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EXISTENCE AND UNIQUENESS OF SOLUTIONS CAUCHY PROBLEM FOR NONLINEAR INFINITE SYSTEMS OF PARABOLIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

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Abstract. We consider the Cauchy problem for an infinite system of weakly coupled nonlinear differential-functional equations of parabolic type. The right-hand sides of the system are functionals of unknown functions and these system are thus essentially coupled by the functional argument. To prove the existence and uniqueness of the solution to this problem, we shall apply the well-known Banach fixed point theorem.

1. Introduction. We consider an infinite system of weakly coupled nonlinear differential-functional equations of the form

$$\mathcal{F}^i[u^i](t,x) = f^i(t,x,u),\ \ i \in S, \tag{1}$$

where

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i,\ \mathcal{A}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\frac{\partial^2}{\partial x_j \partial x_k},$$

$(t,x) \in \Omega := \{(t,x) : t > 0, x \in R^m\}$, $\overline{\Omega} := \{(t,x) : t \geq 0, x \in R^m\}$, S is a set of indices (finite or countable) and u stands for the mapping

$$u : S \times \overline{\Omega} \ni (i,t,x) \to u^i(t,x) \in R,$$

composed of unknown functions u^i.

Let $B(S)$ be the space of mappings

$$v : S \ni i \to v^i \in R,$$

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with the finite norm

$$\|v\|_{B(S)} := \sup\{|v^i| : i \in S\}.$$

Remark 1. $\Big(B(S),\ \|\cdot\|_{B(S)}\Big)$ is a Banach space.

In the case of finite systems we have $B(S) = R^r$ and in case of an infinite countable S there is $B(S) = l^\infty$.

Denote by $CB_S(\overline{\Omega})$ the space of mappings

$$w : \overline{\Omega} \ni (t,x) \to \Big(w(t,x) : S \ni i \to w^i(t,x) \in R\Big) \in B(S),$$

where the functions w^i are continuous and bounded in $\overline{\Omega}$ with the finite norm

$$\|w\|_0 := \sup\{|w^i(t,x)| : (t,x) \in \overline{\Omega}, i \in S\}.$$

Remark 2. $\Big(CB_S(\overline{\Omega}),\ \|\cdot\|_0\Big)$ is a Banach space.

A function u is said to be a *classical* (or *regular*) *solution* of the system (1) in $\overline{\Omega}$ if $u \in C(\overline{\Omega})$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_j \partial x_k}$ $(j,k = 1,\ldots,m)$ exist and are continuous in Ω, and u satisfies (1) in Ω.

For system (1) we consider the Cauchy problem:

Find a classical solution u of system (1) in $\overline{\Omega}$ fulfilling the initial condition

$$u(0,x) = \varphi(x) \text{ for } x \in R^m. \tag{2}$$

Now to prove the existence and uniqueness of the solution to this problem, we shall apply the Banach fixed point theorem (see [**7**], [**10**]). Considering mainly Banach spaces of continuous and bounded functions, we give some natural sufficient conditions for the existence and uniqueness.
We notice that finite systems of differential-functional equations with the Hale-type functional was studied by H.Leszczyński [**9**]. Infinite systems of parabolic differential-functional equations with the Fourier first initial-boundary condition have been studied by S.Brzychczy [**2**]–[**5**].

2. Notations, definitions and assumptions. A *fundamental solution* $\Gamma^i(t,x;\tau,\xi)$ of the equation $\mathcal{F}^i[u^i] = 0$ in $\overline{\Omega}$ is a function defined for all $(t,x) \in \overline{\Omega}$, and $(\tau,\xi) \in \overline{\Omega}$, where $t > \tau$, which satisfies the following conditions (see [**7**, p. 3]):

(i) for any fixed $(\tau,\xi) \in \overline{\Omega}$ it satisfies, as a function $(t,x) \in \Omega$ for $t > \tau$, the equation $\mathcal{F}^i[u^i] = 0$,

