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ON NEGATIVE ESCAPE TIME IN SEMIDYNAMICAL SYSTEMS

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Abstract. We present the correction of some incorrectness in the paper [**12**].

The paper [**12**] is of fundamental meaning in the theory of semidynamical systems. In this paper R.C. McCann defined the negative escape time in semidynamical systems which is, intuitively, "the minimum time length of all negative trajectories through x" . This concept is of great importance in many investigations in the theory. In [**12**] several results concerning the negative escape time are presented. Four of them are particularly interesting. The first theorem gives the sufficient condition for the system to be isomorphic to a system (on the same space) which has an infinite negative escape time for each $x \in X$ (Theorem 2.2). Then this theorem is used for the further results. In Chapter 3 and 4 of the paper, semidynamical systems without start points on locally compact spaces are considered. There are shown the theorem about a lower semicontinuity of the negative escape time function (Theorem 3.10) and the theorem on the isomorphism of a system to a system with infinite negative escape time for each x (Theorem 4.1). Finally, the necessary and sufficient condition for the existence of the extension of a semidynamical system to the one-point-compactification space for non-compact phase spaces (Theorem 4.2) is proved. Those results were applied by many authors in many papers; some of them are cited in the references.

However, in the paper [**12**] there are two gaps. Two lemmas which are used in the proofs of main results are false. The first lemma is formulated in the very beginning of the paper (it is left without a proof) and used in the proof of Theorem 2.2. Another false lemma appears in Chapter 3 and is used in the proof of the main theorems of its chapter, and, consequently, is applied also in the proof of the last theorem of the paper.

In this paper we present the correction of those errors. Note that the mistakes are not involved with the most important and difficult parts of the proofs. The main idea of an interesting and stimulating proofs of [**12**] does not change.

We start from basic definitions.

A *semidynamical system* (a *semiflow*) on a topological space X (called a *phase space*) is a pair (X,π) where $\pi : \mathbb{R}_+ \times X \to X$ is a continuous function such that $\pi(0,x) = x$ for any $x \in X$ and $\pi(t,\pi(s,x)) = \pi(t+s,x)$ for every $t,s \in \mathbb{R}, x \in X$. We define a *positive trajectory* of x as $\pi^+(x) = \pi([0,+\infty)\times\{x\})$. For $t \geqslant 0$ and $y \in X$ by $F(t,y)$ we mean $\{z \in X : \pi(t,z) = y\}$. In an analogous way we define $F(\Delta, D)$ for $\Delta \subset [0,+\infty)$ and $D \subset X$. A point $x \in X$ is said to be a *start point* if $F(t,x) = \emptyset$ for $t > 0$. By a *solution through* x we mean a function $\sigma : \Delta \to M$ (where Δ is an interval equal to $[\alpha,0]$ or $(\alpha,0]$, in the second case α may be equal to $-\infty$) such that $\sigma(0) = x$ and $\pi(t,\sigma(u)) = \sigma(t+u)$ for any t,u with $u \in \Delta, t \geqslant 0, t+u \in \Delta$. If a solution σ is maximal (relative to the property of being a solution, with respect to inclusion), we call it a *left maximal solution through* x and its image is called a *negative trajectory through* x. For the basic notions and elementary properties of semidynamical systems, see [**1**], [**11**], [**13**], [**14**], [**15**].

Now we come to the paper [**12**].

In a semidynamical system (X,π) on a Hausdorff space the author introduces the negative escape time of a point $x \in X$ in the following way. The number $m(x)$ is defined as $\inf\{t \geqslant 0 : \pi(t,y) = x$ for some start point $y\}$ if the set of all negative trajectories through x which originate at start points is nonempty and $+\infty$ if this set is empty. Further, $n(x)$ is defined as $\inf\{t \geqslant 0 :$ there exist sequences (t_i) in $\mathbb{R}_+$ and (x_i) in X such that $t_i \to t^-, \pi(t_i,x_i) \to x, x_i \in \pi^+(x_{i+1})$ and (x_i) has no convergent subsequence$\}$ if the set of all negative trajectories through x which do not originate at start points is nonempty and as $+\infty$ if this set is empty ($t_i \to t^-$ means that $t_i \to t$ with each $t_i \leqslant t$). Then the negative escape time $N(x)$ of x is introduced as $N(x) = \min\{n(x), m(x)\}$.

