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ON RIEMANNIAN MANIFOLDS WHOSE TANGENT SPHERE BUNDLES CAN HAVE NONNEGATIVE SECTIONAL CURVATURE

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Abstract. The authors proved a theorem about the sectional curvature of tangent sphere bundles over locally symmetric Riemannian manifolds (see Theorem A below). After a slight generalization of this theorem (Theorem 1) we prove several results which give strong support of the conjecture that the converse of Theorem 1 also holds. The problem still remains open, in general.

1. Introduction. Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and let $(T_rM, \tilde{g})$ denote the tangent sphere bundle of radius $r > 0$ equipped with the induced Sasaki metric. We have started our study on the geometry of tangent sphere bundles $(T_rM, \tilde{g})$ in [**5**] with

Theorem A ([**5**]). *Let (M, g), $\dim M \geq 2$, be either locally symmetric with positive sectional curvature or locally flat. Then, for each sufficiently small positive number r, the tangent sphere bundle $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature.*

As a slight generalization of Theorem A we shall show

Theorem 1. *Let (M, g), $\dim M \geq 3$, be a Riemannian locally symmetric space with nonnegative sectional curvature. Then, for each sufficiently small*

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positive number $r > 0$, the tangent sphere bundle $(T_rM, \tilde{g})$ is a space of non-negative sectional curvature.

Under the hypothesis of Theorem 1, we can see easily from [**7**, Theorem 3.3] that $(T_rM, \tilde{g})$ is never a space of strictly positive sectional curvature. On the other hand, if (M, g) is a two-dimensional standard sphere, then $(T_rM, \tilde{g})$ is a space of positive sectional curvature according to the criterion by Yampolsky [**10**].

The natural problem now is the question whether the conclusion of Theorem 1 may also hold for Riemannian manifolds which are not locally symmetric. This paper does not definitely solve this problem but it gives some new evidence that the converse of Theorem 1 might hold, too.

The first step in this direction has been made in [**5**], where the following result was proved:

Theorem B ([**5**]). *There exist arbitrarily small perturbations of a spherical cap of the standard four-sphere with the following property: if (M, g) is such a perturbation, then $(T_rM, \tilde{g})$ admits negative sectional curvatures for every positive number r.*

Here we shall prove the following modification of Theorem B:

Theorem 2. *Let (M, g), $\dim M \geq 3$, be a Riemannian manifold and let x be a spherical point of M, i.e., such that all sectional curvatures at x are constant. Moreover, let the covariant derivative $(\nabla R)_x$ of the Riemannian curvature tensor R be nonzero. Then in any tangent sphere bundle $(T_rM, \tilde{g})$ over (M, g) there is a point (x, u), $u \in M_x$, such that the tangent space $(T_rM)_{(x,u)}$ admits a two-plane with negative sectional curvature.*

Corollary 3. *Let (M, g) be a Riemannian manifold such that the covariant derivative ∇R of the Riemannian curvature tensor R is nonzero everywhere. If, for some radius $r > 0$, the tangent sphere bundle $(T_rM, \tilde{g})$ has nonnegative sectional curvature, then (M, g) has no spherical points.*

We are now looking for the converse to Theorem 1. We shall first present a "nonstandard" converse of this Theorem.

Proposition 4. *Let (M, g), $\dim M \geq 3$, be a Riemannian manifold with non-negative sectional curvature and let $x \in M$ be a point such that the covariant derivative $(\nabla R)_x$ of the Riemannian curvature tensor R is nonzero. Then for every sufficiently large radius r, the tangent sphere bundle $(T_rM, \tilde{g})$ over (M, g) contains a point (x, u), $u \in M_x$, such that the tangent space $(T_rM)_{(x,u)}$ admits a two-plane with negative sectional curvature.*

Theorem 5. *Let (M, g), $\dim M \geq 3$, be a Riemannian manifold such that, for all sufficiently large radii $r > 0$, the tangent sphere bundles $(T_rM, \tilde{g})$ over*

(M,g) are spaces of nonnegative sectional curvature. Then the space (M,g) is locally symmetric.

Finally, we shall prove the true converse of Theorem 1, but still under an additional assumption. This assumption reads that either $\dim M = 3$, or $\dim M > 3$ and (M,g) is conformally flat.

