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MORE ON THE SHIFT DYNAMICS–INDECOMPOSABLE CONTINUA CONNECTION

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Abstract. If X is a compact, locally connected metric space, $f : X \to X$ is a homeomorphism, and Q is a closed neighborhood of X, then $Z = \{p \in Q : f^n(p) \in Q$ for all integers $n\}$ is the permanent set for f on Q, and $E = \{p \in Q :$ there is some positive integer N_p such that if $n \geq N_p$, then $f^{-n}(p) \in Q\}$ is the entrainment set. In a previous paper, we began a study of the entrainment sets of topological horseshoes, and showed that, under mild conditions, the closure of the entrainment set for a topological horseshoe is "indecomposable–like" in that it admits a continuous map onto an indecomposable continuum. Furthermore, if f denotes the map associated with the topological horseshoe and K denotes the closure of the entrainment set for the horseshoe, then there is a map $\tilde{f}$ on the indecomposable continuum, denoted $\tilde{K}$, and a map $h : K \to \tilde{K}$ such that $h \circ \tilde{f} = f \circ h$, i.e., the dynamics of f on K factors over the dynamics of $\tilde{f}$ on $\tilde{K}$. Here we continue this study of the structure of entrainment sets of topological horseshoes and investigate the presence of invariant indecomposable continua contained in the closure of entrainment sets.

1. Introduction. In a previous paper [**5**], we began a study of the entrainment sets of topological horseshoes, and showed that, under mild conditions, the closure of the entrainment set for a topological horseshoe is "indecomposable–like" in that it admits a continuous map onto an indecomposable continuum. Furthermore, if f denotes the map associated with the topological horseshoe and K denotes the closure of the entrainment set for the horseshoe, then there is a map $\tilde{f}$ on the indecomposable continuum, denoted $\tilde{K}$, and a

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map $h : K \to \tilde{K}$ such that $h \circ \tilde{f} = f \circ h$, i.e., the dynamics of f on K factors over the dynamics of $\tilde{f}$ on $\tilde{K}$. Here we continue this study of the structure of entrainment sets of topological horseshoes and investigate the presence of invariant indecomposable continua contained in the closure of entrainment sets.

A picture such as Figure 1 is often called a topological horseshoe. The figure shows the image of the quadrilateral Q under the action of a diffeomorphism F on the plane, with F being the Poincaré return map obtained from a system of differential equations modelling a fluid flow. Fluid flows (mostly) from left to right in the picture, with a sequence of cylinder obstructions causing the development of topological horseshoes between each pair of cylinders. The term "topological horseshoe" is often used rather loosely in the literature. Here, to make the term mathematically precise, we define it as follows: Suppose X is a metric space, $f : X \to X$ is continuous, Q is a compact subset of X, and A is a closed subset of Q such that $f(A) = A$, and $f \mid A$ factors over the shift on M symbols (with $M \geq 2$). Then we say that f admits a *topological horseshoe.*

We might ask what can be rigorously concluded from a situation such as that depicted in Figure 1 about the points $Z := \{p \in Q : F^n(p) \in Q$ for all integers $n\}$, and the points $E := \{p \in Q :$ there is some positive integer N_p such that if $n \geq N_p$, then $F^{-n}(p) \in Q\}$. We call the set Z the *permanent set* of F on Q, and E the *entrainment set.* (See Figures 2, 3, and 4. Figure 2 shows the permanent set for F, Figure 3 shows the second stage in the development of the entrainment set, and Figure 4 shows the fully developed entrainment set.) The presence of a permanent set Z having the property that the $F|Q$ factors over the shift on M symbols (i.e., F admits a topological horseshoe) occurs frequently when the image of the set Q "crosses Q" more than once in a certain way. The set Z has been extensively studied by many authors. Here, as in [**5**], we focus on the entrainment set E. In the literature it is usually assumed that F is a diffeomorphism and that F is hyperbolic on Q, and in this case we say the example is a Smale horseshoe. It is often easy to verify that F is a diffeomorphism, but it is far more difficult to verify hyperbolicity in a situation such as we have here. Often hyperbolicity is not present.

We are interested in entrainment sets of topological horseshoes for several reasons:

(i) The dynamics on the permanent set in a neighborhood of the topological horseshoe are described "in the large" at least by the dynamics of the shift on M symbols. (See [**15**], [**16**], [**11**], [**9**], [**10**], [**8**], [**2**], [**3**], [**13**], and [**18**].) In particular, although we know there are periodic sets of all periods in the permanent set, we don't know, without further information about the space and the homeomorphism involved, if there are any periodic orbits in those sets. In addition to the usual M–shift dynamics inside the set, interesting behavior and topology can happen *outside* Q

in the entrainment set associated with Q as well. For the Smale horseshoe map the entrainment set consists entirely of points attracted to a fixed point outside Q, but for other examples (even those arising naturally as a result of a model) the entrainment set is much more interesting and complicated, and can contain infinitely many more horseshoes. (See Figure 4.)

(ii) The entrainment sets for a topological horseshoe are physically observable in real experiments in the sense that they can be observed forming. (See [**17**].) Of course, no experiment can reveal the infinitely fine structure of an entrainment set. Nonetheless, the entrainment set can be thought of as the result of pouring dye into a region and then watching it evolve. The entrainment set is the limit as time goes to ∞ of the theoretical position of the dye. Thus, it may well be possible in experiments to measure and compute the entrainment set's fractal dimension, Lyapunov exponents, Hausdorff dimension, etc. (See [**4**].) Cantor sets, or quotient Cantor sets and periodic points, on the other hand, are nearly impossible to observe forming in a real, as opposed to a simulated, flow.

(iii) Similar indecomposable sets often appear as the "strange" sets associated with nonlinear dynamics (e.g., strange attractors, fractal basin boundaries, and closures of unstable and stable manifolds of chaotic saddles, as well as entrainment sets), and, when present, they provide a useful characterization of these phenomena. (See [**16**].)

Why would we say that a continuum that has continuous image an indecomposable continuum is "indecomposable–like"? (Definitions of terms used are given in the next section.) If X is a locally connected continuum, its image must also be locally connected. Thus, a continuum with an indecomposable image cannot be locally connected; in fact, the local connectedness must somehow be "squeezed out" by the associated continuous map. The locally connected part of the topology on X is information lost by the continuous map. This, however, means that it *is* possible to squeeze out the local connectedness via a continuous map. However, while squeezing out the the local connectedness, some of the interesting topology of the entrainment set may also be squeezed out. The boundary of the entrainment set is likely to retain more of the interesting topology of the entrainment set. Thus, having the boundary of the entrainment set, or even an invariant closed subset of that boundary, in a space be indecomposable is an even greater indicator of the presence of a "certain amount of indecomposability" in the set; in particular, it means that the boundary of the entrainment set is quite "fractal" in nature. If the boundary of the entrainment set is indecomposable, then any open set about a boundary point whose closure does not contain the entrainment set would intersect the boundary in an uncountable number of components.

Marcy Barge proved the following theorem in [**1**]. (It is slightly re-worded here.)