First of all notice that in this definition some formal problems may appear. We mean that it may happen that there are negative trajectories through x which do not originate at start points, however $n(x)$ cannot be defined as we had to take the infimum of the empty set. This is, in particular, in the case where the phase space is compact. Then we cannot find any sequence (x_i) satisfying the required conditions, as any such a sequence contains a convergent subsequence. This is not a serious problem, as is it obvious from the intuitive introduction that in such a case we want $n(x)$ to be equal to infinity, however, the definition should be stated formally with all details pointed out precisely. Thus, we suggest the following definition.

Definition 1. Let $x \in X$. Denote by M_x the set of all α such that $[-\alpha, 0]$ is the domain of a left maximal solution through x (of course, $M_x \neq \emptyset$ if and only if there exists a negative trajectory through x which originates at a start point). Define $N_x = \{t > 0 :$ there exist sequences (t_i) in $\mathbb{R}_+$ and (x_i) in X such that $t_i \to t^-, \pi(t_i, x_i) \to x, x_i \in \pi^+(x_{i+1})$ and (x_i) has no convergent subsequence$\}$. Then $m(x)$ is defined as $\inf M_x$ if $M_x \neq \emptyset$ and as $+\infty$ if $M_x = \emptyset$; $n(x)$ is defined as $\inf N_x$ if $N_x \neq \emptyset$ and as $+\infty$ if $N_x = \emptyset$. By the *negative escape time* $N(x)$ *of* x we mean $N(x) = \min\{n(x), m(x)\}$. For $M \subset X$ we put $n(M) = \inf\{n(x) : x \in M\}$, $m(M) = \inf\{m(x) : x \in M\}$ and $N(M) = \min\{n(M), m(M)\}$.

In [**12**], after the basic definitions are introduced, the following Lemma is stated.

> ***Lemma*** ([**12**, Lemma 1.1]). *Let $x \in X$ and $(x_i), (t_i)$ be sequences in X, $\mathbb{R}_+$, respectively, such that $t_i \to t^-$, $\pi(t_i, x_i) = x$, $x_i \in \pi^+(x_{i+1})$, and (x_i) has no convergent subsequence. Then $n(x_i) \leqslant t - t_i$.*

This Lemma is not true. Consider the following

Example 2. Let X be equal to the interval $(0, 2]$ and define $\pi(t, x) = \min\{t+x, 2\}$. Then 2 is a stationary point, each positive trajectory is eventually stationary. Take $x = 2, x_n = \frac{1}{n}, n \geqslant 1$ and $t_n = 2 - \frac{1}{n+1}$. We have: $t_n \to 2^-, \pi(t_n, x_n) = 2, x_n \in \pi^+(x_{n+1})$ and $n(x_n) = \frac{1}{n} > \frac{1}{n+1} = t - t_n$.

The lemma is important for the further reasoning in [**12**]. Below we present the correction.

First we show

Lemma 3. *Let $x \in X$ and $(x_i), (s_i)$ be sequences in X, $\mathbb{R}_+$, respectively, such that (x_i) has no convergent subsequence and $\pi(s_i, x_i) = x$, $x \notin \pi([0, s_i), x_i)$, $x_i \notin \pi((0, +\infty), x_i)$ and $x_i \in \pi^+(x_{i+1})$ for all i. Then $s_n \leqslant s_{n+1}$ and $\pi(s_{n+1} - s_n, x_{n+1}) = x_n$ for all n.*

Proof. Put $\tau_n = \inf\{\tau \geqslant 0 : \pi(\tau, x_{n+1}) = x_n\}$. The number τ_n is well defined as $x_i \in \pi^+(x_{i+1})$; we have $\tau_n \geqslant 0$. By the continuity of π we have $\pi(\tau_n, x_{n+1}) = x_n$; indeed, if (τ^j) is a sequence such that $\tau^j \to \tau_n$ when $j \to +\infty$ and $\pi(\tau^j, x_{n+1}) = x_n$, then $\pi(\tau_n, x_{n+1}) = x_n$.