(ii) for any continuous function $h = h(x)$ in $x \in R^m$ there is

$$\lim_{t \searrow \tau} \int_{R^m} \Gamma^i(t,x;\tau,\xi) h(\xi) d\xi = h(x).$$

We assume that:

($\mathbf{H}_\alpha$) the coefficients $a^i_{jk}(t,x)$ $(i \in S,\ j,k = 1,\dots,m)$, $a^i_{jk}(t,x) = a^i_{kj}(t,x)$ satisfy the following *Hölder continuous condition with exponent* α $(0 < \alpha \le 1)$ in $\overline{\Omega}$:

$$\exists\ H > 0\ \forall\ i \in S\ \forall\ j,k = 1,\dots,m\ \forall\ (t,x) \in \overline{\Omega}\ \forall\ (t',x') \in \overline{\Omega} :$$

$$|a^i_{jk}(t,x) - a^i_{jk}(t',x')| \le H(\|x-x'\|^\alpha + |t-t'|^{\frac{\alpha}{2}})$$

We also assume that the operators $\mathcal{F}^i$ $(i \in S)$ are uniformly parabolic in $\overline{\Omega}$ (the operators $\mathcal{A}^i$ are uniformly elliptic in $\overline{\Omega}$), i.e.

$$\exists\ \mu_1,\mu_2 > 0\ \forall\ \xi = (\xi_1,\dots,\xi_m) \in R^m\ \forall\ (t,x) \in \overline{\Omega}\ \forall\ i \in S :$$

$$\mu_1 \sum_{j=1}^m \xi_j^2 \ge \sum_{j,k=1}^m a^i_{jk}(t,x)\xi_j\xi_k \ge \mu_2 \sum_{j=1}^m \xi_j^2$$

and the functions

$$f^i : \overline{\Omega} \times CB_S(\overline{\Omega}) \ni (t,x,s) \to f^i(t,x,s) \in R,\ i \in S,$$

are continuous and satisfy:

($\mathbf{L}$) *the Lipschitz condition* in s uniformly with respect to (t,x), i.e.

$$\forall\ i \in S\ \exists\ L > 0\ \forall\ (t,x) \in \overline{\Omega}\ \forall s, \tilde{s} \in CB_S(\overline{\Omega}) :$$

$$|f^i(t,x,s) - f^i(t,x,\tilde{s})| \le L\|s-\tilde{s}\|_0.$$

($\mathbf{V}$) *the Volterra condition*, i.e. for arbitrary $(t,x) \in \Omega$ and arbitrary $\eta, \tilde{\eta} \in CB_S(\overline{\Omega})$ such that $\eta^j(\bar{t},x) = \tilde{\eta}^j(\bar{t},x),\ j \in S$ for $0 \le \bar{t} \le t$ is $f^i(t,x,\eta) = f^i(t,x,\tilde{\eta})$ $(i \in S)$.

Let $\eta \in CB_S(\overline{\Omega})$. We define the nonlinear Nemytskiĭ operator $\mathbf{F} = (\mathbf{F}^1, \mathbf{F}^2, ...)$

$$\mathbf{F} : \eta \to \mathbf{F}[\eta],$$

setting

$$\mathbf{F}^i[\eta](t,x) := f^i(t,x,\eta), \quad i \in S.$$

Using Nemytskiĭ operator we can rewrite the Cauchy problem (1), (2) in the form

$$\begin{cases} \mathcal{F}^i[u^i](t,x) = \mathbf{F}^i[u](t,x), & (t,x) \in \Omega,\ i \in S, \\ u(0,x) = \varphi(x), & x \in R^m. \end{cases} \tag{3}$$