Theorem 1.1. Barge's Theorem. *Let F be a C^1 diffeomorphism on an m–manifold $\mathbf{M}$ with p a saddle fixed point of F, one-dimensional unstable manifold $W^u(p)$, and stable manifold $W^s(p)$. In addition, suppose that*

- **(A1)** *the closure K of a branch W' of the unstable manifold $W^u(p)$ is compact;*
- **(A2)** *there is an arc α in W' such that $\alpha \cap W^s(p) \neq \emptyset$ but α is not contained in $W^s(p)$;*
- **(A3)** *there is an essential $m-1$ sphere, denoted by S, contained in $W^s(p)\backslash\{p\}$ such that $S \cap \overline{W'} = \emptyset$.*

Then K is an indecomposable continuum.

The results in this paper are related to Barge's Theorem. When the stable and unstable manifolds of a saddle point intersect transversely, there must be, for some n, an invariant Cantor set C for F^n containing the saddle point p on which the dynamics are those of the two–shift. (See [**14**] for why this is so.) Barge requires hyperbolicity for the fixed point p and a smooth diffeomorphism on a manifold for his results, but he does not require the lockout property for any neighborhood of his Cantor set, nor that any neighborhood of C be isolated. Our results hold for compact locally connected metric spaces and require no differentiability assumptions, but do require the lockout property and an isolated set.

An example of a dynamical system to which both Barge's Theorem and results given here apply is the classical Smale horseshoe. (See [**14**] for a complete discussion.) The rectangle associated with the Smale horseshoe contains the invariant Cantor set Λ in its interior, has the lockout property, and also isolates Λ. Hence, by our results, the closure of the entrainment set of Λ is an indecomposable continuum. (This particular continuum, by the way, is well known to continuum theorists, and has been studied since 1920 or so. It is often called a Knaster bucket handle.) On the other hand, one branch of the unstable manifold of p intersects the stable manifold of p in a point $q \neq p$. The other assumptions of Barge's Theorem are also satisfied. Thus, it follows from Barge's theorem that the closure of this branch of the unstable manifold of p is an indecomposable continuum. Since the closure of this branch of the unstable manifold contains the entire unstable manifold of p and also contains the entire unstable set of Λ, Barge's theorem also yields the fact that the closure of the entrainment set of Λ is an indecomposable continuum.

2. Notation, terminology, and background. If X is a metric space, and A is a subset of X, then we use the notation A°, $\overline{A}$, and ∂A to denote

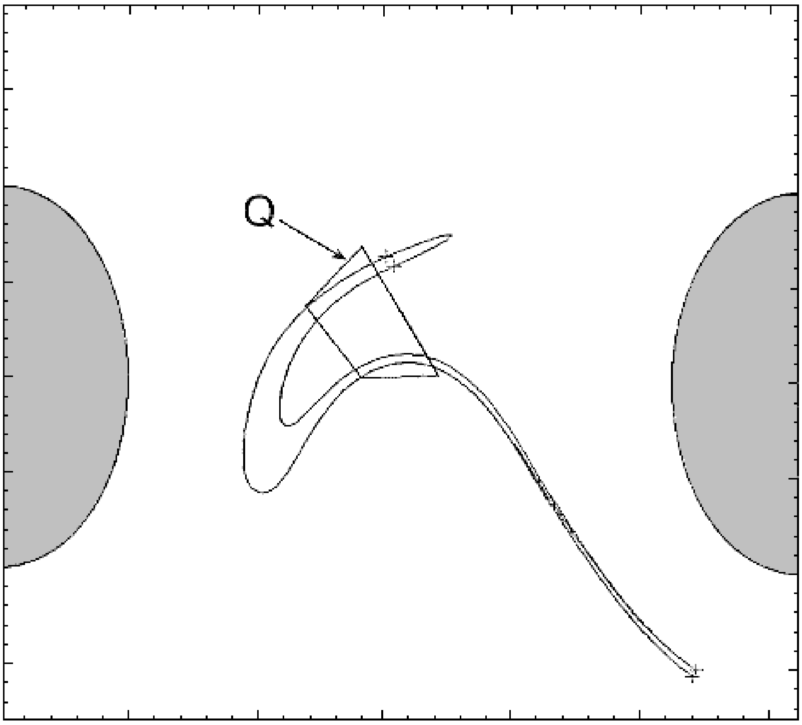

FIGURE 1. The figure, from a study of a model of fluid flow past a sequence of cylinders [**15**], shows a carefully chosen quadrilateral Q, and a horseshoe. The model is studied via its Poincaré map F, a plane diffeomorphism. Numerical evidence strongly suggests that F is hyperbolic on Q, but this would be difficult to verify rigorously. The cylinder obstacles are shaded in the figure. The "horseshoe" is the image of the quadrilateral Q; vertices of the quadrilateral Q are mapped to the crosses indicated.

the interior, closure, and boundary of A in X, respectively. If Y is a subspace of X (with the inherited topology), $A \subset Y$, and we wish to discuss the interior, closure, and boundary of A in the subspace Y, we use the notation $Int_Y(A)$, $Cl_Y(A)$, and $Bdy_Y(A)$, respectively, to avoid confusion. The symbols $\mathbf{Z}$, $\mathbf{N}$, and $\widetilde{\mathbf{N}}$ are used to denote the integers, the positive integers, and the nonnegative integers, respectively. We use d to denote a metric on X (which is compatible with its topology), unless this leads to confusion. If $\epsilon > 0$ and $x \in X$, let $D_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$. If $\epsilon > 0$ and

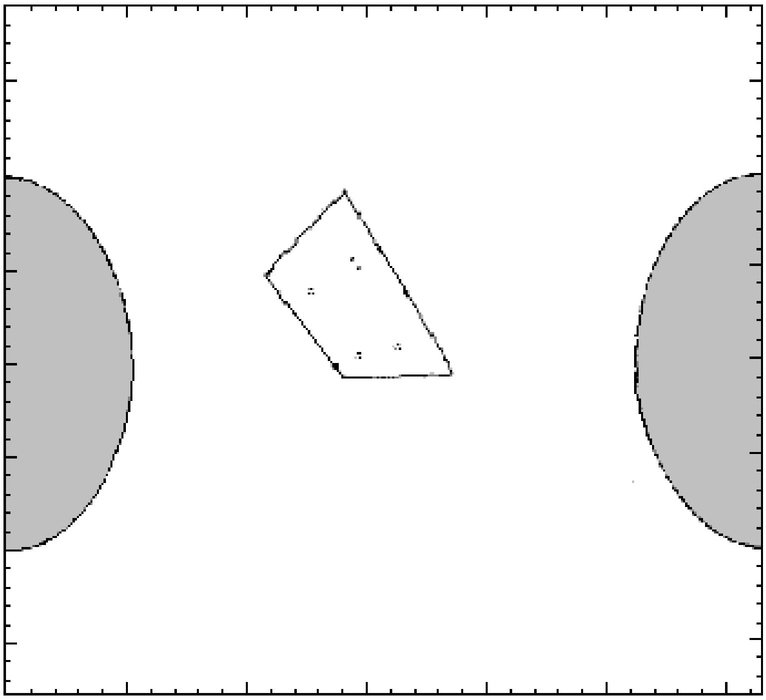

FIGURE 2. The permanent set in the quadrilateral Q for the fluid flow model is depicted.

A, $B \subset X$, let $D_\epsilon(A) = \{y \in X : d(x,y) < \epsilon$ for some $x \in A\}$, and let $d(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$.

