $g_u^\perp$ *is independent of the choice of the tangent frame* u*.*

Proof. Let $\{\xi_1, \xi_2\}$ and $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ be two transversal frames (spanning a transversal bundle σ), associated with null frames $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$, respectively, and a the function satisfying Lemma 2.2. Then we have

$$\begin{aligned} &g_u^\perp(\tilde{\xi}_1, \tilde{\xi}_1) = g_u^\perp(a^2\xi_k, a^2\xi_k) = 0 = g_v^\perp(\tilde{\xi}_1, \tilde{\xi}_1), \\ &g_u^\perp(\tilde{\xi}_1, \tilde{\xi}_2) = g_u^\perp(a^2\xi_k, a^{-2}\xi_j) = g_u^\perp(\xi_k, \xi_j) = -2 = g_v^\perp(\tilde{\xi}_1, \tilde{\xi}_2), \\ &g_u^\perp(\tilde{\xi}_2, \tilde{\xi}_2) = g_u^\perp(a^{-2}\xi_j, a^{-2}\xi_j) = 0 = g_v^\perp(\tilde{\xi}_2, \tilde{\xi}_2), \end{aligned}$$

for certain $j, k \in \{1, 2\}$ such that $j \neq k$. This completes the proof. □

Due to the last lemma we will denote the obtained metric by $g^\perp$.

Lemma 2.4. *Let* M *be a complex surface in* $\mathbf{C}^4$*,* $\{X_1, X_2\}$ *– a null frame and* σ*,*$\tilde{\sigma}$ *– two transversal plane bundles. If* $\{\xi_1, \xi_2\}$ *and* $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ *are the* *transversal frames associated with* $\{X_1, X_2\}$ *and spanning* σ *and* $\tilde{\sigma}$*, respectively, then*

$$\xi_1 = \tilde{\xi}_1 + Z_1, \tag{2.21}$$

$$\xi_2 = \tilde{\xi}_2 + Z_2 \tag{2.22}$$

for some tangent vector fields Z_1, Z_2*.*

Proof. Let h^1, h^2 be the second fundamental forms induced by $\{\xi_1, \xi_2\}$ and $\tilde{h}^1, \tilde{h}^2$ – induced by $\{\tilde{\xi}_1, \tilde{\xi}_2\}$. Then there exist complex valued functions a, b, c, d and tangent vector fields Z_1, Z_2 such that

$$\xi_1 = a\tilde{\xi}_1 + b\tilde{\xi}_2 + Z_1,$$
$$\xi_2 = c\tilde{\xi}_1 + d\tilde{\xi}_2 + Z_2.$$

We know by (1.15) and (1.16) that

$$\tilde{h}^1 = ah^1 + ch^2,$$
$$\tilde{h}^2 = bh^1 + dh^2.$$

Substituting (X_1, X_1) and (X_2, X_2) in the last two equalities we get

$$a = 1, b = 0, c = 0, d = 1.$$

□

3. The construction of the unique equiaffine transversal bundle. In this section we will always assume that g is a fixed affine metric in a neighbourhood of a point $x_0 \in M$. We introduce here a complex valued, skew-symmetric $\mathbf{C}$ bilinear 2-form ω_g, defined up to the sign in the following way. For a g-null frame $\{X_1, X_2\}$ we set $\omega_g(X_1, X_2) = 1$. In the paper we will consider the condition $\nabla\omega_g = 0$ which is independent of the sign of ω_g. We shall call a transversal bundle satisfying this condition an equiaffine transversal bundle. In the following we will determine a unique equiaffine bundle adding one more condition as it was done in the real case and prove that that bundle is holomorphic. First we give some conditions which are equivalent to the fact that a bundle is equiaffine. In the following lemma ∇ will denote the connection induced by a given transversal bundle σ and τ_j^k – 1-forms induced by a fixed transversal frame.

Lemma 3.1. *Let* σ *be a transversal bundle over the complex surface* M *in* $\mathbf{C}^4$*. Then the following conditions are equivalent:*

1) σ *is an equiaffine bundle;*
2) $\tau_1^1 + \tau_2^2 = 0$ *for* τ_j^k *induced by any transversal* $g^\perp$*-null frame;*

3) *for any holomorphic tangent null frame $\{X_1, X_2\}$ and associated null transversal frame $\{\xi_1, \xi_2\}$ (as determined by Theorem 2.1) the following equations hold:*

$$(\nabla g)(X_1, X_2, X_1) = 0, \tag{3.1}$$

$$(\nabla g)(X_2, X_1, X_2) = 0. \tag{3.2}$$

PROOF. First we prove the equivalence between 1) and 3). Let $\{X_1, X_2\}$ be a null holomorphic tangent frame. We introduce functions a_1 up to a_8 such that:

$$\begin{aligned} \nabla_{X_1} X_1 &= a_1 X_1 + a_2 X_2, \\ \nabla_{X_1} X_2 &= a_3 X_1 + a_4 X_2, \\ \nabla_{X_2} X_1 &= a_5 X_1 + a_6 X_2, \\ \nabla_{X_2} X_2 &= a_7 X_1 + a_8 X_2. \end{aligned} \tag{3.3}$$