LEMMA 1. *If the continuous function* $f = (f^1, f^2, ...)$ *generating the Nemytskiĭ operator* $\mathbf{F}$ *satisfies the Lipschitz condition* ($\mathbf{L}$) *and*

$$\exists\ M_0 > 0\ \forall\ i \in S\ \forall\ (t,x) \in \overline{\Omega}\ |f^i(t,x,0)| \le M_0 \tag{$\mathbf{B}$}$$

then

$$\mathbf{F} : CB_S(\overline{\Omega}) \to CB_S(\overline{\Omega}).$$

Proof. For any $z \in CB_S(\overline{\Omega})$, there is

$$|\mathbf{F}^i[z](t,x)| \leq |\mathbf{F}^i[z](t,x) - \mathbf{F}^i[0](t,x)| + |\mathbf{F}^i[0](t,x)| \leq L\|z\|_0 + M_0.$$

□

Lemma 2. (*A.Friedman* [**7**, Theorem 10, p. 23] *or O.A.Ladyzhenskaya, V.A.Solonnikov, N.N.Uraltseva* [**8**, p. 405])
If the operators $\mathcal{F}^i$ $(i \in S)$ are uniformly parabolic in $\overline{\Omega}$ and the coefficients $a^i_{jk}(t,x)$ $(i \in S,\ j,k = 1,\dots,m)$ satisfy the condition $(\mathbf{H}_\alpha)$ and are bounded in $\overline{\Omega}$, then there exists a fundamental solution $\Gamma^i(t,x;\tau,\xi)$ of the equation

$$\mathcal{F}^i[u^i](t,x) = 0.$$

□

Using fundamental solutions $\Gamma^i(t,x;\tau,\xi)$ $(i \in S)$ for the equation $\mathcal{F}^i[u^i](t,x) = 0$, we consider the following infinite system of nonlinear integral equations

$$(4)\qquad u^i(t,x) = \int\limits_{R^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi)\ \mathbf{F}^i[u](\tau,\xi)d\xi\ , \qquad (i \in S).$$

We define a *C-solution* of differential problem (1), (2) in $\overline{\Omega}$ as a function from the space $CB_S(\overline{\Omega})$ which satisfies the system of integral equations (4) in $\overline{\Omega}$.

In this sense the system of integral equations (4) is equivalent to differential problem (1), (2) in $\overline{\Omega}$.

3. Theorem on the existence and uniqueness.

Theorem 1. *Let all the above assumptions hold, f^i $(i \in S)$ satisfy the* $(\mathbf{B})$*-condition and $\varphi \in CB_S(R^m)$. Then there exists a C-solution u of problem (1), (2) in $\overline{\Omega}$.*

Proof. For arbitrary function $z \in CB_S(\overline{\Omega})$ we define a mapping $\mathbf{T}$ setting

$$u = \mathbf{T}[z],$$

where

$$(5)\qquad u^i(t,x) = \int\limits_{R^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[z](\tau,\xi)d\xi\ , \qquad (i \in S).$$

Owing to Lemma 1 it is easily seen that the mapping $\mathbf{T}$ maps the space $CB_S(\overline{\Omega})$ into itself.

Now, we show that this mapping $\mathbf{T}$ is a contraction. To this end in the space $CB_S(\overline{\Omega})$ we introduce the following Bielecki's norm (see A.Bielecki [**1**] or J.Dugundji, A.Granas [**6**, p. 25])

$$\|w\|_{0,\psi} := \sup\left\{ \frac{|w^i(t,x)|}{\psi(t)} : (t,x) \in \overline{\Omega}, i \in S \right\},$$

where

$$\psi(t) = \exp\left\{ \frac{Lt}{\theta} \right\}, \qquad 0 < \theta < 1$$

and we remark that the norms $\|\cdot\|_0$ and $\|\cdot\|_{0,\psi}$ are equivalent in the space $CB_S(\overline{\Omega})$.