Suppose M is a positive integer greater than 1. Then $\sum_M$ denotes the set of all bi-infinite sequences $s = (\ldots s_{-1} \bullet s_0 s_1 \ldots)$ such that $s_i \in \{1, 2, \ldots, M\}$. If for $s = (\ldots s_{-1} \bullet s_0 s_1 \ldots)$ and $t = (\ldots t_{-1} \bullet t_0 t_1 \ldots)$ in $\sum_M$, we define $d(s,t) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}$, then d is a distance function on $\sum_M$. The topological space $\sum_M$ generated by the metric function d is a Cantor set. A natural homeomorphism on the space $\sum_M$ is the *shift homeomorphism* σ defined by $\sigma(s) = \sigma(\ldots s_{-1} \bullet s_0 s_1 \ldots) = (\ldots s_{-1} s_0 \bullet s_1 \ldots) = s'$ for $s = (\ldots s_{-1} \bullet s_0 s_1 \ldots) \in \sum_M$, i.e., $\sigma(s) = s'$, where $s'_i = s_{i+1}$. More specifically, the map σ is called the *shift on M symbols.*

A *continuum* is a compact, connected metric space. A subset of a continuum which is itself a continuum is a *subcontinuum.* The continuum C' is a *proper subcontinuum* of the continuum C if $C' \subset C$, but $C' \neq C$. A continuum

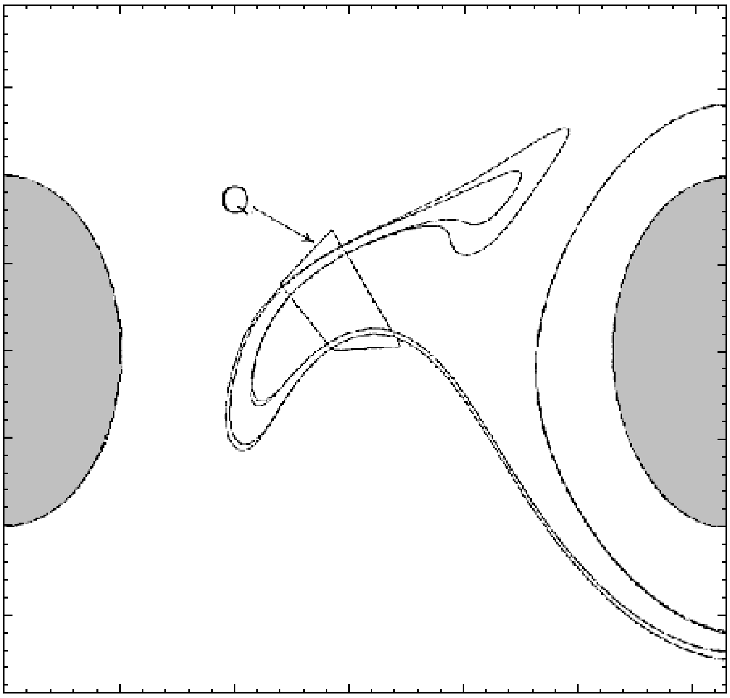

FIGURE 3. The image of Q under the second iterate of the fluid flow map is shown. Note that this image now wraps completely around the downstream cylinder.

is *indecomposable* if it is not the union of two (necessarily overlapping) proper subcontinua. Equivalently, a continuum is indecomposable if and only if every proper subcontinuum has empty interior (relative to the continuum). If x is a point of the continuum X, then the *composant* $Cps(x)$ in X containing x is the set of all points y in X such that there is a proper subcontinuum in X that contains both x and y. The collection $C(X)$ of all composants of an indecomposable continuum X partitions X into c many mutually disjoint, first category, connected F_σ–sets. (For more information and references concerning indecomposable continua and how they arise in dynamical systems, see [**6**], [**7**], and [**12**].)

A space is *irreducible between the points* p *and* q (*closed sets* A *and* B) provided that it is connected and the points p and q (closed sets A and B) cannot be joined by any closed connected set which is different from the whole space. If X is a continuum possessing three distinct points x_0, x_1, and x_2 such

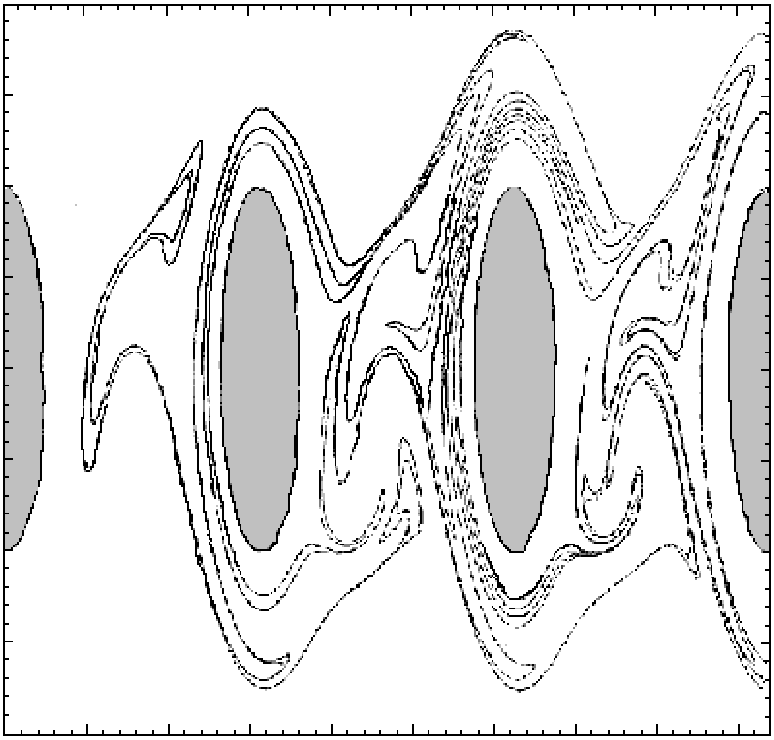

FIGURE 4. The figure shows the fully developed entrainment set. This entrainment set is much more complicated than that for the Smale horseshoe. Its closure contains an infinite collection of nested indecomposable continua (when the plane is compactified). Note how the entrainment set "wraps" around downstream horseshoes and downstream cylinders, making a quite topologically–complicated set.

that X is irreducible between each pair of those points, then X is indecomposable. In fact, if $\mathcal{P}$ is a subset of an indecomposable continuum with the property that no two points of $\mathcal{P}$ are contained in a single composant of $\mathcal{P}$, then $\mathcal{P}$ must be irreducible between each pair of distinct points of $\mathcal{P}$. Thus, irreducibility and indecomposability are related concepts.

A different type of irreducibility plays a big role in our results here: Suppose X and Y are continua, and f is a continuous map from X onto Y. If C is a proper subcontinuum of X implies $f(C) \neq Y$, then X is *irreducible with respect to* f. We need the following theorem from Kuratowski's book:

Theorem 2.1. [**12**, p.208] *If X is a continuum and f is a continuous mapping from X onto the indecomposable continuum Y, then X contains an indecomposable continuum. If X is irreducible with respect to f, then X is indecomposable.*

If X is a space and A is a subset of X, then $\partial A = A \cap (\overline{X \backslash A})$. If $B \subset A$, then B is a *boundary set* in A if $Cl_A(A \backslash B) = A$. Another needed theorem from Kuratowski's book follows. (The theorem in Kuratowski's book is more general; it is re-worded here to reflect our situation.)