The condition $\nabla \omega_g = 0$ is equivalent to

$$\nabla \omega_g(X_1, X_1, X_2) = 0,$$
$$\nabla \omega_g(X_2, X_1, X_2) = 0$$

and, after applying (3.3), to

$$\omega_g(\nabla_{X_1} X_1, X_2) + \omega_g(X_1, \nabla_{X_1} X_2) = 0,$$
$$\omega_g(\nabla_{X_2} X_1, X_2) + \omega_g(X_1, \nabla_{X_2} X_2) = 0.$$

This gives immediately

$$a_1 + a_4 = 0, \tag{3.4}$$

$$a_5 + a_8 = 0. \tag{3.5}$$

On the other hand the equations (3.1) and (3.2) are also equivalent to (3.4) and (3.5). Next we shall prove the equivalence between 1) and 2). It is easy to verify that a $g^{\perp}$-null transversal frame is associated to exactly one g-null tangent frame. We can thus say that they are associated to each other. Take a $g^{\perp}$-null frame $\{\xi_1, \xi_2\}$ spanning σ, and the associated tangent frame $\{X_1, X_2\}$. We have $\theta(X_1, X_2) = [X_1, X_2, \xi_1, \xi_2] = -2$, so $\theta = -2\omega_g$ up to the sign.

Since the determinant is parallel with respect to the connection D, we have

$$\begin{aligned} 0 =& D_X[X_1, X_2, \xi_1, \xi_2] - [D_X X_1, X_2, \xi_1, \xi_2] - [X_1, D_X X_2, \xi_1, \xi_2] \\ &- [X_1, X_2, D_X \xi_1, \xi_2] - [X_1, X_2, \xi_1, D_X \xi_2], \end{aligned}$$

where X is an arbitrary vector field on M. Due to (1.1) and (2.2), the last equality is equivalent to the following ones:

$$\begin{aligned} &- [\nabla_X X_1, X_2, \xi_1, \xi_2] - [X_1, \nabla_X X_2, \xi_1, \xi_2] \\ &- [X_1, X_2, \tau_1^1(X)\xi_1, \xi_2] - [X_1, X_2, \xi_1, \tau_2^2(X)\xi_2] = 0 \end{aligned}$$

and

$$(\nabla_X \theta)(X_1, X_2) + 2(\tau_1^1(X) + \tau_2^2(X)) = 0.$$

But the condition $\nabla \omega_g = 0$ is equivalent to $\nabla \theta = 0$. Thus the last equality is equivalent to

$$(\nabla_X \omega_g)(X_1, X_2) - (\tau_1^1(X) + \tau_2^2(X)) = 0.$$

This gives the equivalence between 1) and 2) and completes the proof of the lemma. □

Let $\nabla^\perp$ denote the normal connection induced on a transversal bundle σ such that $\nabla^\perp_X \xi$ is the transversal component of $D_X \xi$. We are going to consider the tensor $\nabla^\perp g^\perp(X, \xi, \eta)$ where X is a tangent vector field and ξ, η – transversal ones. Notice that $\nabla^\perp g^\perp$ is 3-linear over the module of complex-valued functions, if X, ξ and η are holomorphic.

Definition 3.2. Let σ be an equiaffine transversal bundle. We say that $\nabla^\perp g^\perp$ is symmetric if there is a null holomorphic tangent frame $u = \{X_1, X_2\}$ such that the following equalities hold for the associated transversal frame $\{\xi_1, \xi_2\}$:

$$\nabla^\perp g^\perp(X_1, \xi_2, \xi_1) + \nabla^\perp g^\perp(X_2, \xi_1, \xi_1) = 0, \tag{3.6}$$

$$\nabla^\perp g^\perp(X_1, \xi_2, \xi_2) + \nabla^\perp g^\perp(X_2, \xi_1, \xi_2) = 0. \tag{3.7}$$

We shall show that if equations (3.6) and (3.7) hold for a frame u then they hold for every null holomorphic frame v. We need the following lemma.