Note that:

$$x = \pi(s_n, x_n) = \pi(s_n + \tau_n, x_{n+1}), \tag{3.1}$$

$$x \notin \pi([0, s_n), x_n) = \pi([\tau_n, \tau_n + s_n), x_{n+1}), \tag{3.2}$$

$$x \notin \pi([0, \tau_n), x_{n+1}). \tag{3.3}$$

The properties (3.1) and (3.2) are obvious. To show (3.3) note that if $x \in \pi([0,\tau_n], x_{n+1})$ then there exists a $\lambda_n \in [0,\tau_n)$ with $\pi(\lambda_n, x_{n+1}) = x$. From this we have $\pi(s_n+\tau_n-\lambda_n, x_n) = \pi(\tau_n-\lambda_n, x) = \pi(\tau_n-\lambda_n+\lambda_n, x_{n+1}) = x_n$ which contradicts the assumption that $x_n \notin \pi((0,+\infty, x_n)$.

From (3.1), (3.2) and (3.3) we have $x = \pi(s_n+\tau_n, x_{n+1})$ and $x \notin \pi([0, s_n+\tau_n), x_{n+1})$. On the other hand, $x = \pi(s_{n+1}, x_{n+1})$ and $x \notin \pi([0, s_{n+1}), x_{n+1})$. Thus $s_{n+1} \leqslant s_n+\tau_n \leqslant s_{n+1}$, so $s_n \leqslant s_{n+1}$ and $\pi(s_{n+1}-s_n, x_{n+1}) = \pi(\tau_n, x_{n+1}) = x_n$. We have proved Lemma 3. □

Let us state

Definition 4. We put:
$N_x^* = \{s \geqslant 0 :$ there exist sequences $(x_i), (s_i)$ contained in X and $\mathbb{R}_+$, respectively, such that (x_i) has no convergent subsequence, $\pi(s_i, x_i) = x$, $x_i \in \pi^+(x_{i+1})$, and $\pi(\lambda, x_i) \neq x$ for any $\lambda \in [0, s_i)\}$.

Remark. In the above definition the condition $\pi(\lambda, x_i) \neq x$ for any $\lambda \in [0, s_i)$ is put in the purpose of a suitable change of Lemma 1.1 from [**12**].

Now we show

Lemma 5. *The set N_x is empty if and only if the set N_x^* is empty. Moreover, if any of those sets is nonempty we have* $\inf N_x = \inf N_x^*$.

Proof. We show that

(5.1) $\qquad$ for any $t \in N_x$ there exists an $s \in N_x^*$ such that $s \leqslant t$.

Let $t \in N_x$. There exist sequences (t_i) in $\mathbb{R}_+$ and (x_i) in X such that $t_i \to t^-, \pi(t_i, x_i) \to x, x_i \in \pi^+(x_{i+1})$ and (x_i) has no convergent subsequence. Define $s_n = \inf\{s \geqslant 0 : \pi(s, x_n) = x\}$. Since $\pi(t_n, x_n) = x$, a number s_n is well defined and $s_n \leqslant t_n$ for any n. If any sequence s_n^k converges to s_n when $k \to +\infty$ and $\pi(s_n^k, x_n) = x$ for any k, then $\pi(s_n, x_n) = x$. Thus the sequences (x_n) and (s_n) satisfy the assumptions of [**12**, Lemma 2.2], so $x_n \notin \pi((0,+\infty), x_n)$ for n large enough. Applying Lemma 3 we conclude that (s_n) is an increasing sequence, so it is convergent to s (where s may be also equal to $+\infty$); we have $s_n \to s^-$. Thus $s \leqslant t$ as $s_n \leqslant t_n$ for any n. We have proved (5.1).

Obviously $N_x^* \subset N_x$, so if $N_x^* \neq \emptyset$ then $N_x \neq \emptyset$ and $\inf N_x \leqslant \inf N_x^*$. From (5.1) we conclude that if $N_x \neq \emptyset$ then $N_x^* \neq \emptyset$ and $\inf N_x^* \leqslant \inf N_x$. This finishes the proof of the Lemma. □

Now we may define $n^*(x)$ as $\inf N_x^*$ if $\inf N_x^* \neq \emptyset$ and as $+\infty$ if $\inf N_x^* = \emptyset$. According to Lemma 5 we have $n^*(x) = n(x)$.