Theorem 2.2. [**12**, p. 213] *Every continuum containing a composant which is a boundary set is indecomposable. Consequently, a continuum is indecomposable if and only if it contains a composant which is a boundary set.*

Suppose X is a compact metric space, and $\mathcal{F}$ is a partition of X into closed subsets having the property that if $F_1, F_2, \dots$ is a sequence of members of $\mathcal{F}$, $F \in \mathcal{F}$, $x_1, x_2, \dots$ is a sequence of points with $x_i \in F_i$ converging to the point $x \in F$, then whenever $y_{p_1}, y_{p_2}, \dots$ is a sequence of points with $y_{p_i} \in F_{p_i}$ converging to the point y, then y must also be an element of F. Then $\mathcal{F}$ is an *upper semicontinuous decomposition* of X, and $\mathcal{F}$ is a compact metric space when $\mathcal{F}$ is endowed with the *quotient topology*: a basis for this topology is the collection $\mathcal{B} := \{u : u \text{ is open in } X \text{ and } u \text{ is a union of members of } \mathcal{F}\}$. The map $p : X \to \mathcal{F}$ defined by $p(x) = C_x$ for $x \in X$, where C_x denotes that member of $\mathcal{F}$ that contains x is called the *projection* of X to $\mathcal{F}$. The projection map is continuous, closed and onto. If $\mathcal{F}$ is a Cantor set, we say X is a *quotient Cantor set*; if $\mathcal{F}$ is an indecomposable continuum, we say that X is a *quotient indecomposable continuum.*

Another topology on the collection of closed subsets of a compact metric space that we need is the *Vietoris topology.* For X a compact metric space, $2^X = \{C : C \text{ is a closed subset of } X\}$. Define $\nu(A, B) = \min\{\inf\{\epsilon : B \subset D_\epsilon(A)\}, \inf\{\epsilon : A \subset D_\epsilon(B)\}\}$ for $A, B \in 2^X$. Then ν is a metric on 2^X and 2^X is a compact, metric space. The metric ν is called the *Hausdorff metric* on 2^X.

When two topological spaces X and Y are homeomorphic, they have the same topological properties. If there is a continuous map from X onto Y, then, since some information may be lost, the topologies are related, but not necessarily equivalent. Analogously, if $f : X \to X$ and $g : Y \to Y$ are two dynamical systems, then

(i) f is *conjugate* to g if there is a homeomorphism $h : X \to Y$ such that $g \circ h = h \circ f$; and

(ii) f *factors over* g if there is a continuous map $h : X \to Y$ such that $g \circ h = h \circ f$.

If f and g are conjugate, then the dynamics on the two systems are equivalent. If f factors over g, then information may be lost because h is only continuous, and the dynamics on the two systems are not necessarily equivalent, although they are related.

If $F : X \to X$ is a homeomorphism, the closed set B satisfies the *lockout property* if when $q \in B$ and $F^k(q) \notin B$ for some $k > 0$, then further iterates of q remain outside B, i.e., $F^n(q) \notin B$ if $n \geq k$. A closed, invariant set A is *isolated* if there exists a neighborhood Q of A in X such that for each $x \in Q \backslash A$, there is some integer n such that $F^n(x) \notin Q$.

We defined permanent sets and entrainment sets for systems defined on the plane (or a 2–manifold) in the introduction, but those notions can be extended to any dynamical system. If $f : X \to X$ is a continuous map on a metric space X and Q is a closed subset of X, then the set $Z := \{p \in Q : f^n(p) \in Q$ for all integers $n\}$ is the *permanent set* of f on Q, and the set $E := \{p \in Q :$ there is some positive integer N_p such that if $n \geq N_p$, then $f^{-n}(p) \in Q\}$ is the *entrainment set* of f on Q.

3. Needed prior results. Proofs of the following lemmas and theorems are given in [**5**]. Some of the lemmas leading up to the two main theorems are included here because they reveal important aspects of the structure of the entrainment set, aspects needed for the next section. Note, in particular, that the entrainment set of Q does not depend on the choice of Q. It is necessary to choose a Q that is an isolating neighborhood of A, that has the lockout property, and satisfies the technical condition that $F(Q) \cap F^{-1}(Q) \subset Q^{\circ}$, but any such Q will do.

Definition. Suppose B is a closed subset of the compact metric space X, and $F : X \to X$ is a homeomorphism. Let $B^j = \cup_{i \geq j} F^i(B)$, $B_j = \cap_{i \geq j} F^i(B)$, $B^{\infty} = \cap_{j=0}^{\infty} \overline{B^j}$, and $B_{\infty} = \overline{\cup_{j=0}^{\infty} B_j}$.

Lemma 3.1. *Suppose $\tilde{X}$ is a compact, locally connected metric space, $\tilde{F} : \tilde{X} \to \tilde{X}$ is a homeomorphism, $\tilde{A}$ is a closed invariant subset of $\tilde{X}$ such that $\tilde{F}|\tilde{A}$ is conjugate to the shift on M symbols, the closed neighborhood $\tilde{Q}$ of $\tilde{A}$ has the lockout property, and $\tilde{A}$ is isolated in $\tilde{Q}$. Then the closure L of the entrainment set for the permanent set $\tilde{A}$ in $\tilde{Q}$ is an invariant continuum in $\tilde{X}$ which contains $\tilde{A}$. If U is an open subset of $\tilde{Q}$ that contains $\tilde{A}$, then the entrainment set $E_{\overline{U}}$ of $\overline{U}$ is the entrainment set $\tilde{E}$ of $\tilde{Q}$. Furthermore, $L = \tilde{Q}_{\infty} = \overline{U}^{\infty} = \overline{U}_{\infty} = \tilde{Q}^{\infty}$, and the sequence $\overline{U}, \tilde{F}(\overline{U}), \tilde{F}^2(\overline{U}), \ldots$ converges (in the Hausdorff metric) on 2^X to L.*

Notation. For $i \in \widetilde{\mathbf{N}}$ and $x \in \cap_{j=i}^{\infty} \tilde{F}^j(\tilde{Q})$, let $PR_{x,i}$ denote the component of $\cap_{j=i}^{\infty} \tilde{F}^j(\tilde{Q})$ that contains x, and let $PR_{x,\infty} = \cup_{j=0}^{\infty} PR_{x,j}$. Since $PR_{x,m} \subset PR_{x,m+1} \subset \cdots$, $PR_{x,\infty}$ is connected. Moreover, if $x \in \tilde{Q}_0$, $\tilde{F}(PR_{x,0}) = PR_{\tilde{F}(x),1}$, or more generally, for $x \in \tilde{Q}_j$, $\tilde{F}^n(PR_{x,j}) = PR_{\tilde{F}^n(x),n+j}$ for $n > j$. Let $\mathcal{PR} = \{PR_{x,\infty} : x \in \tilde{Q}_0\}$. The collection $\mathcal{PR}$ partitions the entrainment set of $\tilde{Q}$ into an uncountable collection of mutually disjoint connected sets, and $\tilde{F}$ respects the partition. Hence, $PR_{x,\infty}$ is the member of the equivalence class $\mathcal{PR}$ that contains x. Since each $z \in \tilde{E}$ is contained in some $PR_{z',\infty}$, $\cup\mathcal{PR} = \cup_{i=0}^{\infty} \tilde{Q}_i = \tilde{E}$.