Lemma 3.3. *Let ∇ and $\nabla^\perp$ be the connections induced by an equiaffine transversal plane bundle σ. $\nabla^\perp g^\perp$ is symmetric if and only if there is a holomorphic null frame $\{X_1, X_2\}$ satisfying the following equations:*

$$(\nabla g)(X_1, X_1, X_1) = 0, \tag{3.8}$$

$$(\nabla g)(X_2, X_2, X_2) = 0. \tag{3.9}$$

Proof. Let equations hold for a tangent frame $\{X_1, X_2\}$. Let functions a_j be like in (3.3) for $\{X_1, X_2\}$. Let $\{\xi_1, \xi_2\}$ be the transversal frame associated to $\{X_1, X_2\}$. Putting (X_1, X_2, X_1) and (X_1, X_2, X_2) to the Codazzi equations (1.5) and (1.4), respectively, we obtain

$$a_2 = -\tau_2^1(X_2),$$
$$a_7 = -\tau_1^2(X_1),$$

where τ_j^k are induced by $\{\xi_1, \xi_2\}$. From (3.6) and (3.7) we get

$$-g^\perp(\nabla^\perp_{X_1}\xi_2, \xi_1) - g^\perp(\xi_2, \nabla^\perp_{X_1}\xi_1) - 2g^\perp(\nabla^\perp_{X_2}\xi_1, \xi_1) = 0,$$
$$-g^\perp(\nabla^\perp_{X_2}\xi_1, \xi_1) - g^\perp(\xi_2, \nabla^\perp_{X_2}\xi_1) - 2g^\perp(\nabla^\perp_{X_1}\xi_2, \xi_2) = 0,$$

which is equivalent to

$$2\tau_2^2(X_1) + 2\tau_1^1(X_1) + 4\tau_2^1(X_2) = 0,$$
$$2\tau_2^2(X_2) + 2\tau_1^1(X_2) + 4\tau_1^2(X_1) = 0.$$

Since ∇ comes from an equiaffine transversal bundle, $\tau_1^1 + \tau_2^2 = 0$ and we get

$$\tau_2^1(X_2) = \tau_1^2(X_1) = 0.$$

We then make the following computations

$$(\nabla g)(X_1, X_1, X_1) = -2g(\nabla_{X_1} X_1, X_1) = -2a_2 = 2\tau_2^1(X_2),$$
$$(\nabla g)(X_2, X_2, X_2) = -2g(\nabla_{X_2} X_2, X_2) = -2a_7 = 2\tau_1^2(X_1),$$

which complete the proof of the lemma. □

We finally prove that the notion of symmetry of $\nabla^\perp g^\perp$ does not depend on the choice of a tangent null frame $\{X_1, X_2\}$.

LEMMA 3.4. *The tensor* $\nabla^\perp g^\perp$ *is symmetric if and only if equations* (3.6) *and* (3.7) *are true for an arbitrary null holomorphic tangent frame* $\{X_1, X_2\}$ *and the associated frame* $\{\xi_1, \xi_2\}$.

PROOF. Let $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ be two null holomorphic tangent frames, $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ the corresponding transversal frames spanning an equiaffine bundle σ. Let equations (3.6) and (3.7) hold for $\{X_1, X_2\}$. Then by Lemma 3.3 equations (3.8) and (3.9) are also true. We will show that they are true for $\{Y_1, Y_2\}$. Using Lemma 2.2, we have

$$\nabla g(Y_1, Y_1, Y_1) = \nabla g(aX_j, aX_j, aX_j) = a^3 \nabla g(X_j, X_j, X_j) = 0,$$
$$\nabla g(Y_2, Y_2, Y_2) = \nabla g(a^{-1}X_k, a^{-1}X_k, a^{-1}X_k) = a^{-3} \nabla g(X_k, X_k, X_k) = 0,$$

for some $j, k \in \{1, 2\}$ such that $j \neq k$. The above equalities are true, because ∇g is 3-linear over complex-valued functions if it acts on holomorphic vector fields. This completes the proof. □

We shall prove a theorem about the existence of a unique transversal bundle over M.

THEOREM 3.5. *There is a unique equiaffine transversal bundle* σ *over a complex, non-degenerate surface* M *in* $\mathbf{C}^4$ *with symmetric* $\nabla^\perp g^\perp$, *where* $g^\perp$ *is an arbitrary transversal metric.*

PROOF. Let $\tilde{\sigma}$ be an arbitrary transversal bundle. Take $\{X_1, X_2\}$ – a null holomorphic tangent frame and $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ – the associated transversal frame. Then, by Lemma 2.4, for another transversal bundle σ we have

$$\xi_1 = \tilde{\xi}_1 + Z_1,$$
$$\xi_2 = \tilde{\xi}_2 + Z_2,$$

where $\{\xi_1, \xi_2\}$ is the transversal frame associated with $\{X_1, X_2\}$ in σ and Z_1, Z_2 are tangent vector fields which we can decompose as follows:

$$Z_1 = aX_1 + bX_2, \tag{3.10}$$

$$Z_2 = cX_1 + dX_2. \tag{3.11}$$

By Lemma 3.1 and Lemma 3.2 (3), σ satisfies the requirements of the theorem if and only if