Now we have the following

Lemma 6. *Let $x \in X$ and $(x_i), (t_i)$ be sequences in X, $\mathbb{R}_+$, respectively, such that $\pi(t_i, x_i) = x, x_i \in \pi^+(x_{i+1})$, $x \notin \pi([0, t_i), x_i)$ for all i, $t_i \to t^-$ and (x_i) has no convergent subsequence. Then $n(x_i) \leqslant t - t_i$ for n large enough.*

Proof. The sequence (t_i) satisfies the assumptions of Lemma 1.2 in [**12**], so we can find an i_0 such that $x_i \notin \pi((0, +\infty), x_i)$ for $i \geqslant i_0$. The sequences $(x_i)_{i=i_0}^{+\infty}, (t_i)_{i=i_0}^{+\infty}$ satisfy the assumptions of Lemma 3, so $(t_i)_{i=i_0}^{+\infty}$ is increasing and

$$\pi(t_{n+1} - t_n, x_{n+1}) = x_n. \tag{6.1}$$

Let $i \geqslant i_0$. If we find sequences $(y_n), (s_n)$ satisfying the conditions from the definition of $n(x)$ and such that $s_n \to (t - t_i)^-$, we will have $n(x_i) \leqslant t - t_i$ and finish the proof.

Obviously, $y_n \in \pi^+(y_{n+1})$, (y_n) has no convergent subsequence and $s_n = t_{n+i} - t_i \to (t - t_i)^-$ as $n \to +\infty$. We only need to show that $\pi(s_n, y_n) = x_i$ for any n. We prove this by induction. By (6.1), $\pi(t_{i+1} - t_i, x_{i+1}) = x_i$. Now let $\pi(t_{i+k} - t_i, x_{i+k}) = x_i$. Then $\pi(t_{i+k+1} - t_i, x_{i+k+1}) = \pi(t_{i+k+1} - t_{i+k} + t_{i+k} - t_i, x_{i+k+1})$ which by (6.1) is equal to $\pi(t_{i+k} - t_i, x_{i+k}) = x_i$. $\square$

Now, if in the reasoning in [**12**] we use Lemma 6 instead of [**12**, Lemma 1.1], and apply the equality between $n(x)$ and $n^*(x)$, the proofs in Chapter 1 and Chapter 2 follow.

Now we state

Lemma 7. *Put $\widetilde{n}(x) = \inf\{\alpha : (-\alpha, 0]$ is the domain of a left maximal solution through $x\}$. Then $\widetilde{n}(x) = n(x)$.*

Proof. Denote by D_x the set $\{\alpha : (-\alpha, 0]$ is the domain of a left maximal solution through $x\}$. According to [**1**, Lemma 2.2], if $\alpha \in D_x$ and $\alpha < +\infty$, then $\alpha \in N_x$. Thus, if $\widetilde{n}(x) = \inf D_x < +\infty$, then $\widetilde{n}(x) \geqslant n(x)$. This obviously yields to the inequality $\widetilde{n}(x) \geqslant n(x)$ in any case.

If $\widetilde{n}(x) < +\infty$, then applying the definition of N_x and again Lemma 2.2 from [**1**] we deduce that for any $\varepsilon > 0$ there is a $\beta \in (-n(x), n(x)+\varepsilon)$ such that $(\beta, 0]$ is the domain of a left maximal solution through x. Hence $\widetilde{n}(x) \leqslant n(x)$. On the other hand, if $\widetilde{n}(x) = +\infty$ then each left maximal solution through x is defined on the interval $(-\infty, 0]$, so either $N_x = \emptyset$ or $N_x = \{-\infty\}$ and, consequently, $n(x) = +\infty$. We have finished the proof. $\square$

According to the above results, we get immediately

Proposition 8. *The negative escape time $N(x)$ of x is equal to*
$\inf\{\alpha : (-\alpha, 0]$ *or* $[-\alpha, 0]$ *is the domain of a left maximal solution through* $x\}$.

Remark. For many applications, it is more convenient to take as the definition of the negative escape the condition formulated in Proposition 8.