Lemma 3.2. *Suppose $\tilde{X}$ is a compact, locally connected metric space, $\tilde{F} : \tilde{X} \to \tilde{X}$ is a homeomorphism, $\tilde{A}$ is a closed invariant subset of $\tilde{X}$ such that $\tilde{F}|\tilde{A}$ is conjugate to the shift on M symbols, the closed neighborhood $\tilde{Q}$ of $\tilde{A}$ has the lockout property, and $\tilde{A}$ is isolated in $\tilde{Q}$. If $\tilde{E}$ denotes the entrainment set of $\tilde{A}$ in $\tilde{Q}$, then $\tilde{E}$ is connected. If z and z' are points of $\tilde{A}$, $z' \in \overline{PR_{z,\infty}}$. If U is an open set such that $\tilde{A} \subset U \subset \overline{U} \subset \tilde{Q}$ and $x \in \tilde{E} \cap U$ and $z \in \tilde{A}$, then the component C of $\tilde{E} \cap \overline{U}$ that contains x intersects $\overline{PR_{z,\infty}}$ and the component C' of $\tilde{E} \cap \overline{U}$ that contains z intersects $\overline{PR_{x,\infty}}$.*

Theorem 3.3. *Suppose $\tilde{X}$ is a compact, locally connected metric space, $\tilde{F} : \tilde{X} \to \tilde{X}$ is a homeomorphism, $\tilde{A}$ is a closed invariant subset of $\tilde{X}$ such that $\tilde{F}|\tilde{A}$ is conjugate to the shift on M symbols, the closed neighborhood $\tilde{Q}$ of $\tilde{A}$ has the lockout property, and $\tilde{A}$ is isolated in $\tilde{Q}$. Suppose also that $F(\tilde{Q}) \cap F^{-1}(\tilde{Q}) \subset \tilde{Q}^{\circ}$. Then the closure L of the entrainment set for the permanent set $\tilde{A}$ in $\tilde{Q}$ is an invariant continuum, and there is an indecomposable continuum $\widetilde{L}$ containing $\sum_M$ such that*

- **(a)** *there is a homeomorphism g from $\widetilde{L}$ onto $\widetilde{L}$ extending the M–shift on $\sum_M$,*
- **(b)** *there is a continuous map $\tilde{p}$ from L onto $\widetilde{L}$ extending $\tilde{h} : \tilde{A} \to \sum_M$, and*
- **(c)** *$(\tilde{F}|L) \circ \tilde{p} = \tilde{p} \circ g$.*

Remark 3.1. In Theorem 3.3, if $PR_{z,\infty} \in \mathcal{PR}$, $\tilde{p}(PR_{z,\infty})$ is dense in a composant of $\tilde{L}$. If $PR_{z,\infty}$ and $PR_{z',\infty}$ are different members of $\mathcal{PR}$, $\overline{PR_{z,\infty}} \cap PR_{z',\infty} \neq \emptyset$.

Up to this point, the prior results have addressed *only* the case where $\tilde{F}|\tilde{A}$ is conjugate to the shift on M symbols. The next lemma shows that a version of the previous theorem holds for the case where $F|A$ factors over the shift on M symbols.

LEMMA 3.4. *Suppose X is a locally connected, compact metric space, A is a quotient Cantor set in X, $F : X \to X$ is a homeomorphism with $F(A) = A$, and $F|A$ factors over the M–shift $\sigma : \sum_M \to \sum_M$ via the continuous map $h : A \to \sum_M$. There is an upper semicontinuous decomposition $\widetilde{X}$ of X with associated projection map $\pi : X \to \widetilde{X}$ such that*

- **(1)** *$\widetilde{X}$ is a locally connected, compact metric space,*
- **(2)** *$\widetilde{A} := \pi(A)$ is a Cantor set in $\widetilde{X}$, and*
- **(3)** *$F : X \to X$ preserves the decomposition and therefore induces a homeomorphism $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ such that $\widetilde{F}(\widetilde{A}) = \widetilde{A}$, and $\widetilde{F}|\widetilde{A}$ is conjugate to the M–shift $\sigma : \sum_M \to \sum_M$.*

THEOREM 3.5. *Suppose X is a compact, locally connected metric space, $F : X \to X$ is a homeomorphism, A is a closed invariant subset of X such that $F|A$ factors over the shift on M symbols via the continuous map $h : A \to \sum_M$, the closed neighborhood Q of A has the lockout property, and A is isolated in Q. Suppose also that $F(Q) \cap F^{-1}(Q) \subset Q^\circ$. Then the closure K of the entrainment set for the permanent set A in Q is an invariant closed set, and there is an indecomposable continuum $\widetilde{K}$ containing $\sum_M$ such that*

- **(a)** *there is a homeomorphism $\widetilde{f}$ from $\widetilde{K}$ onto $\widetilde{K}$ extending the M–shift on $\sum_M$,*
- **(b)** *there is a continuous map p from K onto $\widetilde{K}$ extending $h : A \to \sum_M$, and*
- **(c)** *$F|K \circ p = p \circ \widetilde{f}$.*

NOTATION. Now that $F \mid A$ factors over the shift rather than being conjugate to the shift, the members of the equivalence class $\mathcal{R}$ which corresponds to the equivalence class $\mathcal{PR}$ for the conjugate-to-the-shift case need not be connected. We use the upper semicontinuous decomposition and projection map π from Lemma 3.4 to define the new equivalence classes: For $i \in \widetilde{\mathbf{N}}$ and $x \in \cap_{j=i}^\infty F^j(Q)$, let $R_{x,i} = \pi^{-1}(PR_{\pi(x),i})$, and let $R_{x,\infty} = \pi^{-1}(PR_{\pi(x),\infty}) = \cup_{j=0}^\infty \pi^{-1}(PR_{\pi(x),j})$. Note that $R_{x,m} \subset R_{x,m+1} \subset \cdots$. Moreover, if $x \in Q_0$, $F(R_{x,0}) = R_{F(x),1}$, or more generally, for $x \in Q_j$, $F^n(R_{x,j}) = R_{F^n(x),n+j}$ for $n > j$. Let $\mathcal{R} = \{R_{x,\infty} : x \in Q_0\}$. The collection $\mathcal{R}$ partitions the entrainment set of Q into an uncountable collection of mutually disjoint sets, and F respects the partition. Hence, $R_{x,\infty}$ is the member of the equivalence class $\mathcal{R}$ that contains x. Since each $z \in E$ is contained in some $R_{z',\infty}$, $\cup\mathcal{R} = \cup_{i=0}^\infty Q_i = E$.

REMARK 3.2. In Theorem 3.5, if $R_{z,\infty} \in \mathcal{R}$, $p(R_{z,\infty})$ is dense in a composant of $\widetilde{K}$. If $R_{z,\infty}$ and $R_{z',\infty}$ are different members of $\mathcal{R}$, $\overline{R_{z,\infty}} \cap R_{z',\infty} \neq \emptyset$.

4. The results. We use the notation and terminology of the previous section. Theorem 3.5 guarantees that the closure of the entrainment set E of

A is an invariant quotient indecomposable continuum. However, $\overline{E}$ need not be a continuum.

The following results have been used implicitly, or proved for special cases, in a number of papers of the author, including the one that precedes this one. We prove them here for the general case. These results may appear elsewhere in the literature, but we don't know where. For completeness, we include them here.