$$\begin{aligned} &\nabla g(X_1, X_2, X_1) = 0,\\ &\nabla g(X_2, X_1, X_2) = 0,\\ &\nabla g(X_1, X_1, X_1) = 0,\\ &\nabla g(X_2, X_2, X_2) = 0, \end{aligned} \tag{3.12}$$

where ∇ is induced by σ. By (1.14), this is equivalent to

$$\begin{aligned} &- g(\tilde{\nabla}_{X_1} X_2, X_1) - g(X_2, \tilde{\nabla}_{X_1} X_1) + g(X_2, Z_1) = 0,\\ &- g(\tilde{\nabla}_{X_2} X_1, X_2) - g(X_1, \tilde{\nabla}_{X_2} X_2) + g(X_1, Z_2) = 0,\\ &- 2g(\tilde{\nabla}_{X_1} X_1, X_1) + 2g(Z_1, X_1) = 0,\\ &- 2g(\tilde{\nabla}_{X_2} X_2, X_2) + 2g(Z_2, X_2) = 0, \end{aligned}$$

where $\tilde{\nabla}$ is the connection induced by $\tilde{\sigma}$. Since $\{X_1, X_2\}$ is a null frame with respect to the metric g, we compute the functions a, b, c, d using (3.10) and (3.11) as follows:

$$\begin{aligned} &\tilde{\nabla} g(X_1, X_2, X_1) = -a,\\ &\tilde{\nabla} g(X_2, X_1, X_2) = -d,\\ &\tilde{\nabla} g(X_1, X_1, X_1) = -2b,\\ &\tilde{\nabla} g(X_2, X_2, X_2) = -2c. \end{aligned} \tag{3.13}$$

Thus we have determined the bundle σ uniquely, because if $\tilde{\sigma}$ satisfies (3.12) for $\tilde{\nabla}$, then $a = b = c = d = 0$, which implies $Z_1 = Z_2 = 0$ and, in consequence, $\sigma = \tilde{\sigma}$. It is also easy to verify that the bundle σ does not depend on the choice of an affine metric g. The proof is now complete. □

We will call the bundle satisfying the above theorem the affine normal bundle induced by a given immersion.

COROLLARY 3.6. *A transversal bundle σ is the affine normal bundle if and only if the equations* (3.12) *hold, where ∇ is the connection induced by σ, $\{X_1, X_2\}$ – a holomorphic null tangent frame and g – an affine metric on the surface.*

Using Lemma 3.1 and the proof of Lemma 3.3 we get the following

Corollary 3.7. *Let σ be a transversal bundle, $\{X_1, X_2\}$ – a holomorphic null tangent frame and $\{\xi_1, \xi_2\}$ – the associated transversal frame. σ is the affine normal bundle if and only if*

$$\tau_1^1 + \tau_2^2 = 0,$$
$$\tau_1^2(X_1) = 0,$$
$$\tau_2^1(X_2) = 0.$$

The following theorem shows that the unique transversal bundle has to be holomorphic. This fact is analogous to the codimension-one case.

Theorem 3.8. *The equiaffine normal bundle over a complex surface in $\mathbf{C}^4$ is holomorphic.*

Proof. We choose a holomorphic null tangent frame$\{X_1, X_2\}$. Let $\{\xi_1, \xi_2\}$ be the associated normal frame in the normal bundle σ. We prove that the forms τ_k^j and the shape operators induced by the normal bundle are complex (compare the introduction). From the Codazzi equation (1.4) (where X is an arbitrary vector field) we get the following equations:

$$\begin{aligned} &- 2h^1(\nabla_X X_1, X_1) + \tau_1^1(X) \\ &= X_1\big(h^1(X, X_1)\big) - h^1(\nabla_{X_1} X, X_1) - h^1(X, \nabla_{X_1} X_1) \\ &+ \tau_1^1(X_1)h^1(X, X_1) + \tau_1^2(X_1)h^2(X, X_1) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &- 2h^1(\nabla_{JX} X_1, X_1) + \tau_1^1(JX) \\ &= X_1\big(h^1(JX, X_1)\big) - h^1(\nabla_{X_1} JX, X_1) - h^1(JX, \nabla_{X_1} X_1) \\ &+ \tau_1^1(X_1)h^1(JX, X_1) + \tau_1^2(X_1)h^2(JX, X_1), \end{aligned} \tag{3.15}$$

which is equivalent to

$$\begin{aligned} &- 2ih^1(\nabla_X X_1, X_1) + \tau_1^1(JX) \\ &= i\big[X_1\big(h^1(X, X_1)\big)\big] - ih^1(\nabla_{X_1} X, X_1) - ih^1(X, \nabla_{X_1} X_1) \\ &+ i\tau_1^1(X_1)h^1(X, X_1) + i\tau_1^2(X_1)h^2(X, X_1), \end{aligned} \tag{3.16}$$