Let $z, \ \overline{z} \in CB_S(\overline{\Omega})$ and $u = \mathbf{T}[z], \ \overline{u} = \mathbf{T}[\overline{z}]$. Owing to the definition of the mapping $\mathbf{T}$ and condition ($\mathbf{L}$), there is

$$|u^i(t,x) - \overline{u}^i(t,x)| =$$

$$= \Bigg| \int\limits_{R^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[z](\tau,\xi)d\xi -$$

$$- \int\limits_{R^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[\overline{z}](\tau,\xi)d\xi \Bigg| \le$$

$$\le \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi) \ |\mathbf{F}^i[z](\tau,\xi) - \mathbf{F}^i[\overline{z}](\tau,\xi)| \ d\xi =$$

$$= \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi) \ |f^i(\tau,\xi,z) - f^i(\tau,\xi,\overline{z})| \ d\xi \le$$

$$\le \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi) \ L \ \|z - \overline{z}\|_0 \ d\xi =$$

$$= \int\limits_0^t d\tau \int\limits_{R^m} \Gamma^i(t,x;\tau,\xi) \ L \ \frac{\|z - \overline{z}\|_0}{\psi(\tau)} \ \psi(\tau) \ d\xi \le$$

$$\le \|z - \overline{z}\|_{0,\psi} \int\limits_0^t L \ \psi(\tau) \ d\tau = \ \|z - \overline{z}\|_{0,\psi} \int\limits_0^t L \ \exp\left\{ \frac{L\tau}{\theta} \right\} d\tau \le$$

$$\le \theta \|z - \overline{z}\|_{0,\psi} \ \psi(t).$$

Finally

$$\|u-\overline{u}\|_{0,\psi} \leq \theta \ \|z-\overline{z}\|_{0,\psi}$$

which means that operator $\mathbf{T}$ is a contraction.

Therefore, from the Banach contraction principle it follows that there exists the unique fixed point $u \in CB_S(\overline{\Omega})$ of the mapping $\mathbf{T}$. This means that the infinite system of integral equations (4) has the unique solution $u \in CB_S(\overline{\Omega})$, e.i. there exists the C-solution of the problem (1), (2) in $\overline{\Omega}$. □

REMARK 3. The C-solution of the problem (1), (2) in $\overline{\Omega}$ has continuous first order derivatives in Ω.

Indeed if a function u is a C-solution of the problem (1), (2) in $\overline{\Omega}$, then u is bounded in $\overline{\Omega}$ and, by virtue of Lemma 1, $\mathbf{F}[u]$ is a bounded function too. Therefore $\int\limits_{R^m} \frac{\partial}{\partial x_j}\Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi$ $(j=1,\ldots,m,\ i\in S)$ are uniformly convergent for all t,x,τ, where $(t,x)\in\Omega, \tau<t$.

Thus by differentiating (4) we obtain

$$\frac{\partial}{\partial x_j}u^i(t,x) =$$

$$= \frac{\partial}{\partial x_j}\int\limits_{R^m}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \frac{\partial}{\partial x_j}\int\limits_0^t d\tau \int\limits_{R^m}\Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi =$$

$$= \int\limits_{R^m}\frac{\partial}{\partial x_j}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t d\tau\int\limits_{R^m}\frac{\partial}{\partial x_j}\Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi$$

$$i\in S,\ j=1,\ldots,m.$$

□

Now, under additional assumptions, we prove that a C-solution of the problem (1), (2) in $\overline{\Omega}$ is a classical solution of this problem in $\overline{\Omega}$.

THEOREM 2. *Apart from the assumption of Theorem 1, let:*

(i) *the derivatives* $D_{x_j}f^i$ *and* $D_s f^i(\ j=1,\ldots,m,\ i\in S)$ *of the functions* $f^i : \overline{\Omega}\times CB_S(\overline{\Omega}) \ni (t,x,s) \to f^i(t,x,s)\in R$, $(i\in S)$ *exist and be continuous in* $\overline{\Omega}\times CB_S(\overline{\Omega})$,

(ii) $\forall\ j=1,\ldots,m\ \forall\ s\in CB_S(\overline{\Omega})\ \exists\ N\geq 0\ \|D_{x_j}\mathbf{F}[s]\|_0 \leq N$,

then the C-solution is a classical solution of the problem (1), (2) *in* $\overline{\Omega}$.