In [**12**], to prove important Theorems 4.1 and 4.2, the author defines $t(x) = \sup\{F([0,t],x)$ is compact$\}$ (for a compact set M we have $t(M) = \sup\{F([0,t],M)$ is compact$\})$ and proves that $N(M) = t(M)$ for a compact set M (Th.3.8). However, earlier the following Lemma is stated and used in the proof of Theorem 3.8.

> ***Lemma*** ([**12**, Lemma 3.7]). *Let (X,π) be a semidynamical system without start points on a locally compact space X and $M \subset X$ be compact. Assume that $t(M) < +\infty$. Then $F(t(M),M)$ is not compact.*

This lemma is not true, as can be shown in the following

Example 9. Consider $X = (\mathbb{R} \times \{0\}) \cup (\{0\} \times (-2,0]) \subset \mathbb{R}^2$ and define π as follows:

$$\pi(t,(x,0)) = (t+x,0)$$

$$\pi(t,(0,y)) = \begin{cases} (0,t+y) & \text{for } t+y \leqslant 0 \\ (t+y,0) & \text{for } t+y \geqslant 0 \end{cases}$$

Obviously X is locally compact and the system has no start points.

Consider $p = (0,0)$. We have $N(p) = t(p) = 2$, $F(2,p) = \{(-2,0)\}$, so $F(t(p),p)$ is compact. In fact, for any s the set $F(s,p)$ is compact as it is either a singleton set or it contains two elements.

Below we present the Lemma in corrected version and the correct proof of Theorem 3.8 in [**12**].

We have

Lemma 10. *Let (X,π) be a semidynamical system without start points on a locally compact space X and $M \subset X$ be compact. Assume that $t(M) < +\infty$. Then $F([0,t(M)],M)$ is not compact.*

Proof. Suppose for the contrary that $F([0,t(M)],M)$ is compact. Then also $F(t(M),M)$ is compact as it is a closed subset of a compact set (we use [**12**, Lemma 3.3],). Take a compact neighbourhood U of $F(t(M),M)$. According to [**1**, Prop. 4.4] there is an $\varepsilon > o$ such that $F([0,\varepsilon],F(t(M),M)) \subset U$. Thus $F([0,\varepsilon],F(t(M),M))$ is compact as a closed subset of U and $F([0,t(M)+\varepsilon],M)) = F([0,t(M)],M) \cup F([0,\varepsilon],F(t(M),M))$ (see [**12**, Lemma 3.5]) is compact. This contradicts the definition of $t(M)$. □

Let us recall [**12**, Theorem 3.8].

Theorem 11. *Let (X,π) be a semidynamical system without start points on a locally compact space X and $M \subset X$ be compact. Then $N(M) = t(M)$.*

Proof. The system has no start points, so $N(M) = n(M)$. For the proof, two inequalities are to be shown: $t(M) \leqslant N(M)$ and $N(M) \leqslant t(M)$. The proof of the first inequality follows as in [**12**, 3.8]. To prove the second inequality suppose to the contrary that $t(M) < n(M)$. Thus $t(M) < +\infty$. According to Lemma 10, the set $F([0, t(M)], M)$ is not compact. Thus there exists a net (z_i) contained in $F([0, t(M)], M)$ which has no convergent subnet. For any i there are an $s_i \in [0, t(M)]$ and a $z_i \in M$ such that $\pi(s_i, z_i) \in M$. Now according to Proposition 8, for any i there is an $y_i \in F(t(M), M)$ such that $\pi(t(M)-s_i, y_i) = z_i$. We claim that (y_i) has no convergent subnet. Indeed, for any i the number $t(M) - s_i$ belongs to the compact interval $[0, t(M)]$, so if (y_{i_k}) is a convergent subnet of (y_i) we may assume without loss of generality that $t(M) - s_{i_k}$ is a convergent subnet of $t(M)-s_i$. Then $z_{i_k} = \pi(t(M)-s_{i_k}, y_{i_k}) \to \pi(t(M)-s, y)$ for some $s \in [0, t(M)]$ and $y \in F(t(M), M)$, so (z_i) has a convergent subnet, a contradiction.

We have shown that there exists a net (y_i) contained in $F(t(M), M)$ which has no convergent subnet. Now the proof follows further as in the paper [**12**]. $\square$

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