THEOREM 6. *Let (M,g) be a Riemannian manifold such that the conformal Weyl tensor W vanishes (in particular, let $\dim M = 3$). If the tangent sphere bundle $(T_rM,\tilde{g})$ is a space of nonnegative sectional curvature for some radius $r > 0$, then (M,g) is locally symmetric.*

From this theorem we shall deduce the following

COROLLARY 7. *Let (M,g) be a Riemannian manifold of dimension n such that the conformal Weyl tensor W vanishes (in particular, let $\dim M = 3$). Then the tangent sphere bundle $(T_rM,\tilde{g})$ is a space of nonnegative sectional curvature for all sufficiently small radii $r > 0$ if, and only if, (M,g) is locally isometric to one of the following spaces:*

$$\mathbb{R}^n, \quad S^n(c), \quad \text{or} \quad S^{n-1}(c) \times \mathbb{R}^1,$$

where $\mathbb{R}^n$ is the Euclidean n-space and $S^n(c)$ is the n-sphere of radius $1/\sqrt{c}$.

The references in this paper will be limited to a necessary minimum. For more references concerning related topics, see [**1**].

2. Tangent sphere bundles — a short review. Let M be a smooth and connected manifold of dimension $n \geq 2$. Then the tangent bundle TM over M consists of all pairs (x,u), where x is a point of M and u is a vector from the tangent space M_x of M at x. We denote by p the natural projection of TM to M defined by $p(x,u) = x$.

Let g be a Riemannian metric on the manifold M and ∇ its Levi-Civita connection. Then the tangent space $(TM)_{(x,u)}$ of TM at (x,u) splits into the horizontal and vertical subspaces $H_{(x,u)}$ and $V_{(x,u)}$ with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For a vector $X \in M_x$, the *horizontal lift* of X to a point $(x,u) \in TM$ is the unique vector $X^h \in H_{(x,u)}$ such that $p_* X^h = X$. The *vertical lift* of X to (x,u) is the unique vector $X^v \in V_{(x,u)}$ such that $X^v(\mathrm{d}f) = Xf$ for all smooth functions f on M. Here we consider a 1-form $\mathrm{d}f$ on M as a function on TM. The map $X \longmapsto X^h$ is an isomorphism between M_x and $H_{(x,u)}$; and the map $X \longmapsto X^v$ is an isomorphism between M_x and $V_{(x,u)}$. In an obvious way we can define horizontal and vertical lifts of vector fields on M. These are uniquely defined vector fields on TM.

For each system of local coordinates $(x^1, x^2, \ldots, x^n)$ in M, one defines, in the standard way, the system of local coordinates $(x^1, x^2, \ldots, x^n; u^1, u^2, \ldots, u^n)$

in TM. The *canonical vertical vector field* on TM is a vector field $\mathcal{U}$ defined, in terms of local coordinates, by $\mathcal{U} = \sum_i u^i \partial/\partial u^i$. Here $\mathcal{U}$ does not depend on the choice of local coordinates and it is defined globally on TM. For a vector $u = \sum_i u^i (\partial/\partial x^i)_x \in M_x$, we see that $u^h_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^h_{(x,u)}$ and $u^v_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^v_{(x,u)} = \mathcal{U}_{(x,u)}$.

The *Sasaki metric* on the tangent bundle TM of a Riemannian manifold (M, g) is determined, at each point $(x, u) \in TM$, by the formulas

$$\begin{cases} \bar{g}_{(x,u)}(X^h, Y^h) = g_x(X, Y), \\ \bar{g}_{(x,u)}(X^h, Y^v) = 0, \\ \bar{g}_{(x,u)}(X^v, Y^v) = g_x(X, Y), \end{cases} \tag{2.1}$$

where X and Y are arbitrary vectors from M_x.
Evidently, we have $\bar{g}_{(x,u)}(X^h, \mathcal{U}) = 0$ and $\bar{g}_{(x,u)}(X^v, \mathcal{U}) = g_x(X, u)$. Let $\bar{\nabla}$ be the Levi-Civita connection of $(TM, \bar{g})$, and let X and Y be vector fields on M, then we have at each *fixed* point $(x, u) \in TM$,