because ∇ is a complex connection, X_1 is holomorphic and h^j are $\mathbf{C}$-bilinear. Multiplying the equation (3.14) by i and comparing with (3.16) we obtain

$$\tau_1^1(JX) = i\tau_1^1(X).$$

Using the Codazzi equations (1.4) and (1.5) in an analogous way we prove similarly that

$$\tau_j^k(JX) = i\tau_j^k(X)$$

for $j, k \in \{1.2\}$. Using this fact and the fact that $[JX, Y] = J[X, Y]$ for holomorphic Y we can observe that

$$id\tau_j^k(X, Y) - d\tau_j^k(JX, Y) = \frac{1}{2}\big(iX\tau_j^k(Y) - (JX)\tau_j^k(Y)\big) \tag{3.17}$$

for arbitrary X and holomorphic Y. In the following only the vector field Y is holomorphic. Adding up the Ricci equations (1.8) and (1.10) and using the fact that $\tau_1^1 + \tau_2^2 = 0$, we obtain

$$\begin{aligned} &h^1(X, S_1Y) - h^1(Y, S_1X) + h^2(X, S_2Y) - h^2(Y, S_2X) \\ &\quad = \tau_2^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_2^1(X) + \tau_1^2(Y)\tau_2^1(X) - \tau_2^1(Y)\tau_1^2(X) = 0. \end{aligned} \tag{3.18}$$

Putting JX instead of X in this equation and comparing the result with (3.18) multiplied by i we get

$$h^1(Y, S_1(JX) - JS_1X) + h^2(Y, S_2(JX) - JS_2X) = 0. \tag{3.19}$$

Putting $Y = X_1$ in (3.19), we obtain

$$h^1(X_1, S_1(JX) - JS_1X) = 0,$$

which gives

$$S_1(JX) - JS_1X = \alpha(X)X_2 \tag{3.20}$$

for some $\alpha(X) \in \mathbf{C}$. Putting $Y = X_2$ in the equation (3.19), we get

$$h^2(X_2, S_2(JX) - JS_2X) = 0$$

and, in consequence,

$$S_2(JX) - JS_2X = \beta(X)X_1 \tag{3.21}$$

for $\beta(X) \in \mathbf{C}$. We now use the Ricci equation (1.9) and apply (3.17):

$$\begin{aligned} &h^2(JX, S_1Y) - h^2(Y, S_1(JX)) - ih^2(X, S_1Y) + ih^2(Y, S_1X) \\ &\quad = \frac{1}{2}\big(iX\tau_2^1(Y) - (JX)\tau_2^1(Y)\big). \end{aligned}$$

Putting $Y = X_2$ in this equation yields

$$h^2(X_2, S_1(JX) - JS_1X) = 0,$$

whence $\alpha(X) = 0$. Similarly, applying the Ricci equation (1.11) we get

$$h^1(X_1, S_2(JX) - JS_2X) = 0,$$

whence $\beta(X) = 0$. Thus both S_1 and S_2 are complex. This completes the proof. □

Remark. We could prove the last theorem more easily, using the construction of the associated normal frame (Lemma 2.2). Here we wanted to use only the conditions imposed on the affine normal bundle, as in Theorem 3.5.

Theorem 3.9. *Assume that there is an equiaffine plane bundle* $\tilde{\sigma}$ *such that* $\tilde{\nabla} g$ *is totally symmetric. Then*

1) *the normal bundle* σ *satisfies the condition* $\nabla g = 0$*;*
2) $\sigma = \tilde{\sigma}$ *if and only if* $\tilde{\nabla} g = 0$.

Proof. Let $u = \{X_1, X_2\}$ be a holomorphic null frame. Let $\{\xi_1, \xi_2\}$ and $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ be the corresponding transversal frames that span σ and $\tilde{\sigma}$, respectively. Let also

$$\xi_1 = \tilde{\xi}_1 + Z_1,$$
$$\xi_2 = \tilde{\xi}_2 + Z_2.$$

We have

$$\begin{aligned}
&\nabla g(X_1, X_2, X_2) = -2g(\nabla_{X_1} X_2, X_2) \\
&= -2g(\tilde{\nabla}_{X_1} X_2 + \tilde{h}^1(X_1, X_2) Z_1 + \tilde{h}^2(X_1, X_2) Z_2, X_2) \\
&= \tilde{\nabla} g(X_1, X_2, X_2) = \tilde{\nabla} g(X_2, X_1, X_2) \\
&= -2g(\tilde{\nabla}_{X_2} X_1 + \tilde{h}^1(X_2, X_1) Z_1 + \tilde{h}^2(X_2, X_1) Z_2, X_1) \\
&= -2g(\nabla_{X_2} X_1, X_2) = \nabla g(X_2, X_1, X_2) = 0
\end{aligned}$$

due to the properties of σ (Corollary 3.6). In the analogous way we obtain $\nabla g(X_2, X_1, X_1) = 0$. Taking into account also (3.12) we can see that $\nabla g = 0$. Then it suffices to prove the 'if' part of the second assertion. Let $\tilde{\sigma}$ satisfy $\tilde{\nabla} g = 0$. But, in particular, it satisfies (3.12) then, so it coincides with the affine normal bundle σ. It completes the proof of the lemma. □

4. Other canonically determined transversal bundles. In this section we adopt the constructions of Burstin, Mayer and Klingenberg to the complex case (see [**2**], [**4**]).