LEMMA 4.1. *Suppose X is a locally compact, separable, connected metric space and X is indecomposable, but not compact. Then there is a sequence $Q_0, Q_1, \ldots$ of compact subsets of X such that $Q_i \subset Q_{i+1}^{\circ}$, $X = \cup_{i=0}^{\infty} Q_i$, and $X \neq \cup_{i=0}^{n} Q_i$ for any n.*

PROOF. Since X is second countable and locally compact, but not compact, it has a countable basis $\mathcal{B} = \{b_1, b_2, \ldots\}$ of open sets such that $\overline{b_i}$ is compact for each i, and no finite subcollection of $\mathcal{B}$ covers X. We may assume that each b_i is nonempty. Let $Q_0 = \overline{b_1}$. Let $\mathcal{B}_1'$ denote the collection of all members of $\mathcal{B} \setminus \{b_1\}$ that intersect b_1. Since Q_0 is compact, some finite subcollection $\mathcal{B}_1$ of $\mathcal{B}_1'$ covers Q_0. Let $Q_1 = \{\overline{b_2}\} \cup (\overline{\cup \mathcal{B}_1})$. Then $Q_0 \subset Q_1^{\circ}$. We continue this process: Since Q_1 is compact, some finite subcollection $\mathcal{B}_2$ of $\mathcal{B}_2'$, the collection of all members of $\mathcal{B} \setminus (\{b_1, b_2\} \cup \mathcal{B}_1)$ that intersect Q_1, covers Q_1. Let $Q_2 = \{\overline{b_3}\} \cup (\overline{\cup \mathcal{B}_2})$. Then, by construction, $Q_2 \subset Q_1^{\circ}$.

At each step, the set Q_n will be chosen so that $Q_n \subset Q_{n-1}^{\circ}$. Also, $\overline{b_{n+1}} \subset Q_n$. Then $X = \cup_{i=0}^{\infty} Q_i$, and $X \neq \cup_{i=0}^{n} Q_i$ for any n. □

Suppose X is a space. If $x \in X$, the *continuum component* of x is the union of all continua contained in X that contain x. in Q_j.

THEOREM 4.2. *Suppose X is a locally compact, separable, connected metric space and X is indecomposable, but not compact. Suppose further that each continuum component of a point in X is dense in X. Then if X^* is a metric compactification of X, X^* is an indecomposable continuum.*

PROOF. Suppose, to the contrary, that X^* is not indecomposable. Then there is a proper subcontinuum H of X^* that has nonempty interior in X^*. Since $X^* \setminus X$ is closed and nowhere dense in X^*, H must contain an open set o such that $\overline{o}$ is compact, and $\overline{o} \subset Int_{X^*}(H)$.

Suppose $Q_0, Q_1, \ldots$ is a sequence of compact subsets of X such that $Q_i \subset Q_{i+1}^{\circ}$, $X = \cup_{i=0}^{\infty} Q_i$, and for each n, $X \neq \cup_{i=0}^{n} Q_i$. Note that $Q_i \cap (X^* \setminus X) = \varnothing$ for each i. Choose $x_0 \in X$ such that $x_0 \notin H$. For some j_1, $x \in Q_{j_1}$ and $o \subset Q_{j_1}^{\circ}$. Since C_{x_0} is dense in X, there is some $y \in C_{x_0} \cap o$. Then there is a continuum C_1 containing x_0 and y and contained in C_{x_0}, and C_1 is a subset of Q_{j_2} for some $j_2 > j_1$. Let C_2 denote the component of Q_{j_2} containing x_0, and let C_3 denote the component of $Q_{j_2} \setminus o$ containing x_0. Then $C_3 \subset C_2 \subset C_{x_0}$, and $C_3 \cap \partial o \neq \varnothing$. Moreover, C_2 is nowhere dense in Q_{j_2}, so there is an open set u contained in $Q_{j_2}^{\circ}$ such that $\overline{u} \cap (C_2 \cup H) = \varnothing$.

Again, since C_{x_0} is dense in X, there is some point $z \in u$ such that $z \in C_{x_0}$. Then there is a continuum C_4 containing x_0 and z in X. There is some $j_3 > j_2$ such that $C_4 \subset Q_{j_3}$. Then $C_4 \subset C_5$, the component of x_0 in Q_{j_3}, and if C_6 denotes the component of $Q_{j_3} \backslash u$ that contains x_0, then C_6 must intersect ∂u. Furthermore, C_6 contains C_2, and C_6 is nowhere dense in Q_{j_3}.

There is $\epsilon > 0$ such that $D_\epsilon(C_6)$ does not contain o. For each w in $C_6 \backslash \overline{u}$, there is $\epsilon/2 > \epsilon_w > 0$ such that $D_{2\epsilon_w}(w)$ does not intersect $\overline{u}$ and $D_{2\epsilon_w}(w) \subset Q_{j_3+1}$. Let $D = Cl_X(\cup\{D_{\epsilon_w}(w) : w \in C_6 \backslash \overline{u}\}) \subset Q_{j_3+1}$. Suppose that $\mathcal{D}$ denotes the upper semicontinuous decomposition of D into its components. That is, we are considering the space $\mathcal{D}$ whose points are the components of D endowed with the quotient topology. Note that C_6 is a component of D, so $C_6 \in \mathcal{D}$. Because X is an indecomposable, connected, completely metrizable space (although it is not complete in X^*), there must be an uncountable number of continuum components comprising X, and each of these continuum components is dense in X. Then $\mathcal{D}$ must be totally disconnected, and is a perfect, compact metric space. Then $\mathcal{D}$ is a Cantor set. Let $P : D \to \mathcal{D}$ denote the projection map associated with the decomposition. The map P is continuous and onto. Also, D does not contain o but $D \cap o \neq \varnothing$, since $C_6 \cap o \neq \varnothing$. There is a closed and open set $\mathcal{O}$ of $\mathcal{D}$ that contains the point C_6 of $\mathcal{D}$, and does not contain any point of $\mathcal{D}$ that is the image of a component of D that intersects $H \cap \partial D$. (Since C_6 does not intersect $H \cap \partial D$, there cannot be members of $\mathcal{D}$ which do intersect $H \cap \partial D$ and also contain points arbitrarily close to a point of C_6. Since $\mathcal{D}$ is an upper semicontinuous decomposition of D, this would mean that C_6 also contained points of $H \cap \partial D$.) Then $P^{-1}(\mathcal{O})$ is both open and closed relative to D, and $C_6 \subset P^{-1}(\mathcal{O})$. But then $H = (H \cap P^{-1}(\mathcal{O})) \cup (H \backslash P^{-1}(\mathcal{O}))$, neither of which is empty. Since $P^{-1}(\mathcal{O})$ is closed in D, it is closed in X and X^*. Since $P^{-1}(\mathcal{O})$ is open in D and does not intersect $H \cap \partial D, P^{-1}(\mathcal{O}) \cap H$ is open in H, and $H \backslash P^{-1}(\mathcal{O})$ is closed in H. Then H is not connected. This is a contradiction. $\square$

Next, we use the Smale horseshoe to construct several examples that illustrate possible properties of $\overline{E}$. (See [**14**, p. 277–281] for a detailed discussion of this example. Robinson calls the example a geometric horseshoe.) Suppose D is a stadium-shaped region in the plane, and so that the space is compact, consider the one-point compactification of the plane, which gives us S^2. Let $f : S^2 \to S^2$ be a homeomorphism such that $f|D$ is a Smale horseshoe map (with two or more crossings). Let $\Lambda = \cap_{i=-\infty}^{\infty} f^i(D)$, let D' denote the rectangle contained in the stadium that contains Λ in its interior, and let $S = \cap_{i=0}^{\infty} f^i(D)$.