$$\begin{cases} (\bar{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} - \dfrac{1}{2} \left(R_x(X, Y)u\right)^v, \\ (\bar{\nabla}_{X^h} Y^v)_{(x,u)} = \dfrac{1}{2} \left(R_x(u, Y)X\right)^h + (\nabla_X Y)^v_{(x,u)}, \\ (\bar{\nabla}_{X^v} Y^h)_{(x,u)} = \dfrac{1}{2} \left(R_x(u, X)Y\right)^h, \\ (\bar{\nabla}_{X^v} Y^v)_{(x,u)} = 0, \end{cases} \tag{2.2}$$

where R is the Riemannian curvature tensor of (M, g) defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. As concerns the canonical vertical vector field $\mathcal{U}$, we have

$$\begin{cases} \bar{\nabla}_{X^h} \mathcal{U} = 0, \quad \bar{\nabla}_{X^v} \mathcal{U} = X^v, \\ \bar{\nabla}_{\mathcal{U}} X^h = 0, \quad \bar{\nabla}_{\mathcal{U}} X^v = 0, \\ \bar{\nabla}_{\mathcal{U}} \mathcal{U} = \mathcal{U} \end{cases} \tag{2.3}$$

for each vector field X on M.

Let r be a positive number. Then the *tangent sphere bundle of radius* r over a Riemannian manifold (M, g) is the hypersurface $T_r M = \{(x, u) \in TM \mid g_x(u, u) = r^2\}$. The canonical vertical vector field $\mathcal{U}$ is normal to $T_r M$ in $(TM, \bar{g})$ at each point $(x, u) \in T_r M$. Also, $\bar{g}(\mathcal{U}, \mathcal{U}) = r^2$ along $T_r M$. For any vector field X tangent to M, the horizontal lift X^h is always tangent to $T_r M$ at each point $(x, u) \in T_r M$. Yet, in general, the vertical lift X^v is not tangent to $T_r M$ at (x, u). The *tangential lift* of X (see [**1**]) is a vector field X^t tangent to $T_r M$ and defined by

$$X^t = X^v - \frac{1}{r^2} \bar{g}(X^v, \mathcal{U}) \mathcal{U}.$$

Thus, at each point $(x,u)\in T_rM$, we have

$$X^t_{(x,u)} = X^v_{(x,u)} - \frac{1}{r^2} g_x(X,u)\mathcal{U}_{(x,u)}.$$

Now we endow the hypersurface $T_rM \subset (TM,\bar g)$ with the induced Riemannian metric $\tilde g$, which is uniquely determined by the formulas

$$\left\{\begin{array}{l} \tilde g(X^h,Y^h) = \bar g(X^h,Y^h),\\ \tilde g(X^h,Y^t) = 0,\\ \tilde g(X^t,Y^t) = \bar g(X^v,Y^v) - \frac{1}{r^2}\bar g(X^v,\mathcal{U})\bar g(Y^v,\mathcal{U}), \end{array}\right. \tag{2.4}$$

where X and Y are arbitrary vector fields on M. In the following we shall use the symbol $\langle\cdot,\cdot\rangle$ for the scalar product g_x on M_x. Then (2.4) can be rewritten, at each fixed point $(x,u)\in T_rM$, in the form

$$\left\{\begin{array}{l} \tilde g_{(x,u)}(X^h,Y^h) = \langle X,Y\rangle,\\ \tilde g_{(x,u)}(X^h,Y^t) = 0,\\ \tilde g_{(x,u)}(X^t,Y^t) = \langle X,Y\rangle - \frac{1}{r^2}\langle X,u\rangle\langle Y,u\rangle, \end{array}\right. \tag{2.5}$$

where X and Y are arbitrary vectors from M_x.

We notice that $u^t_{(x,u)} = 0$ for $(x,u)\in T_rM$ and hence the tangent space $(T_rM)_{(x,u)}$ coincides with the set $\{X^h + Y^t | X\in M_x, Y\in\{u\}^\perp\subset M_x\}$.

In [**5**] all basic formulas for the *curvature operators* on the tangent sphere bundle T_rM have been derived by calculating first the shape operator and then using the Gauss equation. We shall not reproduce them here.