Let M be a nondegenerate complex surface in $\mathbf{C}^4$ and g be an affine metric defined locally on M. Let σ be a complex transversal bundle over M with the induced connection ∇. If $\hat{\nabla}$ defines the Levi-Civita connection for g, then we define a tensor K by the formula:

$$K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y, \tag{4.1}$$

which is symmetric because both connections are torsion-free. We see that the connection $\hat{\nabla}$ is independent of the choice of an affine metric g, then so is K. Let $\{X_1, X_2\}$ be a null holomorphic frame on M and $\{\xi_1, \xi_2\}$ – the associated transversal frame in σ. The Laplacian of the immersion x of the surface is now given by the following formula:

$$\Delta_g x = D_{X_1}X_2 + D_{X_2}X_1 - \hat{\nabla}_{X_1}X_2 - \hat{\nabla}_{X_2}X_1. \tag{4.2}$$

Using the Gauss formula and the tensor K as well as the form of h^j, we see that $\Delta_g x = 2K(X_1, X_2)$. For another affine metric $\bar{g} = cg$, we have $\Delta_{\bar{g}} x = c^2 \Delta_g x$.

We define a tensor η over $\mathbf{R}$, as follows:

$$\eta(X, Y) = D_X Y - \hat{\nabla}_X Y - \frac{1}{2} g(X, Y)\Delta_g x. \tag{4.3}$$

Let σ be a transversal bundle. Using K and a given transversal frame $\{\xi_1, \xi_2\}$, we write

$$\eta(X, Y) = K(X, Y) + h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2 - \frac{1}{2} g(X, Y)\Delta_g x. \tag{4.4}$$

We can see that η is also a tensor over $\mathbf{C}$. Since $\eta(X_1, X_2) = 0$, where $\{X_1, X_2\}$ is the frame associated with $\{\xi_1, \xi_2\}$, the image of η is spanned by two vectors $\eta(X_1, X_1)$ and $\eta(X_2, X_2)$. Moreover we notice that it does not depend on the choice of an affine metric g. We define the Burstin-Mayer transversal bundle as the transversal bundle spanned by the image of the tensor η. We denote it by σ_{BM}. Notice that if we use a frame of σ_{BM} in formula (4.4), we get

$$\nabla_X Y = \hat{\nabla}_X Y + \frac{1}{2} g(X, Y)\Delta_g x, \tag{4.5}$$

taking its tangent component and considering (4.1), where ∇ denotes the connection induced by the bundle σ_{BM}. It is a holomorphic connection because $\hat{\nabla}$ is holomorphic and the Laplacian is also a holomorphic vector field.

The following theorem shows when the Burstin–Mayer bundle is equiaffine.

THEOREM 4.1. *Let ∇ be the connection induced by the transversal bundle σ_{BM}, $\hat{\nabla}$ – the Levi-Civita connection for an affine metric g and ω_g – the complex volume element for g. Then the following conditions are equivalent:*

1) $\nabla = \hat{\nabla}$;
2) $\nabla \omega_g = 0$;
3) $\Delta_g x = 0$.

PROOF. Equation (4.5) shows that 1) and 3) are equivalent. Condition 1) implies that $\nabla g = 0$, whence $\nabla \omega_g = 0$ by its definition. Assume now that $\nabla \omega_g = 0$ and we will use the notation $K_X Y$ instead of $K(X, Y)$. We have thus $K_X \omega_g = 0$. But by equation (4.5) $K_X Y = \frac{1}{2} g(X, Y)\Delta_g x$, which implies

$$\begin{aligned}(K_X \omega_g)(Y, Z) =& X\omega_g(Y, Z) - X\omega_g(Y, Z) - \omega_g(K_X Y, Z) - \omega_g(Y, K_X Z) \\ =& \frac{1}{2} g(X, Y)\omega_g(\Delta_g x, Z) - \frac{1}{2} g(X, Z)\omega_g(\Delta_g x, Y).\end{aligned} \tag{4.6}$$

Taking a null frame $\{X_1, X_2\}$, put $X = Z = X_j$, $Y = X_k$ in (4.6), where $j, k \in \{1, 2\}$ and $j \neq k$. We get $\omega_g(\Delta_g x, X_j) = 0$ for $j = 1, 2$, but this means that $\Delta_g x = 0$. The proof is then completed. □

In the following we present the next construction following an idea of Klingenberg ([**4**]). We fix a transversal bundle σ and its local frame $\{\xi_1, \xi_2\}$. Then the cubic forms C_1 and C_2 are defined in the following way:

$$C_j(X, Y, Z) = (\nabla_X h^j)(Y, Z) + \tau_j^1(X) h^1(Y, Z) + \tau_j^2(X) h^2(Y, Z), \tag{4.7}$$

where h^j and τ_j^k are induced by the frame $\{\xi_1, \xi_2\}$.