Then Λ is the Cantor set with M–shift dynamics, and S is the invariant indecomposable continuum associated with a Smale horseshoe. We refer to it as the *Smale horseshoe continuum*.

Example 4.1. For some $k > 1$, let $T = \{t_1, t_2, \ldots t_k\}$ denote a finite space endowed with the discrete topology and let $\beta : T \to T$ denote the homeomorphism defined by $\beta(t_i) = t_{i+1}$ for $1 \leq i < k$, $\beta(t_k) = t_1$. Then $f \times \beta : S^2 \times T \to S^2 \times T$ is a homeomorphism. Furthermore, $\Lambda \times T$ is an invariant subset of $D \times T$ under the action of $f \times \beta$, and there is a map $h : \Lambda \times T \to \sum_M$ such that $h \circ ((f \mid \Lambda) \times \beta) = \sigma \circ h$. Thus, this dynamical system satisfies the conditions of Theorem 3.5 (with "X" being $S^2 \times T$ and "Q" being $D' \times T$). However, the closure of the entrainment set of $D' \times T$ is $S \times T$, which is disconnected.

Example 4.2. Now consider the previous example, and make one modification: Choose a fixed point p from Λ, and identify the points (p, t_i) for $1 \leq i \leq k$. Since $f \times \beta$ takes the set $\{(p, t_i)\}$ to itself, a new dynamical system arises, which again satisfies the conditions of Theorem 3.5. In this case the closure of the entrainment set is a continuum, but it is not indecomposable. It is the union of k distinct indecomposable continua intersecting at the point $\{(p, t_i)\}$.

Example 4.3. For some $k > 1$, let $T = \{t_1, t_2, \ldots t_k\}$ denote a finite space endowed with the discrete topology and let $id : T \to T$ denote the identity homeomorphism on T. For $x \in \Lambda$, let $\Psi_x = \{(x, t_i)\}$. Then $\mathcal{M} := \{\Psi_x : x \in \Lambda\} \cup \{\{z\} : z \in (S^2 \backslash \Lambda) \times T\}$ is an upper semicontinuous decomposition of $S^2 \times T$. Furthermore, the homeomorphism $f \times id$ on $S^2 \times T$ respects the decomposition, so it induces a homeomorphism $\tilde{f}$ on the quotient space $\mathcal{M}$, which is a compact, locally connected metric space. Let π_0 denote the natural projection from $S^2 \times T$ onto $\mathcal{M}$. Since $\pi_0(\Lambda \times T)$ is a copy of Λ, this system satisfies the conditions of Theorem 3.3 (with "Q" being $\pi_0(D' \times T)$ and "K" being $\pi_0(S \times T)$). The closure of the entrainment set here is connected, but it is not an indecomposable continuum. Again, it is the union of k distinct indecomposable continua. Each equivalence class $R_{z,\infty}$ is connected and dense in $K = \overline{E} = \pi_0(S \times T)$.

Note that if $\beta : T \to T$ denotes the homeomorphism defined by $\beta(t_i) = t_{i+1}$ for $1 \leq i < k$, $\beta(t_k) = t_1$, then $f \times \beta : S^2 \times T \to S^2 \times T$ is a homeomorphism which also respects the members of the decomposition $\mathcal{M}$. Thus, $f \times \beta$ also induces a homeomorphism $g : \mathcal{M} \to \mathcal{M}$ which satisfies the conditions of Theorem 3.3.

Example 4.4. This example is from a talk given by Marcy Barge several years ago: Choose a composant C from the Smale horseshoe continuum that contains a fixed point and is not accessible from $D \backslash S^2$. This composant is either a folded ray or a folded line. Then "split" along the folded line or folded ray and insert a canal. If this is done carefully (the canal must get very thin as one travels down the line in either direction or down the ray, and the resulting space should still be S^2). Denote the modified stadium region (D with the inserted canal) as $\hat{D}$, and the modified contained rectangular region (D' with the inserted canal) as $\hat{D}'$. Each x in C has now been replaced by a line segment l_x extending from one side of the canal to the other. The homeomorphism f can be modified so that a homeomorphism $\hat{f}$ results and $\hat{f}(l_x) = f(l_y)$ when $f(x) = y$ for $x \in C$. Then $\hat{f}(\hat{D}) \subset \hat{D}$, and $\hat{\Lambda} = \cap_{i=-\infty}^{\infty} \hat{f}^i(\hat{D})$ is an invariant closed set which factors over $\sum_M$. (Note that the components of $\hat{\Lambda}$ are either line segments or points.) In this case, the closure of the entrainment set will have interior in S^2.

Example 4.5. Choose a family $\mathcal{C}$ of countably many composants from the Smale horseshoe continuum such that no member of $\mathcal{C}$ is accessible from $D \backslash S^2$, and choose the family $\mathcal{C}$ so that $f(\cup\mathcal{C}) = \cup\mathcal{C}$. Each member of $\mathcal{C}$ is either a folded ray or a folded line. Again split carefully along each composant of $\mathcal{C}$ and insert a canal, doing this so that the resulting new space is still S^2. Modify the example as in the previous example, with modifications necessary along each split apart composant. The result is a dynamical system satisfying the conditions of Theorem 3.5. The boundary of each "canal" is equal to the boundary of any other "canal" in this example is equal to the boundary of all the canals in this example. Moreover, this boundary is invariant and is a Lakes of Wada continuum, and it is indecomposable. In this case countably many members of $\mathcal{R}$ have interior relative to S^2.

Example 4.6. Let $rot : S^1 \to S^1$be an irrational rotation of the unit circle. Then $f \times rot : S^2 \times S^1 \to S^2 \times S^1$ is a homeomorphism. Furthermore, $\Lambda \times S^1$ is an invariant subset of $D' \times S^1$ under the action of $f \times rot$, and there is a map $h : \Lambda \times S^1 \to \sum_M$ such that $h \circ ((f \mid \Lambda) \times rot) = \sigma \circ h$. Thus, this dynamical system satisfies the conditions of Theorem 3.5 (with "X" being $S^2 \times S^1$ and "Q" being $D' \times S^1$). Here the closure of the entrainment set of $D' \times S^1$ is $S \times S^1$, which is the product of an indecomposable continuum and the unit circle. Note that there are no fixed points or other periodic points in this example, and no invariant indecomposable continua. (Since the product of an indecomposable continuum (S) with another continuum (S^1) is decomposable, the entrainment set here fails to be indecomposable.)

The closure $\overline{E}$ of the entrainment set must however contain an indecomposable continuum whose image is $\widetilde{K}$:

Theorem 4.3. *Suppose X is a compact, locally connected metric space, $F : X \to X$ is a homeomorphism, A is a closed invariant subset of X such that*

$F|A$ factors over the shift on M symbols via the continuous map $h : A \to \sum_M$, the closed neighborhood Q of A has the lockout property, and A is isolated in Q. Then the closure K of the entrainment set for the permanent set A in Q is an invariant closed set and $\overline{K}$ contains an indecomposable continuum K' such that $p(K') = \widetilde{K}$. (Recall that p is the extension of $h : A \to \sum_M$, and $p : K \to \tilde{K}$.)