It is obvious that each tangent two-plane $\tilde P\subset (T_rM)_{(x,u)}$ is spanned by an orthonormal basis of the form $\{{X_1}^h + {Y_1}^t, {X_2}^h + {Y_2}^t\}$. For such a basis we have $\|X_i\|^2 + \|Y_i\|^2 = 1$, $i=1,2$, and $\langle X_1,X_2\rangle + \langle Y_1,Y_2\rangle = 0$. Moreover, we can assume $\langle X_1,X_2\rangle = \langle Y_1,Y_2\rangle = 0$. This can be realized easily by a convenient rotation of the given basis. As usual, Y_1 and Y_2 are supposed to be orthogonal to u. From the formulas for the curvature operators one obtains as in [**5**] the following formula for the sectional curvature of the two-plane $\tilde P$:

$$\begin{aligned} \tilde K(\tilde P) = {}& \langle R_x(X_1,X_2)X_2,X_1\rangle + 3\langle R_x(X_1,X_2)Y_2,Y_1\rangle + \frac{1}{r^2}\|Y_1\|^2\|Y_2\|^2\\ & - \frac{3}{4}\|R_x(X_1,X_2)u\|^2 + \frac{1}{4}\|R_x(u,Y_2)X_1\|^2 + \frac{1}{4}\|R_x(u,Y_1)X_2\|^2\\ & + \frac{1}{2}\langle R_x(u,Y_1)X_2, R_x(u,Y_2)X_1\rangle - \langle R_x(u,Y_1)X_1, R_x(u,Y_2)X_2\rangle\\ & + \langle(\nabla_{X_1}R)_x(u,Y_2)X_2,X_1\rangle + \langle(\nabla_{X_2}R)_x(u,Y_1)X_1,X_2\rangle. \end{aligned} \tag{2.6}$$

Now, there are orthonormal pairs $\{\hat{X}_1, \hat{X}_2\}$ and $\{\hat{Y}_1, \hat{Y}_2\}$, and angles $\alpha, \beta \in [0, \pi/2]$ such that

$$\begin{cases} X_1 = \cos\alpha\,\hat{X}_1, & Y_1 = \sin\alpha\,\hat{Y}_1; \\ X_2 = \cos\beta\,\hat{X}_2, & Y_2 = \sin\beta\,\hat{Y}_2. \end{cases}$$

We also put $\hat{u} = u/\|u\| = u/r$. This notation will be used in the sequel.

3. The proof of main results.

Proof of Theorem 1. Because (M, g) is locally isometric to a globally symmetric space and because the statement of the Theorem is purely local, we can assume that (M, g) itself is globally symmetric and simply connected. Then we have the de Rham decomposition

$$(M, g) = (M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_s, g_s),$$

where (M_0, g_0) is the Euclidean part and all (M_i, g_i) for $i = 1, 2, \ldots, s$ are irreducible symmetric spaces of compact type.

Fix a point $x = (x_0, x_1, \ldots x_s) \in M$ and denote by $N_i = M_i \times \{(x_0, \ldots, \widehat{x_i}, \ldots, x_s)\}$, $i = 0, 1, \ldots, s$, the corresponding leaf in M, where the symbol $\widehat{x_i}$ indicates that the component x_i is omitted. Let us recall that, if U, V and W are vectors tangent to the leaves N_i at x, and if at least two of them are tangent to different leaves, then $R_x(U, V)W = 0$. Also recall that if W is tangent to some leaf N_j, $j = 1, 2, \ldots, s$, then, for any choice of tangent vectors U and V at x, the vector $R_x(U, V)W$ is either a null vector or it is tangent to the leaf N_j, as well. Finally, recall that the tangent spaces to the leaves form an orthogonal decomposition of the tangent space M_x.