PROOF. Let u be the C-solution of the problem (1), (2) in $\overline{\Omega}$, obtained in Theorem 1. It is easy to see that u satisfies the initial condition.

For $x\in R^m,\ t>0$ there exist $\frac{\partial\Gamma^i}{\partial t}$, $\frac{\partial\Gamma^i}{\partial x_j}$, $\frac{\partial^2\Gamma^i}{\partial x_j\partial x_k}$, $(\ i\in S,\ j,k=1,\ldots,m)$; furthermore, for all $i\in S,\ j=1,\ldots,m,\quad \Gamma^i(t,x;\tau,\xi)\frac{\partial}{\partial\xi_j}\mathbf{F}^i[u](\tau,\xi)$ and

$\frac{\partial}{\partial x_k}\Gamma^i(t,x;\tau,\xi)\frac{\partial}{\partial \xi_j}\mathbf{F}^i[u](\tau,\xi)$ are continuous for all (t,x,τ,ξ), where $(t,x)\in\Omega$, $\tau<t$, $\xi<x$, as well as the integrals $\int\limits_{R^m}\Gamma^i(t,x;\tau,\xi)\frac{\partial}{\partial \xi_j}\mathbf{F}^i[u](\tau,\xi)d\xi$ and $\int\limits_{R^m}\frac{\partial}{\partial x_k}\Gamma^i(t,x;\tau,\xi)\frac{\partial}{\partial \xi_j}\mathbf{F}^i[u](\tau,\xi)d\xi$ are uniformly convergent for all t,x,τ, where $(t,x)\in\Omega,\tau<t$; moreover, the following estimate ([**7**, p. 263] or [**8**]) holds:

$$|\frac{\partial}{\partial x_j}\Gamma^i(t,x;\tau,\xi)|\leq\frac{C}{(t-\tau)^{\frac{m+1}{2}}}\exp\left\{-C'\frac{(x-\xi)^2}{t-\tau}\right\}\ (i\in S,\ j=1,\ldots,m)$$

where $C,\ C'$ are positive constants.
Therefore, we get

$$\frac{\partial^2 u^i}{\partial x_j\partial x_k}=$$

$$=\frac{\partial}{\partial x_k}\left[\int\limits_{R^m}\frac{\partial}{\partial x_j}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi+\int\limits_0^t d\tau\int\limits_{R^m}\frac{\partial}{\partial x_j}\Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi\right]=$$

$$=\int\limits_{R^m}\frac{\partial^2}{\partial x_j\partial x_k}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi+\frac{\partial}{\partial x_k}\int\limits_0^t d\tau\int\limits_{R^m}-\frac{\partial}{\partial \xi_j}\Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi=$$

$$=\int\limits_{R^m}\frac{\partial^2}{\partial x_j\partial x_k}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi+\frac{\partial}{\partial x_k}\int\limits_0^t d\tau\int\limits_{R^m}\Gamma^i(t,x;\tau,\xi)\frac{\partial}{\partial \xi_j}\mathbf{F}^i[u](\tau,\xi)d\xi=$$

$$=\int\limits_{R^m}\frac{\partial^2}{\partial x_j\partial x_k}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi+\int\limits_0^t d\tau\int\limits_{R^m}\frac{\partial}{\partial x_k}\Gamma^i(t,x;\tau,\xi)\frac{\partial}{\partial \xi_j}\mathbf{F}^i[u](\tau,\xi)d\xi.$$

Hence the C-solution is a classical solution of the problem (1), (2) in $\overline{\Omega}$.

□

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