PROOF. Choose a fixed point q in $\sum_M$. Then $h^{-1}(q)$ is a closed subset of A, and there is some component Q' of Q intersecting $h^{-1}(q)$ such that $h(Q' \cap A)$ contains an open subset of $\sum_M$. Consider the sequence $Q', F(Q'), F^2(Q'), \ldots$ of continua in X. Since X is compact, some subsequence $F^{n_1}(Q'), F^{n_2}(Q'), \ldots$ converges (in the Hausdorff metric) to a continuum $\hat{Q} \subset K$.

Suppose $q \in o$, which is open, and $\overline{o} \subset h(Q' \cap A) \subset \sum_M$. If $\epsilon > 0$, there is an integer N such that if $n \geq N$, each point x of $\sum_M$ is less than ϵ in distance from $\sigma^n(o)$. Then $\sigma^{n_1}(\overline{o}), \sigma^{n_2}(\overline{o}), \ldots$ is a sequence of closed sets that converges to $\sum_M$ (in the Hausdorff metric). Since $h \circ F^{n_i}(Q' \cap A) = \sigma^{n_i} \circ h(Q' \cap A) \supset \sigma^{n_i}(\overline{o})$, $h(\hat{Q} \cap A) = \sum_M$. It follows that $p(\hat{Q}) = \widetilde{K}$ (because not all of the points of $\sum_m$ are in the same composant of $\sum_m$).

Thus, K contains a continuum $\hat{Q}$ which maps onto $\widetilde{K}$. Then it contains a continuum C irreducible with respect to this property, i.e., C has the property that $F(C) = \widetilde{K}$, but if C' is a proper subcontinuum of C, then $F(C') \neq \widetilde{K}$. Then by Theorem 2.1, C is an indecomposable continuum. $\square$

Note that the indecomposable continuum C in K need not be invariant. (It is not invariant in Example 4.1.) However, if we add some fairly natural conditions involving irreducibility, there is an invariant indecomposable continuum in ∂E.

We need to go back to the structure of K first, though. Consider the collection $\mathcal{R} = \{R_{x,\infty} : x \in Q\}$, which partitions the entrainment set E. Each $R_{x,\infty}$ maps into a composant of $\tilde{K}$ under p, and $p(R_{x,\infty})$ is dense in that composant. The collection $\mathcal{R}$ is uncountable, and more importantly, so is $\mathcal{PR}$. The members of $\mathcal{R}$ may not be connected, as is the case in Example 4.1, but the members of $\mathcal{PR}$ are connected. Form a subcollection $\mathcal{R}'$ of $\mathcal{R}$ as follows: Choose a basis $B = \{b_0, b_1, \ldots\}$ for K. Let $\mathcal{B}' = \{b_{\gamma 1}, b_{\gamma 2}, \ldots\}$ denote the subcollection of $\mathcal{B}$ that has the property that each $b_{\gamma i}$ intersects only countably many members of $\mathcal{R}$. (It is possible that $\mathcal{B}' = \emptyset$.) For each $b_{\gamma i} \in \mathcal{B}'$, let $\mathcal{R}_{\gamma i} = \{R_{x,\infty} : R_{x,\infty} \cap b_{\gamma i} \neq \emptyset\}$. Then each $\mathcal{R}_{\gamma i}$ is a countable collection of members of $\mathcal{R}$. Let $\mathcal{R}' = \mathcal{R} \backslash (\cup_{i=1}^{\infty} \mathcal{R}_{\gamma i})$. Then $\mathcal{R}'$ is an uncountable subcollection of $\mathcal{R}$. Note that it follows from results in the previous section that $\overline{\cup \mathcal{R}'}$ must be an invariant closed set, and $\overline{\cup \mathcal{R}'} \subset \partial E = \partial K$.

Theorem 4.4. *Suppose X is a compact, locally connected metric space, $F : X \to X$ is a homeomorphism, A is a closed invariant subset of X such that $F|A$ factors over the shift on M symbols via the continuous map $h : A \to \sum_M$, the closed neighborhood Q of A has the lockout property, and A is isolated in Q. (Again, $p : K \to \tilde{K}$ is the extension of $h : A \to \sum_M$ whose existence is guaranteed in Theorem 3.5.)*

- **(a)** *If some member $R_{z,\infty}$ of $\mathcal{R}'$ is dense in $\cup\mathcal{R}'$, and is connected, then $p(\overline{R_{z,\infty}}) = \tilde{K}$ and $K' := \overline{\cup\mathcal{R}'}$ is an invariant indecomposable continuum in ∂K.*
- **(b)** *If some member $R_{z,\infty}$ of $\mathcal{R}'$ is a composant of $\overline{\cup\mathcal{R}'}$, then $K' := \overline{\cup\mathcal{R}'}$ is an invariant indecomposable continuum in ∂K.*
- **(c)** *If ∂K is a continuum such that no proper subcontinuum maps onto $\tilde{K}$ under p, then ∂K is an invariant indecomposable continuum.*
- **(d)** *If $h^{-1}(x)$ is connected for each $x \in \sum_M$, then K is a continuum. If, in addition, no proper subcontinuum of K maps onto $\tilde{K}$, then K is an indecomposable continuum.*

Proof. (a) Note that by the construction of $\mathcal{R}'$, $K' = \overline{K' \backslash R_{z,\infty}}$. Hence $R_{z,\infty}$ is a boundary set in K'. Also, $\overline{R_{z,\infty}} = K'$, which is therefore an invariant continuum. Also, $\overline{R_{z,\infty}} \cap R_{x,\infty} \neq \emptyset$ for any $R_{x,\infty} \in \mathcal{R}'$. Then $p(\overline{R_{z,\infty}})$ intersects more than one composant of $\tilde{K}$, and $p(\overline{R_{z,\infty}}) = K$. Furthermore, since by construction, no subcontinuum of $\overline{R_{z,\infty}}$ maps onto $\tilde{K}$, $\overline{R_{z,\infty}}$ is an irreducible continuum (i.e., no proper subcontinuum of $\overline{R_{z,\infty}}$ maps onto $\tilde{K}$). Then by Theorem 2.1, K' is an indecomposable continuum, and it is invariant, and contained in ∂K.

(b) If $R_{z,\infty}$ of $\mathcal{R}'$ is a composant of $\overline{\cup\mathcal{R}'}$, then it is dense in K' and connected. The result follows from part (a).

(c) Since $F(\partial K) = \partial K$, it is an invariant continuum. Since it is also irreducible with respect to p, it follows from Theorem 2.1 that ∂K is indecomposable.

(d) If $h^{-1}(x)$ is connected for each $x \in \sum_M$, h must be a homeomorphism from A onto $\sum_m$, and it follows from the results in Section 3 that K is a continuum. Since K is irreducible with respect to p, Theorem 2.1 again implies that K is indecomposable. □

5. Questions. Does Marcy Barge's Theorem hold in higher dimensions? (Recall that his theorem required a one–dimensional unstable manifold.) Might it be true that if X is a compact, smooth manifold, $F : X \to X$ is a hyperbolic diffeomorphism, A is a closed invariant subset of X such that $F|A$ is conjugate to the shift on M symbols, the closed neighborhood Q of A has the lockout property, and A is isolated in Q, then the closure of the entrainment set of Q is an indecomposable continuum?

These questions are interesting, but they will have to wait for a later paper.

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