Now, consider an orthonormal pair $\{\hat{X}_1, \hat{X}_2\}$ in M_x. If both $\hat{X}_1$ and $\hat{X}_2$ are tangent to N_0, then we see at once from formula (2.6) that $\tilde{K}(\tilde{P}) \geq 0$ for any two-plane $\tilde{P}$ defined in the last part of Section 2. If $\hat{X}_1$ and $\hat{X}_2$ are tangent to an irreducible factor N_i, $i = 1, 2, \ldots, s$, then we have $K(\hat{X}_1 \wedge \hat{X}_2) \geq \delta_i > 0$ where δ_i is the minimum of sectional curvature on (M_i, g_i). Now we can use the same argument as in the proof of Theorem 4 in [**5**] (*i.e.,* Theorem A in this paper) to show that $\tilde{K}(\tilde{P}) \geq 0$ holds for every choice of an orthonormal triplet $\{\hat{Y}_1, \hat{Y}_2, \hat{u}\}$ in M_x and for all radii $r > 0$ such that $r \leq r_i$, where $r_i > 0$ depends only on the geometry of (M, g). Finally, let $\hat{X}_1$ and $\hat{X}_2$ be tangent to two different leaves N_i and N_j, $i \neq j$; then $R_x(\hat{X}_1, \hat{X}_2) = 0$. Moreover $R_x(U, V)X_1$ and $R_x(U, V)X_2$ are tangent to the leaves N_i and N_j, respectively, for any choice of $U, V \in M_x$. Hence, for every choice of an orthonormal triplet $\{\hat{Y}_1, \hat{Y}_2, \hat{u}\}$ in M_x, the right-hand side of formula (2.6) reduces to three terms, which are all nonnegative.

This completes the proof. □

We start the *proof* of Theorem 2 with an algebraic lemma:

LEMMA 8. *Let x be a fixed point of a Riemannian manifold (M, g). Then either there is an orthonormal triplet $\{X, Y, Z\}$ of M_x such that $\langle(\nabla_X R)_x(X, Y)Y, Z\rangle \neq 0$ or $(\nabla R)_x = 0$ identically.*

PROOF. Let us denote, for the sake of brevity, $\langle(\nabla_X R)_x(Y, Z)U, V\rangle$ by $B(Y, Z, U, V; X)$.

Suppose that

$$B(X, Y, Y, Z; X) = 0 \tag{3.1}$$

for all orthonormal triplets $\{X, Y, Z\}$. Then the second Bianchi identity applied to the last three arguments gives

$$B(X, Y, Y, X; Z) = 0 \tag{3.2}$$

for all orthonormal triplets $\{X, Y, Z\}$.

Further, we get

$$B(X, Y, U, Z; X) = 0 \tag{3.3}$$

for each orthonormal quadruplet $\{X, Y, Z, U\}$. Indeed, we have

$$B(X, Y + U, Y + U, Z; X) = 0.$$

Also because $B(X, Y, Y, Z; X) = 0$ and $B(X, U, U, Z; X) = 0$, we get

$$B(X, Y, U, Z; X) + B(X, U, Y, Z; X) = 0.$$

If we apply the first Bianchi identity to the first three arguments in the second term, we get, by the standard symmetries of B, that

$$2B(X, Y, U, Z; X) = B(U, Y, X, Z; X). \tag{3.4}$$

After the transposition between Y and Z we get, by the symmetry of B, that

$$2B(U, Y, X, Z; X) = B(X, Y, U, Z; X).$$

Hence and from (3.4) we get

$$B(U, Y, X, Z; X) = B(X, Y, U, Z; X) = 0, \tag{3.5}$$

which is equivalent to (3.3).

Now, from (3.2) we get $B(X, Y + U, Y + U, X; Z) = 0$ and, by the standard symmetries of B, we get finally

$$B(X, Y, U, X; Z) = 0 \tag{3.6}$$

for each orthonormal quadruplet $\{X, Y, Z, U\}$.

Let us consider now an orthonormal quintuplet $\{X, Y, Z, U, V\}$. First we have from (3.3) that $B(X + V, U, Y, Z; X + V) = 0$ and then

$$B(X, U, Y, Z; V) + B(V, U, Y, Z; X) = 0,$$

which can be rewritten as

$$B(X, U, Y, Z; V) + B(Y, Z, V, U; X) = 0.$$

Applying the second Bianchi identity to the second term, we obtain

$$2B(X, U, Y, Z; V) = B(Y, Z, X, V; U).$$

After the transposition between U and V we hence get

$$2B(X, V, Y, Z; U) = B(Y, Z, X, U; V),$$

and from the last two equalities we have

$$B(X, V, Y, Z; U) = B(Y, Z, X, U; V) = 0.$$

Finally, we obtain

$$B(X, Y, Z, U; V) = 0 \tag{3.7}$$

for any orthonormal quintuplet $\{X, Y, Z, U, V\}$.