Let $\{X_1, X_2\}$ be a null frame on M. We will show that there is exactly one transversal bundle σ such that the following equations are satisfied:

$$\begin{aligned} C_1(X_1, X_1, X_1) &= 0, \\ C_1(X_1, X_2, X_2) &= 0, \\ C_2(X_2, X_2, X_2) &= 0, \\ C_2(X_2, X_1, X_1) &= 0, \end{aligned} \tag{4.8}$$

where the cubic forms are defined with respect to the frame associated to $\{X_1, X_2\}$. First we notice that the above equations do not depend on the choice of an affine metric. Then we show that on an arbitrary nondegenerate surface there exists a desired bundle.

Lemma 4.2. *Let $\{X_1, X_2\}$ be a null frame on M. Then there exists exactly one transversal bundle σ such that equations* (4.8) *are satisfied, where the cubic forms are induced by the frame associated to $\{X_1, X_2\}$ in σ.*

Proof. Let σ, $\tilde{\sigma}$ be two arbitrary transversal bundles and $\{\xi_1, \xi_2\}$, $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ – their frames associated to $\{X_1, X_2\}$. By Lemma 2.4 there are such tangent vector fields Z_1 and Z_2 that $\xi_1 = \tilde{\xi}_1 + Z_1$ and $\xi_2 = \tilde{\xi}_2 + Z_2$. Using formulas (1.15) and (1.16) we get $\tilde{h}^j = h^j$ for $j = 1, 2$. Comparing $D_X \xi_j$ with $D_X \tilde{\xi}_j$, we obtain

$$\tau_j^k(X) = \tilde{\tau}_j^k(X) + \tilde{h}^j(X, Z_k)$$

for all j, k. The cubic forms induced by $\{\xi_1, \xi_2\}$ and $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ are related in the following way:

$$\begin{aligned} \tilde{C}_j(X_j, X_j, X_j) &= C_j(X_j, X_j, X_j) - h^j(X_j, Z_j), \\ \tilde{C}_j(X_j, X_k, X_k) &= C_j(X_j, X_k, X_k) - h^j(X_j, Z_k), \end{aligned} \tag{4.9}$$

where $j, k = 1, 2$ and $j \neq k$. If $Z_1 = aX_1 + bX_2$ and $Z_2 = cX_1 + dX_2$, then equations (4.9) give:

$$\begin{aligned}
\tilde{C}_1(X_1, X_1, X_1) + a &= C_1(X_1, X_1, X_1),\\
\tilde{C}_1(X_1, X_2, X_2) + c &= C_1(X_1, X_2, X_2),\\
\tilde{C}_2(X_2, X_2, X_2) + d &= C_2(X_2, X_2, X_2),\\
\tilde{C}_2(X_2, X_1, X_1) + b &= C_2(X_2, X_1, X_1).
\end{aligned}$$

Having the bundle $\tilde{\sigma}$, choose functions a, b, c, d such that the right hand sides of the above equations vanish. We see that the bundle σ satisfying these conditions is unique. This completes the proof of the lemma. □

Since the fundamental forms are **C**-linear and the cubic forms are symmetric in each two arguments, the cubic forms are also complex tensors. We prove that the transversal bundle constructed in the previous lemma does not depend on the choice of a null frame $\{X_1, X_2\}$.

LEMMA 4.3. *Let $\{X_1, X_2\}, \{\tilde{X}_1, \tilde{X}_2\}$ be two tangent null frames and $\{\xi_1, \xi_2\}$, $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ be the associated transversal frames in the same transversal bundle σ. If equations* (4.8) *are true for X_1, X_2 and the cubic forms induced by $\{\xi_1, \xi_2\}$, then they are true for $\{\tilde{X}_1, \tilde{X}_2\}$ and the cubic forms induced by $\{\tilde{\xi}_1, \tilde{\xi}_2\}$. Thus both tangent frames induce the same transversal bundle because of the unqueness in Lemma 4.2.*

PROOF. The null frames are related by $\tilde{X}_1 = \gamma X_1$, $\tilde{X}_2 = \gamma^{-1} X_2$ whereas the associated transversal frame satisfy the relations $\tilde{\xi}_1 = \gamma^2 \xi_1$ and $\tilde{\xi}_2 = \gamma^{-2}\xi_2$, where γ is a function. Formulas (1.15) and (1.16) imply that $\tilde{h}^1 = \gamma^{-2} h^1$ and $\tilde{h}^2 = \gamma^2 h^2$. Computing $D_X \tilde{\xi}_1$ and $D_X(\gamma^2 \xi_1)$ we get $\tilde{\tau}_1^1(X) = \tau_1^1(X) + 2\gamma^{-1} X(\gamma)$ and $\tilde{\tau}_2^1(X) = \gamma^4 \tau_2^1(X)$. Analogously, we get $\tilde{\tau}_1^2(X) = \gamma^{-4}\tau_1^2(X)$ and $\tilde{\tau}_2^2(X) = \tau_2^2(X) - 2\gamma^{-1}X(\gamma)$. Having $\tilde{\nabla} = \nabla$, we obtain

$$\begin{aligned}
\tilde{C}_1 &= \gamma^{-2} C_1,\\
\tilde{C}_2 &= \gamma^2 C_2.
\end{aligned}$$

Now we easily check that equations (4.8) are satisfied for $\tilde{C}_j$ and X_j for $j = 1, 2$. The proof is completed. □

The last two lemmas allow us to define the Klingenberg transversal bundle as the bundle satisfying Lemma 4.2 for any null frame $\{X_1, X_2\}$. We will denote this bundle by σ_K.

THEOREM 4.4. *Let ∇ denote the connection induced on a nondegenerate complex surface M in $\mathbf{C}^4$ by the Klingenberg transversal bundle σ_K. Consider* *the following conditions:*

1) $\nabla\omega_g = 0$;
2) $\nabla g = 0$;
3) $\Delta_g x = 0$.

Then 1) implies 2) and 2) implies 3). Moreover, condition 2) implies that $\sigma_K = \sigma_{BM}$.

Proof. First we prove the implication 1) to 2). Take a null frame $\{X_1, X_2\}$ and the associated transversal frame $\{\xi_1, \xi_2\}$. Let functions $a_1, ..., a_8$ satisfy the equalities (3.3) with respect to ∇ and $\{X_1, X_2\}$. By Lemma 3.1 and its proof we get that $a_1 + a_4 = 0$, $a_5 + a_8 = 0$ and $\tau_1^1 + \tau_2^2 = 0$. Equations (4.8) give

$$\begin{aligned} -2a_1 + \tau_1^1(X_1) &= 0, \\ \tau_1^2(X_1) &= 0, \\ -2a_8 + \tau_2^2(X_2) &= 0, \\ \tau_2^1(X_2) &= 0. \end{aligned}$$

Symmetry of the cubic forms leads to the following implications:
$C_1(X_2, X_1, X_2) = 0$ implies $a_7 = 0$,
$C_2(X_1, X_2, X_1) = 0$ implies $a_2 = 0$,
$C_1(X_1, X_2, X_1) = C_1(X_2, X_1, X_1)$ implies $-a_3 = -2a_5 + \tau_1^1(X_2)$,
$C_2(X_2, X_1, X_2) = C_2(X_1, X_2, X_2)$ implies $-a_6 = -2a_4 + \tau_2^2(X_1)$.

Combining the obtained equations, we get $a_3 = a_6 = 0$.
Since the bundle is equiaffine, equations (3.1) and (3.2) are satisfied. The proof of Lemma 3.3 shows that $a_2 = a_7 = 0$ imply equations $\nabla g(X_1, X_1, X_1) = 0$ and $\nabla g(X_2, X_2, X_2) = 0$. Moreover we have $\nabla g(X_2, X_1, X_1) = -2a_6 = 0$ and $\nabla g(X_1, X_2, X_2) = -2a_3 = 0$. Thus $\nabla g = 0$.

We then prove the implication from 2) to 3). Since $\nabla g = 0$ means that $\nabla = \hat{\nabla}$, and further, $K = 0$, we immediately obtain $\Delta_g x = 0$.

To prove the last statement assume that $\nabla g = 0$. Rewrite the definition of the tensor η, using σ_K as the transversal bundle. Then we have $\eta(X, Y) = h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2$, whence $\sigma_{BM} \subset \sigma_K$ which also implies the equality between the two bundles. The proof of the theorem has been completed. □

We see that the bundle σ_K is holomorphic because we can take a holomorphic transversal bundle and holomorphic null frames in its construction done in the proof of Lemma. As a result we get a holomorhic transversal frame spanning σ_K.

At the end we remark that there are nondegenerate surfaces that do not satisfy the condition $\Delta_g x = 0$. The surface $x(u, v) = (u, u^3, uv, uv^2)$ is an example. Thus the Klingenberg equiaffine transversal bundle does not exist on every nondegenerate complex surface in $\mathbf{C}^4$.

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