It remains to show that

$$B(X, Y, X, Y; X) = 0$$

holds for any orthonormal pair $\{X, Y\}$. First, from (3.2) we obtain, for each orthonormal triplet $\{X, Y, U\}$ and for each α,

$$B(\sin\alpha\, X + \cos\alpha\, U, Y, \sin\alpha\, X + \cos\alpha\, U, Y; \cos\alpha\, X - \sin\alpha\, U) = 0,$$

which implies, due to (3.1) and (3.2),

$$\cos\alpha \sin^2\alpha\, B(X, Y, X, Y; X) - \cos^2\alpha \sin\alpha\, B(U, Y, U, Y; U) = 0. \tag{3.8}$$

Now the conclusion follows from (3.8).

This shows that B is a null tensor. □

The *proofs* of Theorem 2 and of Proposition 4 generalize the idea from the proof of Theorem B. First we set as follows: Because $(\nabla R)_x$ is nonzero, then according to the Lemma 8 there is an orthonormal triplet $\{Z_1, Z_2, Z_3\}$ in the tangent space M_x such that $b = \langle (\nabla_{Z_1} R)_x(Z_2, Z_3)Z_2, Z_1 \rangle > 0$. We put

$$X_1 = Z_1,\ Y_1 = 0,\ X_2 = \cos\beta\, Z_2,\ Y_2 = -\sin\beta\, Z_3,\ u = rZ_2,$$

and consider the point $(x, u) \in T_rM$, where $r > 0$ and $\beta \in (0, \pi/2)$ are not specified yet. Further, we put $c = K(Z_1 \wedge Z_2) > 0$. In the proofs we shall estimate the values of the sectional curvature $\tilde{K}(\tilde{P})$ of the tangent two-plane $\tilde{P}$ spanned by ${X_1}^h$ and ${X_2}^h + {Y_2}^t$ in $(T_rM)_{(x,u)}$.

Proof of Theorem 2. Since $x \in M$ is a spherical point, we have $\|R_x(X_1, X_2)u\| = cr\cos\beta$ and $R_x(u, Y_2)X_1 = 0$. Thus, from (2.6), we obtain

$$\tilde{K}(\tilde{P}) = \cos\beta\Big(c\cos\beta - \frac{3}{4}c^2r^2\cos\beta - br\,\sin\beta\Big),$$

which becomes negative for $\beta \in (0,\, \pi/2)$ tending to $\pi/2$. $\square$

Proof of Proposition 4. We write $R_x(Z_1, Z_2)Z_2 = c\, Z_1 + W$, where $W \in M_x$ is orthogonal to Z_1. Hence, putting $C = \|R_x(Z_1, Z_2)Z_2\|$, we get $C \geq c > 0$. Put $D = \|R_x(Z_2, Z_3)Z_1\| \geq 0$. Now, from (2.6), we obtain

$$\tilde{K}(\tilde{P}) = r\, \sin\beta\Big(\frac{1}{4} r D^2 \sin\beta - b\cos\beta\Big) + \cos^2\beta\Big(c - \frac{3}{4}C^2 r^2\Big).$$

The second term is zero for $C = 0$ and every $r > 0$; and it is nonpositive for $C > 0$ and for every $r \geq 2\sqrt{c}/\sqrt{3}\, C$. Let us fix a number $r > 0$ for which this second term is nonpositive. The first term is then negative for all $\beta \in (0, \pi/2)$ such that $\operatorname{ctg}\beta > (1/4) r D^2/b$. Thus a two-plane at $(x, u) \in T_r M$ with negative sectional curvature exists. $\square$

Proof of Corollary 5. Because $(T_r M, \tilde{g})$ is a space of nonnegative sectional curvature, we see that, putting $Y_1 = Y_2 = 0$ in (2.6), (M, g) is also a space of nonnegative sectional curvature. Hence, by Proposition 4, (M, g) is locally symmetric. $\square$

The *proof* of Theorem 6 is based on the following two lemmas. The first one is obvious:

Lemma 9. *Let (M, g), $\dim M \geq 3$, be a Riemannian manifold such that the conformal Weyl tensor W vanishes. Let $\{E_1, E_2, \ldots, E_n\}$ be a basis of M_x which diagonalizes the Ricci tensor Ric_x. Then $R_x(E_i, E_j)E_k = 0$ for every triplet of distinct indices $\{i, j, k\}$.*

Lemma 10. *Let x be a fixed point of a Riemannian manifold (M, g), $\dim M \geq 3$, such that the conformal Weyl tensor W vanishes and let $\langle(\nabla_X R)_x(X, Z)Y, Z\rangle = 0$ holds whenever $\{X, Y, Z\}$ is an orthonormal triplet in M_x such that $R_x(X, Y)Z = 0$. Then $(\nabla R)_x = 0$ identically.*

Proof. We again denote $\langle(\nabla_X R)_x(Y, Z)U, V\rangle$ by $B(Y, Z, U, V; X)$. We also denote by $\{E_1, E_2, \ldots, E_n\}$ a basis of M_x which diagonalizes the Ricci tensor Ric_x.

First, by Lemma 9 and by the assumption of Lemma 10, we have

$$B(E_i, E_k, E_j, E_k; E_i) = 0$$

for each triplet of distinct indices $\{i, j, k\}$. Next we have, according to Lemma 9,

$$R_x(\sin\alpha\, E_i + \cos\alpha\, E_j, \cos\alpha\, E_i - \sin\alpha\, E_j)E_k = 0$$

for each triplet of distinct indices $\{i, j, k\}$ and each α. Hence we get

$$B(\sin\alpha\, E_i + \cos\alpha\, E_j, E_k, \cos\alpha\, E_i - \sin\alpha\, E_j, E_k; \sin\alpha\, E_i + \cos\alpha\, E_j) = 0$$

for each α. Now, we have

$$B(E_j, E_k, E_i, E_k; E_j) = 0 \quad \text{and} \quad B(E_i, E_k, E_j, E_k; E_i) = 0.$$

From the second Bianchi identity we also have

$$B(E_i, E_k, E_i, E_k; E_j) = B(E_j, E_k, E_j, E_k; E_i) = 0.$$

Now, a simple computation gives

$$\sin^2\alpha\, \cos\alpha\, B(E_i, E_k, E_i, E_k; E_i) - \sin\alpha\, \cos^2\alpha\, B(E_j, E_k, E_j, E_k; E_j) = 0$$

for each α and hence $B(E_i, E_k, E_i, E_k; E_i) = B(E_j, E_k, E_j, E_k; E_j) = 0$. This means that $B(E_i, E_j, E_i, E_j; E_i) = 0$ for each pair of indices i and j.

Further, if $\dim M \geq 4$ and i, j, k, l are distinct indices, we have

$$R_x(E_i, E_j)(E_k + E_l) = 0$$

and hence

$$B(E_i, E_k + E_l, E_j, E_k + E_l; E_i) = 0.$$

This implies

$$B(E_i, E_k, E_j, E_l; E_i) + B(E_i, E_l, E_j, E_k; E_i) = 0,$$

and applying the first Bianchi identity to the middle arguments in the second term, we easily obtain

$$2B(E_i, E_k, E_j, E_l; E_i) = B(E_i, E_j, E_k, E_l; E_i).$$

Interchanging the indices j and k, we get an analogous equality and hence we conclude that $B(E_i, E_j, E_k, E_l; E_i) = 0$ for each distinct indices i, j, k, l. Applying the second Bianchi identity to the last three arguments and using also the standard symmetries, we obtain

$$B(E_k, E_i, E_j, E_i; E_l) + B(E_l, E_i, E_i, E_j; E_k) = 0. \tag{3.9}$$

Now, because $R_x(E_k + E_l, E_j)E_i = 0$ holds for all distinct i, j, k, l, we obtain $B(E_k + E_l, E_i, E_j, E_i; E_k + E_l) = 0$. Omitting the terms which vanish identically, we get

$$B(E_k, E_i, E_j, E_i; E_l) + B(E_l, E_i, E_j, E_i; E_k) = 0. \tag{3.10}$$

Equations (3.9) and (3.10) now imply that $B(E_i, E_j, E_i, E_k; E_l) = 0$ for all distinct i, j, k, l.

Let now $\dim M \geq 5$ and let i, j, k, l, m be five distinct indices. Because

$$R_x(E_i + E_m, E_j)(E_k + E_l) = 0,$$

we get

$$B(E_i + E_m, E_k + E_l, E_j, E_k + E_l; E_i + E_m) = 0.$$

Omitting the terms which vanish according to the previous equalities, we get

$$\begin{aligned} &B(E_i, E_k, E_j, E_l; E_m) + B(E_i, E_l, E_j, E_k; E_m) \\ &\quad + B(E_m, E_k, E_j, E_l; E_i) + B(E_m, E_l, E_j, E_k; E_i) = 0. \end{aligned}$$

Now, applying the first Bianchi identity to the middle arguments in the second and the last term, we get

$$\begin{aligned} 2B(E_i, E_k, E_j, E_l; E_m) &- B(E_i, E_j, E_k, E_l; E_m) \\ &+ 2B(E_m, E_k, E_j, E_l; E_i) - B(E_m, E_j, E_k, E_l; E_i) = 0. \end{aligned} \tag{3.11}$$

Interchanging the indices j and k, we hence get

$$\begin{aligned} 2B(E_i, E_j, E_k, E_l; E_m) &- B(E_i, E_k, E_j, E_l; E_m) \\ &+ 2B(E_m, E_j, E_k, E_l; E_i) - B(E_m, E_k, E_j, E_l; E_i) = 0. \end{aligned} \tag{3.12}$$

Now, adding twice the second equality (3.12) to the first equality (3.11), we get by the standard symmetry of B:

$$B(E_k, E_l, E_i, E_j; E_m) + B(E_k, E_l, E_m, E_j; E_i) = 0.$$

Applying the second Bianchi identity to the last three arguments in the second term, we get

$$2B(E_k, E_l, E_i, E_j; E_m) - B(E_k, E_l, E_i, E_m; E_j) = 0.$$

Interchanging the indices j and m, we obtain a new equality and then we finally get

$$B(E_k, E_l, E_i, E_j; E_m) = B(E_k, E_l, E_i, E_m; E_j) = 0.$$

From all this we may conclude that $B(E_i, E_j, E_k, E_l; E_m) = 0$ for any indices i, j, k, l, m and hence $B = 0$, as required. □

Proof of Theorem 6. Let us suppose that the space (M, g) is not locally symmetric. Then, at some point $x \in M$ we have $(\nabla R)_x \neq 0$. According to Lemma 10, there is an orthonormal triplet $\{Z_1, Z_2, Z_3\}$ in M_x such that $\langle (\nabla_{Z_1} R)_x(Z_1, Z_2)Z_2, Z_3) \rangle > 0$ and, at the same time, $R_x(Z_1, Z_2)Z_3 = 0$. Then, using the same procedure as in the proof of Theorem 2, we find for every $r > 0$ a tangent two-plane of T_rM with negative sectional curvature, which is a contradiction. □

In the *proof* of Corollary 7 we shall use the following theorem by Takagi [**9**].

Theorem C ([**9**]). *Let (M, g) be a connected conformally flat Riemannian homogeneous manifold of dimension n. Then (M, g) is locally isometric to*

$$M^n(c), \quad \textit{or} \quad M^s(c) \times M^{n-s}(-c)\ (2 \leq s \leq n-2), \quad \textit{or} \quad M^{n-1}(c) \times \mathbb{R}^1,$$

where $M^n(c)$ is an n-dimensional space of constant curvature $c \neq 0$ and $\mathbb{R}^1$ is the Euclidean 1-space.

Proof of Corollary 7. If $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature for every sufficiently small radius $r > 0$, then, by Theorem 6, (M, g) is locally symmetric and hence locally isometric to a symmetric space, which is globally homogeneous. Hence, for $n > 3$, the result follows from Theorem C.

For $n = 3$, the only simply connected symmetric spaces with nonnegative sectional curvature are $\mathbb{R}^3$, $S^3(c)$ and $S^2(c) \times \mathbb{R}^1$.

The "only if" part follows from Theorem 1. □

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